Casanova's Martingale

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Abstract

In this short exposition, the notion of a martingale is introduced through the example of the martingale betting strategy as played by Girolamo Casanova. The majority of this paper is from [1]. For a more detailed introduction to martingales, consult [3].

1 Preliminaries

In this section we recall some of the basic definitions from measure theory and note their probabilistic analogues. Let Ω be a set and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. Let $\mathbf{P}\colon \mathcal{F} \to [0,1]$ be a measure on Ω such that $\mathbf{P}(\Omega) = 1$.

Definition 1. *$(\Omega, \mathcal{F}, \mathbf{P})$ is called a **probability space**.*

Let $X : \Omega \to \mathbb{R}$ be an $\mathcal{F}$-measurable function. That is to say, $X^{-1}(A) \in \mathcal{F}$ $\forall A \in \mathcal{B}$ where $\mathcal{B}$ are the Borel subsets of $\mathbb{R}$. X is called a *random variable*.

Definition 2. *If $\int |X| d\mathbf{P} < \infty$ (i.e. X is integrable), then define*

$$\mathbf{E}(X) = \int X d\mathbf{P}.$$

An element of $\mathcal{F}$ is called an *event*. Two events A and B are said to be *independent* if $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$.

2 A Gambler's Dilemma

Consider tossing a fair coin and observing whether heads or tails appears. We can construct the following probability space to model this experiment:

Let $\Omega = \{H, T\}$.
Let $\mathcal{F} = \{\phi, H, T, \Omega\}$.
Define a probability on $\mathcal{F}$ by $\mathbf{P}(\phi) = 0$, $\mathbf{P}(H) = \frac{1}{2} = \mathbf{P}(T)$, $\mathbf{P}(\Omega) = 1$.

Define the random variable X to be the outcome of the toss so that $X = x$ where $x \in \{H, T\}$.

Repeatedly toss this coin. Let $\{X_j\}$, $j = 0, 1, 2, \dots$ be a collection of random variables such that $\mathbf{P}(X_j = H) = \frac{1}{2} = \mathbf{P}(X_j = T)$.

Claim. *Heads is bound to turn up sooner or later.*

Proof. Consider the event that no heads have been observed in the first n tosses. Then,

$$\mathbf{P}(X_1 = T, X_2 = T, \dots, X_n = T) = \mathbf{P}(X_1 = T)\mathbf{P}(X_2 = T)\dots\mathbf{P}(X_n = T) = \frac{1}{2^n}$$

by independence and since $\mathbf{P}(X_j = T) = \frac{1}{2}$ for all j.

Thus, the probability that no heads ever turns up is equal to the probability that only tails ever turn up.

So,

$$\begin{aligned}
\mathbf{P}(\text{no heads ever}) &= \mathbf{P}(X_1 = T, X_2 = T, \dots) \\
&= \prod_{j=1}^{\infty} \mathbf{P}(X_j = T) \\
&= \lim_{n\to\infty} \prod_{j=1}^{n} \mathbf{P}(X_j = T) \\
&= \lim_{n\to\infty} \frac{1}{2^n} \\
&= 0
\end{aligned}$$

Hence, $\mathbf{P}(\text{heads eventually turns up}) = 1 - \mathbf{P}(\text{no heads ever}) = 1$. $\square$

Suppose a gambler wagers \$1 on an "evens" bet. That is, the gambler is playing a game in which $\mathbf{P}(\text{win}) = \frac{1}{2} = \mathbf{P}(\text{loss})$ and pays at 1:1 odds.

e.g. If you wager \$1 and win, then you receive your original \$1 plus \$1.

The gambler plays the game using the strategy that if he loses the first play, he wagers \$2 on the second play, and so on until he wins. In general, if he loses on the n^{th} play, then he wagers $\$2^n$ on the next play.

Each wager is calculated so that his inevitable winnings will cover his lost stakes and profit him \$1. For if he wins on the $(n+1)^{st}$ play then he will have lost $\$(1 + 2 + \cdots + 2^{n-1}) = \$(2^n - 1)$ and his winnings will be

$$\$2^n - \$(1 + 2 + \cdots + 2^{n-1}) = \$1.$$

Now if we suppose that the gambler wins on the N^{th} play, then N is a random variable with $\mathbf{P}(N = n) = \frac{1}{2^n}$.

By the claim, we have $\mathbf{P}(N < \infty) = 1$. In other words, with probability one, the gambler is guaranteed a win in the long run.

However, by this time he will have lost an amount $\$L$, with mean

$$\mathbf{E}(L) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n (1 + 2 + \cdots + 2^{n-2}) = \infty.$$

And so the moral is that the gambler must be prepared to lose a lot money, but so too must the casino. Thus, do not follow this strategy unless your initial capital is considerably greater than that of the casino's!

3 The Martingale

Although casinos now outlaw the use of such strategies, they haven't always forbidden it. In his memoirs, Casanova recalls his stay in Venice in 1754:

> Playing the martingale, continually doubling my stake, I won every day during the carnival. I was fortunate enough never to lose the sixth card, and if I had lost it, I should have been without money to play, for I had 2000 gold coins on that card. I congratulated myself on having increased the fortune of my dear mistress.

Unfortunately, for dear Casanova:

> Some days later, I still played the martingale but with such bad luck that I was soon left without a coin. As I shared my property with my mistress, I was obliged to tell her of my losses, and at her request sold all her diamonds, losing what I got for them; she now had only 500 gold coins. There was no more talk of her escaping from the convent, for we had nothing to live on. [1]

As noted by Casanova, the gambling strategy outlined in the preceeding section is called a *martingale betting strategy.*

Now if we write Y_n for the accumulated gain of the gambler after the n^{th} play with losses counting negative, then

- $Y_0 = 0$,
- $|Y_n| \leq 1 + 2 + \cdots + 2^{n-1} = 2^n - 1$

Furthermore, $Y_{n+1} = Y_n$ if the gambler has stopped by time $n + 1$.

$$\therefore Y_{n+1} = \begin{cases} Y_n - 2^n & \text{with probability } \frac{1}{2}, \\ Y_n + 2^n & \text{with probability } \frac{1}{2}. \end{cases}$$

This implies,

$$\begin{aligned} \mathbf{E}(Y_{n+1}|Y_1, Y_2, \ldots, Y_n) &= \frac{1}{2}(Y_n - 2^n) + \frac{1}{2}(Y_n + 2^n) \\ &= Y_n \end{aligned}$$

In this fair game, conditional upon the past information, the gambler will expect no change in his present capital on average. This is an example of what is called a *martingale*.

Definition 3. *A sequence $Y = \{Y_n\}$, $n = 0, 1, 2, \ldots$ of random variables is a **martingale** with respect to the sequence $X = \{X_n\}$, $n = 0, 1, 2, \ldots$ if for all $n \geq 0$,*

- $\mathbf{E}(|Y_n|) < \infty$,
- $\mathbf{E}(Y_{n+1}|X_0, X_1, \ldots, X_n) = Y_n$.

In the gambling example above $\{Y_n\}$ is a martingale with respect to itself.

References

[1] G.R. Grimmett and D.R. Stirzaker. *Probability and Random Processes.* Clarendon Press, Oxford, second edition, 1992.

[2] H.L. Royden. *Real Analysis.* Macmillan, third edition, 1988.

[3] David Williams. *Probability With Martingales.* Cambridge University Press, 1991.