Casanova's Martingale

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Outline

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1. Preliminaries from Measure Theory

- Let Ω be a set.
- Let $\mathcal{F}$ be a σ-algebra of subsets of Ω.
- Let $\mathbf{P}\colon \mathcal{F} \to [0,1]$ be a measure on Ω s.t. $\mathbf{P}(\Omega) = 1$.

Definition: $(\Omega, \mathcal{F}, \mathbf{P})$ is called a **probability space**.

Let $X : \Omega \to \mathbb{R}$ be an $\mathcal{F}$-measurable function.

i.e., $X^{-1}(A) \in \mathcal{F} \ \forall A \in \mathcal{B}$

We say X is a **random variable**.

Definition: If X is integrable (i.e. $\int |X| \, d\mathbf{P} < \infty$), then define

$$\mathbf{E}(X) = \int X \, d\mathbf{P}.$$

An element of $\mathcal{F}$ is called an **event**.

Two events A and B are said to be **independent** if $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$.

2. A Gambler's Dilemma

Example: Toss a fair coin. Observe 1 of 2 outcomes: Heads or Tails.

Construct the following probability space.

- $\Omega = \{H, T\}$
- $\mathcal{F} = \{\phi, H, T, \Omega\}$

Define a probability on $\mathcal{F}$:
$\mathbf{P}(\phi) = 0$, $\mathbf{P}(H) = \frac{1}{2} = \mathbf{P}(T)$, $\mathbf{P}(\Omega) = 1$

Define the rv X to be outcome of toss:
$X = x$ where $x \in \{H, T\}$

Repeatedly toss this coin.

Let $\{X_j\}$, $j = 0, 1, 2, \ldots$ be a collection of rv's s.t. $\mathbf{P}(X_j = H) = \frac{1}{2} = \mathbf{P}(X_j = T)$.

Claim: Heads is bound to turn up sooner or later.

Proof: Consider event that no heads have been observed in the first n tosses.

Then, since $\mathbf{P}(X_j = T) = \frac{1}{2}$ $\forall j$ and by independence of the tosses, we have:

$\mathbf{P}(X_1 = T, X_2 = T, \ldots, X_n = T)$
$= \mathbf{P}(X_1 = T)\mathbf{P}(X_2 = T)\ldots\mathbf{P}(X_n = T) = \frac{1}{2^n}$

So,

$$\begin{aligned}
\mathbf{P}(\text{no head ever turns up}) &= \mathbf{P}(\text{only tails ever turn up}) \\
&= \mathbf{P}(X_1 = T, X_2 = T, \ldots) \\
&= \prod_{j=1}^{\infty} \mathbf{P}(X_j = T) \\
&= \lim_{n\to\infty} \prod_{j=1}^{n} \mathbf{P}(X_j = T) \\
&= \lim_{n\to\infty} \frac{1}{2^n} \\
&= 0
\end{aligned}$$

$\therefore \mathbf{P}(\text{a head eventually turns up})$
$= 1 - \mathbf{P}(\text{no head ever turns up})$
$= 1$ $\square$

Suppose a gambler wagers \$1 on an "evens" bet.
i.e. gambler plays a game with $\mathbf{P}(\text{win}) = \frac{1}{2} = \mathbf{P}(\text{loss})$ and pays at 1:1 odds.

e.g. If you wager \$1 and win, then you receive your original \$1 plus \$1.

The gambler plays using the strategy that if he loses the first play, he wagers \$2 on the second play, and so on until he wins.

In general, if he loses on the n^{th} play, then he wagers $\$2^n$ on the next play.

Each wager is calculated so that his inevitable winnings will cover his lost stakes and profit him \$1.

If he wins on the $(n+1)^{st}$ play then he will have lost
$\$(\mathbf{1+2+\cdots+2^{n-1}}) = \$(\mathbf{2^n - 1})$
and his winnings will be
$\$\mathbf{2^n} - \$(\mathbf{1+2+\cdots+2^{n-1}}) = \$\mathbf{1}.$

Suppose the gambler wins on the N^{th} play.

N is a rv with $\mathbf{P}(N = n) = \frac{1}{2^n}$.

By the claim, $\mathbf{P}(N < \infty) = 1$.

In other words, with probability one, the gambler is guaranteed a win in the long run.

However, by this time he will have lost an amount $\$L$, with mean

$$\mathbf{E}(L) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n (1 + 2 + \cdots + 2^{n-2}) = \infty.$$

And so the moral is that the gambler must be prepared to lose a lot money, but so too must the casino.

Thus, do not follow this strategy unless your initial capital is considerably greater than that of the casino's!

3. The Martingale

Casanova, recalling his stay in Venice in 1754:

> Playing the martingale, continually doubling my stake, I won every day during the carnival. I was fortunate enough never to lose the sixth card, and if I had lost it, I should have been without money to play, for I had 2000 gold coins on that card. I congratulated myself on having increased the fortune of my dear mistress.

Unfortunately, for dear Casanova:

> Some days later, I still played the martingale but with such bad luck that I was soon left without a coin. As I shared my property with my mistress, I was obliged to tell her of my losses, and at her request sold all her diamonds, losing what I got for them; she now had only 500 gold coins. There was no more talk of her escaping from the convent, for we had nothing to live on.

The gambling strategy from the previous section is called **a martingale betting strategy**.

- Y_n = accumulated gain of gambler after n^{th} play. Count losses negative.

- $Y_0 = 0$,

- $|Y_n| \leq 1 + 2 + \cdots + 2^{n-1} = 2^n - 1$

- $Y_{n+1} = Y_n$ if the gambler has stopped by time $n+1$

$$\therefore Y_{n+1} = \begin{cases} Y_n - 2^n & \text{with probability } \frac{1}{2}, \\ Y_n + 2^n & \text{with probability } \frac{1}{2}. \end{cases}$$

This implies,

$$\begin{aligned} \mathbf{E}(Y_{n+1}|Y_1, Y_2, \ldots, Y_n) &= \frac{1}{2}(Y_n - 2^n) + \frac{1}{2}(Y_n + 2^n) \\ &= Y_n \end{aligned}$$

In this fair game, conditional upon the past information, the gambler will expect no change in his present capital on average.

This is an example of what is called a **martingale**.

Definition: A sequence $Y = \{Y_n\}$, $n = 0, 1, 2, \ldots$ of random variables is a **martingale** with respect to the sequence $X = \{X_n\}$, $n = 0, 1, 2, \ldots$ if for all $n \geq 0$,

- $\mathbf{E}(|Y_n|) < \infty$,
- $\mathbf{E}(Y_{n+1}|X_0, X_1, \ldots, X_n) = Y_n$.

Note: In the gambling example above $\{Y_n\}$ is a martingale with respect to itself.