4 Percent of Laura's Dissertation

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1 Setup

Let V be a complex algebraic 3–variety with isolated singular point v. Choose a neighborhood U of v in V so that we have an imbedding $(U, v) \hookrightarrow (\mathbb{C}^n, 0)$. Let

$$(\widetilde{U}, E) \xrightarrow{\pi} (U, v)$$

be any resolution of the singularity v that factors through both the Nash blowup and the blowup of the maximal ideal of the singularity. Note that we can assume that the exceptional divisor E of this resolution is a divisor with normal crossings, $E = \bigcup E_i$; since the components E_i are complex 2-dimensional and meet transversely, a point $e \in E$ can be contained in at most three components (and these "triple" points occur only at isolated points of E).

2 Existence of linear forms

3 Existence of h

4 The sequence is exact

Proposition 4.1. *There is an exact sequence of sheaves on $\widetilde{U}$:*

$$\begin{array}{ccc} \mathcal{N}(Z-E) & \stackrel{\alpha}{\hookrightarrow} & \mathcal{I}_E\Omega^1(\log E) \\ & & \searrow^{\beta} \\ & & \left(\Omega^2(\log E)/\bigwedge^2\mathcal{N}(2Z)\right)\otimes\mathcal{O}(-Z-E) \\ & & \gamma\downarrow \\ & & \Omega^3\otimes\mathcal{O}_{P+N-2Z}(-2Z) \end{array}$$

Proof. See the gigantic diagram for the big picture. Since $\mathcal{O}(R)\approx\mathcal{O}(-Z)$ (by multiplication by h; see below), we have:

$$\begin{aligned} &\left(\Omega^2(\log E)/\textstyle\bigwedge^2\mathcal{N}(2Z)\right)\otimes\mathcal{O}(-Z-E) \\ &\quad\approx\ \Omega^2(\log E)\otimes\mathcal{O}(-Z-E)/\textstyle\bigwedge^2\mathcal{N}(2Z)\otimes\mathcal{O}(-Z-E) \\ &\quad\approx\ \mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R)/\textstyle\bigwedge^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(R), \end{aligned}$$

and since $\bigwedge^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2R)=\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2Z-P-N+2R)$ (this will be shown later in this paper),

$$\begin{aligned} &\Omega^3\otimes\mathcal{O}_{P+N-2Z}(-2Z) \\ &\quad\approx\ \mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2R)\otimes\mathcal{O}/\mathcal{O}(-P-N+2Z) \\ &\quad\approx\ \mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2R)/\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2Z-P-N+2R) \\ &\quad\approx\ \mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2R)/\textstyle\bigwedge^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2R), \end{aligned}$$

the sequence above is equivalent to the sequence:

$$\begin{array}{ccc} \mathcal{N}(Z-E) & \stackrel{\alpha}{\hookrightarrow} & \mathcal{I}_E\Omega^1(\log E) \\ & & \searrow^{\beta} \\ & & \mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R)/\bigwedge^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(R) \\ & & \gamma\downarrow \\ & & \mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2R)/\bigwedge^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2R); \end{array}$$

Thus to prove the proposition it suffices to prove that this new sequence is exact. We will first show that we have an injection $\alpha:\mathcal{N}(Z-E)\hookrightarrow\mathcal{I}_E\Omega^1(\log E)$.

Since the (dual) Nash sheaf $\mathcal{N}$ is generated by $\{d\xi_1, d\xi_2, d\xi_3\}$, we have (this computation assumes we are at a triple point e of E; for the double and simple point cases, simply replace uvw with uv or u, respectively):

$$\begin{aligned}
\mathcal{N}(Z-E) &= \left\{(a\,d\xi_1 + b\,d\xi_2 + c\,d\xi_3)\cdot f \;\middle|\; a,b,c\in\mathcal{O},\ f\in\mathcal{O}(Z-E)\right\} \\
&= \left\{af\xi_1\frac{d\xi_1}{\xi_1'} + bf\xi_2'\frac{d\xi_2}{\xi_2'} + cf\xi_3'\frac{d\xi_3}{\xi_3'} \;\middle|\; a,b,c,\frac{f}{uvw}\xi_1\in\mathcal{O}\right\} \\
&= \left\{k_1\frac{d\xi_1}{\xi_1'} + k_2\frac{d\xi_2}{\xi_2'} + k_2\frac{d\xi_3}{\xi_3'} \;\middle|\; \frac{k_1}{uvw},\ \frac{k_2}{uvw}\frac{\xi_1}{\xi_2'},\ \frac{k_3}{uvw}\frac{\xi_1}{\xi_3'}\in\mathcal{O}\right\} \\
&= \left\{k_1\frac{d\xi_1}{\xi_1'} + k_2\frac{d\xi_2}{\xi_2'} + k_2\frac{d\xi_3}{\xi_3'} \;\middle|\; k_1\in\mathcal{O}(-E),\right. \\
&\qquad\qquad \left. k_2\in\mathcal{O}(Z-N-E),\ k_3\in\mathcal{O}(Z-P-E)\right\} \qquad (1)
\end{aligned}$$

