0.1 Percent of Laura's Dissertation

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1 Smooth Case

Let X and Y be smooth hypersurfaces in a smooth ambient variety M, where $Z = X \cap Y$ is a smooth global complete intersection and X meets Y transversely. If ν_Z, ν_X, ν_Y denote the normal bundles of Z, X, and Y in M, then the transversality of X and Y means that $\nu_Z = \nu_X|_Z \oplus \nu_Y|_Z$. Moreover, the Chern class $c(\nu_Z)$ is given by $1+Z$, where Z denotes the dual of the fundamental class $[Z]$, *i.e.* $Z \cap [M] = [Z]$ (and similarly for $c(\nu_X)$ and $c(\nu_Y)$). Using these facts we can compute the following formula for the MacPherson-Chern class $c_*(Z)$ of Z in terms of X and Y:

$$\begin{aligned}
c_*(Z) &= \text{Dual } (c(TZ)) \\
&= c(TZ) \cap [Z] \\
&= (c(TM|_Z) \cup c(\nu_Z)^{-1}) \cap [Z] \\
&= c(TM|_Z) \cap (c(\nu_Z)^{-1} \cap [Z]) \\
&= c(TM|_Z) \cap ((c(\nu_X|_Z) \cup c(\nu_Y|_Z)) \cap [Z]) \\
&= c(TM|_Z) \cap (i_X^*(1+X) \cup i_Y^*(1+Y)) \cap [Z]) \\
&= c(TM|_Z) \cap \left(\frac{[Z]}{i_X^*(1+X)\, i_Y^*(1+Y)} \right).
\end{aligned}$$

where TZ is the tangent bundle of Z, and i_X, i_Y are the inclusions of Z into X, Y, respectively.

2 Guess for General Formula

Now let $Z = X \cap Y$ be a singular global complete intersection, where $X = \mathcal{V}(f)$ and $Y = \mathcal{V}(g)$ are possibly singular hypersurfaces intersecting transversely in a smooth ambient variety M. Considering the smooth case, and Aluffi's formula for hypersurfaces, we might guess that the general formula is of the form:

$$c_*(Z) = c(TM|_Z) \cap \pi_* \left(\frac{[\mathcal{Z}] - [E_Z]}{i_{\mathcal{X}}^*(1 + \mathcal{X} - E_X)\, i_{\mathcal{Y}}^*(1 + \mathcal{Y} - E_Y)} \right).$$

In the above formula, $\pi: Bl_J M \to M$ is the blowup along some ideal J (which can be made to factor through the blowup along the Jacobian ideal $Jac(f,g)$ consisting of the 2×2 subdeterminants of the Jacobian matrix for f and g, so that the proper transform of Z in the blowup along J will be a generalized Nash blowup of Z). The $\mathcal{Z}$, $\mathcal{X}$, $\mathcal{Y}$ and E_Z, E_X, E_Y then denote the total transforms and exceptional divisors of X, Y, and Z under the blowup, and $i_{\mathcal{X}}$, $i_{\mathcal{Y}}$ denote the inclusions of $\mathcal{Z}$ into $\mathcal{X}$ and $\mathcal{Y}$, respectively.

3 Special Case where Y is a hyperplane H

If the above guess for the formula in the general case is correct, we should have the following formula in the case where $Y = H$ is a hyperplane:

$$c_*(Z) = c(TM|_Z) \cap \pi_* \left(\frac{[\mathcal{Z}] - [E_Z]}{i_{\mathcal{X}}^*(1 + \mathcal{X} - E_X)\, i_{\mathcal{H}}^*(1 + \mathcal{H})} \right). \qquad \mathbf{(1)}$$

To prove the above formula we will need the following lemma:

Lemma *Suppose X is a hypersurface and H a hyperplane in a smooth ambient variety M, meeting transversely not only in the usual sense, but also in the "normal direction". If ν is the normal bundle to $X \cap H$ in X (which exists because of the transversality condition), and $i_{X\cap H}^*$ is the transverse intersection map $H_*(X) \to H_{*-2}(X \cap H)$, Then we have:*

$$c_*(X \cap H) = (1 + i_H^* H)^{-1} \cap i_{X\cap H}^* c_*(X),$$

where $i_H: X \cap H \to H$, $i_H^* H = c_1(\nu)$, and H is the dual of the class $[H]$ in the ambient space M.