Since $\mathcal{O}(Z-P-E)\subset\mathcal{O}(Z-N-E)\subset\mathcal{O}(-E)\approx\mathcal{I}_E$, we have the desired injection α.

To define β, we first define the map

$$\begin{aligned}
\widetilde{\beta}\colon \mathcal{I}_E\Omega^1(\log E) &\longrightarrow \mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R) \\
\omega &\longmapsto \omega\wedge\frac{dh}{h}.
\end{aligned}$$

Take $\omega\in\mathcal{I}_E\Omega^1(\log E)$. Then $\omega=\sum k_i\frac{d\xi_i}{\xi_i'}$, with $k_i\in\mathcal{O}(-E)$. We need to show that $\widetilde{\beta}(\omega)$ is actually in $\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R)$. Recall that there exists a linear function $h\colon\mathbb{C}^n\to\mathbb{C}$ so that $\operatorname{div}(h\circ\pi)=Z+R$ (we will also write $h=h\circ\pi$). Therefore away from R, $h=\xi_1$; in this case we have:

$$\begin{aligned}
\widetilde{\beta}(\omega) &= \omega\wedge\frac{dh}{h} \\
&= \left(\sum k_i\frac{d\xi_i}{\xi_i'}\right)\wedge\frac{d\xi_1}{\xi_1'} \\
&= -k_2\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + k_3\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}; \qquad (2)
\end{aligned}$$

this is clearly in $\mathcal{I}_E\Omega^2(\log E)$ since $\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'}$ and $\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}$ are each nontrivial linear combinations of $\frac{du\,dv}{uv}$, $\frac{dv\,dw}{vw}$, $\frac{dw\,du}{wu}$. At a simple point near R, *i.e.* $e\in E_i\cap R$, we can choose coordinates $\{u,v,w\}$ so that $E_i=\{u=0\}$ and $R=\{v=0\}$. Since $m_i=n_i$ on components E_i that intersect R, $h=u^{m_i}v=u^{n_i}v=\xi_2'v=\xi_2$. In such a case we have:

$$\begin{aligned}\widetilde{\beta}(\omega) &= \omega\wedge\frac{dh}{h}\\ &= \left(\sum k_i\frac{d\xi_i}{\xi_i'}\right)\wedge\frac{d\xi_2}{\xi_2' v}\\ &= \frac{k_1}{v}\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'}-\frac{k_3}{v}\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'},\end{aligned}\tag{3}$$

which is clearly in $\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R)$. Since away from R, $\mathcal{O}(R)\approx\mathcal{O}$, computations (2) and (3) show that $\omega\wedge\frac{dh}{h}$ is always in $\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R)$.

We will define β to the the composition of the map $\widetilde{\beta}$ with the projection

$$\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R)\ \xrightarrow{p}\!\!\!\!\!\rightarrow\ \mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R)\big/\textstyle\bigwedge^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(R);$$

however, first we must show that this projection is well-defined; *i.e.* we must show that $\bigwedge^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(R)$ is a subset of $\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R)$. (The following computation assumes we are at a triple point e of E; for the double and simple point cases, simply replace uvw with uv or u, respectively.)

$$\begin{aligned}&\textstyle\bigwedge^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(R) \qquad (4)\\ &= \Big\{(a\,d\xi_1 d\xi_2+b\,d\xi_2 d\xi_3+c\,d\xi_3 d\xi_1)\cdot g\cdot r\;\Big|\;a,b,c\in\mathcal{O},\\ &\qquad\qquad g\in\mathcal{O}(2Z-E),\ r\in\mathcal{O}(R)\Big\}\\ &= \Big\{agr\xi_1\xi_2'\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'}+bgr\xi_2'\xi_3'\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'}+cgr\xi_3'\xi_1\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\;\Big|\\ &\qquad\qquad a,b,c,\tfrac{g}{uvw}\,\xi_1^2\in\mathcal{O},\ r\in\mathcal{O}(R)\Big\}\\ &= \Big\{Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'}+Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'}+Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\;\Big|\;r\in\mathcal{O}(R)\\ &\qquad\qquad \tfrac{A}{uvw}\,\tfrac{\xi_1}{\xi_2'},\ \tfrac{B}{uvw}\,\tfrac{\xi_1}{\xi_2'}\,\tfrac{\xi_1}{\xi_3'},\ \tfrac{C}{uvw}\,\tfrac{\xi_1}{\xi_3'}\in\mathcal{O}\Big\}\\ &= \Big\{Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'}+Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'}+Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\;\Big|\;A\in\mathcal{O}(Z-N-E),\\ &\qquad B\in\mathcal{O}(2Z-N-P-E),\ C\in\mathcal{O}(Z-P-E),\ r\in\mathcal{O}(R)\Big\} \qquad (5)\end{aligned}$$

Since $\mathcal{O}(2Z-N-P-E)\subset\mathcal{O}(Z-P-E)\subset\mathcal{O}(Z-N-E)\approx\mathcal{I}_E\mathcal{O}(Z-N)$, we see from the above computation that

$$\textstyle\bigwedge^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(R)\subset\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(Z-N+R).$$

Moreover, since $\mathcal{O}(Z-N) \subset \mathcal{O}$, we have shown that $\bigwedge^2 \mathcal{N}(2Z-E) \otimes \mathcal{O}(R)$ is contained in $\mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(R)$. Thus the projection p is well-defined, and we can define $\beta := p \circ \widetilde{\beta}$.

Now we show that the sequence is exact at $\mathcal{I}_E\Omega^1(\log E)$, in other words, that $\ker(\beta) = \mathrm{im}(\alpha)$. Let $\omega = \sum k_i \frac{d\xi_i}{\xi_i'}$ be any element of $\mathcal{I}_E\Omega^1(\log E)$. We have

$$\begin{aligned} \omega \in \ker(\beta) &\iff \widetilde{\beta}(\omega) \in \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(R), \\ \omega \in \mathrm{im}(\alpha) &\iff \quad \omega \in \mathcal{N}(Z-E). \end{aligned}$$

Let us first handle the case where we are away from R. Looking back on our computation of $\bigwedge^2 \mathcal{N}(2Z-E) \otimes \mathcal{O}(R)$ (where r is now equal to 1), and using (2) and the above, we see that $\omega \in \ker(\beta)$ if and only if $-k_2 = A \in \mathcal{O}(Z-N-E)$ and $k_3 = C \in \mathcal{O}(Z-P-E)$. Comparing this with our computation of $\mathcal{N}(Z-E)$, it is clear that this is precisely the condition we need in order to have $\omega \in \mathcal{N}(Z-E)$, *i.e.* $\omega \in \mathrm{im}(\alpha)$.

Near R, and thus at a simple point $e \in E_i \cap R$, we have $Z = N$. From (3) we see that $\omega \in \ker(\beta)$ if and only if $k_1 = A \in \mathcal{O}(Z-N-E) \approx \mathcal{O}(-E)$ and $-k_3 = B \in \mathcal{O}(2Z-N-P-E) \approx \mathcal{O}(Z-P-E)$. Note that since $Z = N$, k_2 is *a priori* in $\mathcal{O}(Z-N-E) \approx \mathcal{O}(-E)$; thus we have exactly the conditions we need in order to have $\omega \in \mathrm{im}(\alpha)$. We have now shown that, in all cases, the sequence is exact at $\mathcal{I}_E\Omega^1(\log E)$.

As a first step towards defining γ, we define the map

$$\begin{aligned} \widetilde{\gamma} \colon \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(R) &\longrightarrow \mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2R) \\ \tau &\longmapsto \tau. \end{aligned}$$

Take $\tau \in \mathcal{I}_E\Omega^2(\log E) \otimes \mathcal{O}(R)$. Then $\tau = Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}$, with $A, B, C \in \mathcal{O}(-E)$ and $r \in \mathcal{O}(R)$. We will first show that the map $\widetilde{\gamma}$ is well-defined, *i.e.* that $\widetilde{\gamma}(\tau)$ is in fact an element of $\mathcal{I}_E\Omega^3(\log E) \otimes \mathcal{O}(2R)$. Away from R (so $r = 1$), we have $h = \xi_1$, and thus:

$$\widetilde{\gamma}(\tau) \quad = \quad \tau \wedge \frac{dh}{h}$$

$$
\begin{aligned}
&= \left(Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right) \wedge \frac{d\xi_1}{\xi_1'} \\
&= B\,\frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'}; \qquad (6)
\end{aligned}
$$