Proof of Lemma (smooth case) A simple calculation; let i_X be the inclusion map for $X \cap H$ in X. Then using the fact that

$$TX|_{X \cap H} = \nu \oplus T(X \cap H), \tag{2}$$

and the definition of the transverse intersection map (in the smooth case), we have:

$$\begin{aligned}
(1 + i_H^* H)^{-1} \cap i_{X \cap H}^* c_*(X) &= (1 + i_H^* H)^{-1} \cap i_X^* (\text{Dual } c_*(X)) \cap [X \cap H] \\
&= (1 + i_H^* H)^{-1} \cap i_X^* c(TX) \cap [X \cap H] \\
&= (i_X^* c(TX) \cup (1 + c_1(\nu))^{-1}) \cap [X \cap H] \\
&= (c(TX|_{X \cap H} \cup c(\nu)^{-1}) \cap [X \cap H] \\
&= c(T(X \cap H)) \cap [X \cap H] \\
&= c_*(X \cap H) \qquad \square
\end{aligned}$$

Proof of Lemma Let $\widetilde{X}$ and $\widetilde{Z}$ be generalized Nash blowups of X and $Z = X \cap H$, respectively, so that we have a diagram:

$$\begin{array}{ccc}
\widetilde{T}_Z & \longrightarrow & \widetilde{T}_X \\
\big\downarrow & & \big\downarrow \\
\widetilde{Z} & \hookrightarrow & \widetilde{X} \\
\big\downarrow \phi & & \big\downarrow \phi \\
Z & \stackrel{i_X}{\hookrightarrow} & X
\end{array}$$

where $\widetilde{T}_Z$ and $\widetilde{T}_X$ denote the respective generalized Nash bundles of Z and X. We are abusing notation by writing ϕ for the restriction of ϕ to $\widetilde{Z}$ on the left hand side of the diagram. Note that we are assuming that we have blown up far enough so that ϕ factors through both the Nash blowup of X, and the Nash blowup of Z (when restricted to $\widetilde{Z}$).

We have a statement analagous to (2) in the singular case, namely:

$$\widetilde{T}_X|_Z = \phi^* \nu \oplus \widetilde{T}_Z,$$

which follows from the transversality conditions. Using this fact, we will show that the Lemma holds when the MacPherson-Chern classes $c_*(X \cap H)$ and $c_*(X)$ are replaced by the Mather-Chern classes $c_M(X \cap H)$ and $c_M(X)$.

If $i_{\widetilde{X}}$ is the inclusion of $\widetilde{Z}$ into $\widetilde{X}$, we have:

$$\begin{aligned}
c_M(X\cap H) &= c_M(Z)\\
&= \phi_*(\text{Dual } c(\widetilde{T}_Z))\\
&= \phi_*(\text{Dual } \left(\frac{c(\widetilde{T}_X|_Z)}{c(\phi^*\nu)}\right)\\
&= \phi_*(\text{Dual } \left(\frac{i^*_{\widetilde{X}}c(\widetilde{T}_X)}{\phi^*c(\nu)}\right)\\
&= \phi_*\left(\frac{i^*_{\widetilde{X}}c(\widetilde{T}_X)}{\phi^*c(\nu)}\cap[\widetilde{Z}]\right)\\
&= \phi_*(\phi^*c(\nu)^{-1}\cap(i^*_{\widetilde{X}}c(\widetilde{T}_X)\cap[\widetilde{Z}]))\\
&= c(\nu)^{-1}\cap\phi_*(i^*_{\widetilde{X}}c(\widetilde{T}_X)\cap[\widetilde{Z}]),
\end{aligned}$$

and thus to prove the Lemma (with the MacPherson-Chern classes replaced by Mather-Chern classes), it suffices to prove that:

$$i^*_{X\cap H}c_M(X)=\phi_*(i^*_{\widetilde{X}}c(\widetilde{T}_X)\cap[\widetilde{Z}]).$$

Assuming that $i^*_{\widetilde{X}\cap\widetilde{H}}[\widetilde{X}]=[\widetilde{X}\pitchfork\widetilde{H}]=[\widetilde{Z}]$ (here $\widetilde{H}$ is the proper transform of H under the blowup given by ϕ), we have:

$$\begin{aligned}
i^*_{X\cap H}c_M(X) &= i^*_{X\cap H}\phi_*(\text{Dual } c(\widetilde{T}_X))\\
&= i^*_{X\cap H}\phi_*(c(\widetilde{T}_X)\cap[\widetilde{X}]\\
&= \phi_*i^*_{\widetilde{X}\cap\widetilde{H}}(c(\widetilde{T}_X)\cap[\widetilde{X}]\\
&= \phi_*(i^*_{\widetilde{X}}c(\widetilde{T}_X)\cap i^*_{\widetilde{X}\cap\widetilde{H}}[\widetilde{X}]\\
&= \phi_*(i^*_{\widetilde{X}}c(\widetilde{T}_X)\cap[\widetilde{Z}],
\end{aligned}$$

Now since the Lemma holds for Mather-Chern classes, and the MacPherson-Chern classes are weighted sums of the Mather-Chern classes, the Lemma as originally stated has been shown. □

We now show that in the special case where $Y=H$ is a hyperplane, X is a (possibly singular) hypersurface, and $Z=X\cap H$, we have equation (1). In what follows we assume that the transverse intersection map satisfies the following "projection formula" for any $\alpha\in H^*(X)$ and $a\in H_*(X)$, where $i_X\colon X\cap H\to X$:

$$i^*_{X\cap H}(\alpha\cap a)=i^*_X\alpha\cap i^*_{X\cap H}a.$$

Using this, Aluffi's formula, and the Lemma above, we prove formula (1):