which is in $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2R)$ since $\frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'}$ is a nowhere-vanishing multiple of $\frac{du\,dv\,dw}{uvw}$ and $\mathcal{O}(R)\approx\mathcal{O}$ away from R. Near R, *i.e.* when $e\in E_i\cap R$, we have $h=\xi_2=\xi_2'v$ (where we have chosen coordinates $\{u,v,w\}$ for $\widetilde{U}$ so that $E_i=\{u=0\}$ and $R=\{v=0\}$), and thus:

$$
\begin{aligned}
\widetilde{\gamma}(\tau) &= \tau\wedge\frac{dh}{h} \\
&= \left(Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right) \wedge \frac{d\xi_2}{\xi_2' v} \\
&= \frac{Cr}{v}\,\frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'}, \qquad (7)
\end{aligned}
$$

which is clearly in $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2R)$ since r and $\frac{1}{v}$ are in $\mathcal{O}(R)$.

As a further step towards defining γ, we will show that we have a well-defined projection

$$
\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2R) \overset{\tilde{p}}{\twoheadrightarrow} \mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2R)\big/{\textstyle\bigwedge^3}\mathcal{N}(3Z-E)\otimes\mathcal{O}(2R).
$$

It suffices to prove that we have an injection of $\bigwedge^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2R)$ into $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2R)$. (Once again, we assume we are at a triple point e of E; for the double and simple point cases, simply replace uvw with uv or u, respectively.)

$$
\begin{aligned}
&{\textstyle\bigwedge^3}\mathcal{N}(3Z-E)\otimes\mathcal{O}(2R) \\
&\quad= \left\{(a\,d\xi_1 d\xi_2 d\xi_3)\cdot f\cdot r^2 \;\middle|\; a\in\mathcal{O}(3Z-E),\; r\in\mathcal{O}(R)\right\} \\
&\quad= \left\{afr^2\xi_1\xi_2'\xi_3'\frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'} \;\middle|\; a,\frac{f}{uvw}\xi_1^3\in\mathcal{O},\;\; r\in\mathcal{O}(R)\right\} \\
&\quad= \left\{Kr^2\frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'} \;\middle|\; \frac{K}{uvw}\,\frac{\xi_1^2}{\xi_1\xi_2}\in\mathcal{O},\; r\in\mathcal{O}(R)\right\} \\
&\quad= \left\{Kr^2\frac{d\xi_1 d\xi_2 d\xi_3}{\xi_1\xi_2'\xi_3'} \;\middle|\; K\in\mathcal{O}(2Z-P-N-E),\; r\in\mathcal{O}(R)\right\} \\
&\quad= \mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2Z-P-N+2R). \qquad (8)
\end{aligned}
$$

Since $\mathcal{O}(2Z-N-P) \approx \mathcal{O}(Z-N)\otimes\mathcal{O}(Z-P) \subset \mathcal{O}$, we have $\bigwedge^3 \mathcal{N}(3Z-E)\otimes\mathcal{O}(2R)$ as a subsheaf of $\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2R)$, and the projection $\widetilde{p}$ is well-defined.

We will define the map γ using the maps $\widetilde{\gamma}$, p, and $\widetilde{p}$ via the diagram:

$$\begin{array}{ccc}
\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R) & \overset{p}{\twoheadrightarrow} & \mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R)\Big/\bigwedge^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(R) \\
{\scriptstyle\widetilde{\gamma}}\Big\downarrow & & {\scriptstyle\gamma}\Big\downarrow \\
\mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2R) & \overset{\widetilde{p}}{\twoheadrightarrow} & \mathcal{I}_E\Omega^3(\log E)\otimes\mathcal{O}(2R)\Big/\bigwedge^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2R)
\end{array}$$

In other words, given $\bar{\tau} \in \mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R)\big/\bigwedge^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(R)$, with representative $\tau \in \mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R)$, $p(\tau)=\bar{\tau}$, we define $\gamma(\bar{\tau}) = \widetilde{p}(\widetilde{\gamma}(\tau))$. This is well-defined because the restriction of $\widetilde{\gamma}$ to $\bigwedge^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(R)$ maps into $\bigwedge^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2R)$; if $\tau = Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}$ is an element of $\bigwedge^2\mathcal{N}(2Z-E)\otimes\mathcal{O}(R)$, then $A\in\mathcal{O}(Z-N-E)$, $B\in\mathcal{O}(2Z-N-P-E)$, $C\in\mathcal{O}(Z-P-E)$, and $r\in\mathcal{O}(R)$. Looking at computations (6), (7), and (8) it is clear that in this case we have $\widetilde{\gamma}(\tau)\in\bigwedge^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2R)$.