$$
\begin{aligned}
c_*(Z) &= c_*(X \cap H) \\
&= (1 + i_H^* H)^{-1} \cap i_{X\cap H}^* c_*(X) \\
&= (1 + i_H^* H)^{-1} \cap i_{X\cap H}^* \left(c(TM|x) \cap \pi_* \left(\frac{[\mathcal{X}] - [E_X]}{1 + \mathcal{X} - E_X} \right) \right) \\
&= (1 + i_H^* H)^{-1} \cap \left(i_X^* c(TM|X) \cap i_{X\cap H}^* \pi_* \left(\frac{[\mathcal{X}] - [E_X]}{1 + \mathcal{X} - E_X} \right) \right) \\
&= c(TM|Z) \cap \left((1 + i_H^* H)^{-1} \cap i_{X\cap H}^* \pi_* \left(\frac{[\mathcal{X}] - [E_X]}{1 + \mathcal{X} - E_X} \right) \right) \\
&= c(TM|Z) \cap \left((1 + i_H^* H)^{-1} \cap \pi_* i_{\mathcal{X}\cap\mathcal{H}}^* \left(\frac{[\mathcal{X}] - [E_X]}{1 + \mathcal{X} - E_X} \right) \right) \\
&= c(TM|Z) \cap \left((1 + i_H^* H)^{-1} \cap \pi_* \left(\frac{i_{\mathcal{X}\cap\mathcal{H}}^*([\mathcal{X}] - [E_X])}{i_{\mathcal{X}}^*(1 + \mathcal{X} - E_X)} \right) \right) \\
&= c(TM|Z) \cap \pi_* \left(\pi^*(1 + i_H^* H)^{-1} \cap \left(\frac{i_{\mathcal{X}\cap\mathcal{H}}^*([\mathcal{X}] - [E_X])}{i_{\mathcal{X}}^*(1 + \mathcal{X} - E_X)} \right) \right) \\
&= c(TM|Z) \cap \pi_* \left(\frac{i_{\mathcal{X}\cap\mathcal{H}}^*([\mathcal{X}] - [E_X])}{i_{\mathcal{X}}(1 + \mathcal{X} - E_X)\, \pi^*(1 + i_H^* H)} \right) \\
&= c(TM|Z) \cap \pi_* \left(\frac{i_{\mathcal{X}\cap\mathcal{H}}^*[X] - i_{\mathcal{X}\cap\mathcal{H}}^*[E_X]}{i_{\mathcal{X}}(1 + \mathcal{X} - E_X)\, i_H^* \pi^*(1 + H)} \right) \\
&= c(TM|Z) \cap \pi_* \left(\frac{[\mathcal{X} \pitchfork \mathcal{H}] - [E_X \pitchfork \mathcal{H}]}{i_{\mathcal{X}}(1 + \mathcal{X} - E_X)\, i_H^*(1 + \mathcal{H})} \right) \\
&= c(TM|Z) \cap \pi_* \left(\frac{[\mathcal{Z}] - [E_Z]}{i_{\mathcal{X}}(1 + \mathcal{X} - E_X)\, i_H^*(1 + \mathcal{H})} \right).
\end{aligned}
$$

We have used the following notations and assumptions:

- $\pi \colon Bl_J M \to M$ is the blowup along the Jacobian ideal $J = (Jac(f), f)$ for X. Note that if we wanted to we could have instead used $J = (Jac(f,g), f, g) \cdot (Jac(f), f)$; this would give us a blowup that factored through both the Nash blowup of Z, and the blowup that gives us Aluffi's formula (which holds under further blowups). $\mathcal{X}$, $\mathcal{H}$ denote the total transforms of X and H under the blowup $Bl_J M$, and E_X denotes the exceptional divisor over X.

- $i^*_{X\cap H}\pi_* = \pi_* i^*_{\mathcal{X}\cap\mathcal{H}}$, where $i^*_{\mathcal{X}\cap\mathcal{H}}$ is the transverse intersection map for $\mathcal{X}\cap\mathcal{H}$.

- $i^*_{\mathcal{X}\cap\mathcal{H}}[\mathcal{X}]$ is equal to $[\mathcal{X}\pitchfork\mathcal{H}]$, and this is the same class as $[\mathcal{Z}]$; similarly for $i^*_{\mathcal{X}\cap\mathcal{H}}[E_X]$.

4 Special Case where Y is a smooth hypersurface

When Y is a smooth hypersurface, we can always find local ambient coordinates so that Y is given by the vanishing of one of the coordinate functions, *i.e.* Y can be locally thought of as a hyperplane. Thus the arguments given in Section 3 above, which are local in nature, can be generalized to prove the case when Y is a smooth hypersurface.

5 The General Case

It seems that, in the general case, $\pi\colon Bl_J M \to M$ is likely to be the blowup along the product of the Jacobian ideals for X and for Y. Perhaps it will be possible to, say, blow up the Jacobian ideal for Y, and then apply the special case at the blow-up level, to arrive at the general formula?

6 Relaxing Transversality

Everything here requires that Y be transverse to all the strata of X; thus in the simple case where X is a hypersurface with an isolated singular point, and thus having a stratum consisting of only that isolated singular point, Y would have to miss the singular point entirely. Perhaps this transversality condition can be relaxed somehow, so that although not true for the intersection of X and Y, it would be true on the blow-up level? Then we might be able apply the above general case (requiring transversality) to the blow-up level to reach an even more general formula.