It now remains only to prove that $\ker(\gamma) = \operatorname{im}(\beta)$. It is easy to show that $\operatorname{im}(\beta)\subseteq\ker(\gamma)$; given ω in $\mathcal{I}_E\Omega^1(\log E)$ we must show that $\gamma(\beta(\omega)) = 0$, *i.e.* that $\widetilde{p}(\widetilde{\gamma}(\widetilde{\beta}(\omega))) = 0$:

$$\widetilde{p}(\widetilde{\gamma}(\widetilde{\beta}(\omega))) \quad = \quad \widetilde{p}\left(\omega\wedge\frac{dh}{h}\wedge\frac{dh}{h}\right) \quad = \quad \widetilde{p}(0) \quad = \quad 0.$$

To show that $\ker(\gamma)\subseteq\operatorname{im}(\beta)$, take $\bar{\tau}=[\tau]$ with τ in $\mathcal{I}_E\Omega^2(\log E)\otimes\mathcal{O}(R)$. If $\bar{\tau}\in\ker(\gamma)$, then τ must be in $\ker(\widetilde{p}\circ\widetilde{\gamma})$; that is to say, $\widetilde{\gamma}(\tau)$ is contained in $\bigwedge^3\mathcal{N}(3Z-E)\otimes\mathcal{O}(2R)$. Suppose $\tau = Ar\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}$ (*a priori* A, B, and C are in $\mathcal{O}(-E)$, and $r\in\mathcal{O}(R)$). Away from R we have $h=\xi_1$ (and $r=1$ in τ), and looking at (6) and (8) we see that if $\tau\in\ker(\widetilde{p}\circ\widetilde{\gamma})$, then $B\in\mathcal{O}(2Z-N-P-E)$. To show that $\bar{\tau}\in\operatorname{im}(\beta)$, we must show that there exists an $\omega\in\mathcal{I}_E\Omega^1(\log E)$ so that $\bar{\tau}=\beta(\omega)=p(\widetilde{\beta}(\omega))$, *i.e.* $p(\tau)=p(\widetilde{\beta}(\omega))$. Choose $\omega=\sum k_i\frac{d\xi_i}{\xi_i'}$ with $k_2=-A$ and $k_3=C$; then by (2),

$$p(\widetilde{\beta}(\omega)) = p\left(A\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + C\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right).$$

On the other hand, since $B \in \mathcal{O}(2Z - N - P - E)$, we have:

$$\begin{aligned} p(\tau) &= p\left(A\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + B\,\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + C\,\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right) \\ &= p\left(A\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + C\,\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right). \end{aligned}$$

Thus we have shown that, away from R, $\ker(\gamma) \subseteq \operatorname{im}(\beta)$. At a simple point $e \in E_i \cap R$ near R we have coordinates $\{u, v, w\}$ on $\widetilde{U}$ so that $E_i = \{u = 0\}$, $R = \{v = 0\}$, and $h = \xi_2 = \xi_2' v$ (recall that $Z = N$ on components E_i that intersect R). By (7) and (8) we see that if $\tau \in \ker(\widetilde{p} \circ \widetilde{\gamma})$, then $C \in \mathcal{O}(Z - P - E)$. Again we must find an $\omega \in \mathcal{I}_E \Omega^1(\log E)$ so that $p(\tau) = p(\widetilde{\beta}(\omega))$; choose $\omega = \sum k_i \frac{d\xi_i}{\xi_i'}$ with $k_1 = A$ and $k_3 = -B$; then by (3),

$$p(\widetilde{\beta}(\omega)) = p\left(\frac{A}{v}\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + \frac{B}{v}\,\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'}\right).$$

On the other hand, since $C \in \mathcal{O}(Z - P - E)$ we have:

$$\begin{aligned} p(\tau) &= p\left(Ar\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\,\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'} + Cr\,\frac{d\xi_3 d\xi_1}{\xi_3'\xi_1'}\right) \\ &= p\left(Ar\,\frac{d\xi_1 d\xi_2}{\xi_1'\xi_2'} + Br\,\frac{d\xi_2 d\xi_3}{\xi_2'\xi_3'}\right). \end{aligned}$$

Since $r = \frac{1}{v}$ this shows that $p(\tau) = p(\widetilde{\beta}(\omega))$, and thus we have shown that, near R, $\ker(\gamma) \subseteq \operatorname{im}(\beta)$. ■