The Calabi-Yau Conjectures for Embedded surfaces

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(Joint work with Bill Minicozzi)
What is a minimal surface?

A surface $\Sigma \subseteq \mathbb{R}^3$ is minimal if the coordinate functions $x_1, x_2, x_3$ are all harmonic on $\Sigma$ (with respect to the intrinsic Laplacian).
Alternative definition of a minimal surface

\[ \Sigma \subseteq \mathbb{R}^3 \]
\[ \vec{n} \text{ - unit normal} \]
\[ \phi \in C^\infty_0 (\Sigma) \]
\[ \Sigma_{\phi, t} = \{ x + t \phi (x) \vec{n} (x) \mid x \in \Sigma \} \]
\[ H \text{ mean curvature} \]
First variation of area

\[ \frac{d}{dt} \text{Area}(\Sigma_{\phi,t}) \bigg|_{t=0} = \int_{\Sigma} \phi \, H \]

\[ \Sigma \text{ minimal} \iff \int_{\Sigma} \phi \, H = 0 \quad \text{for all } \phi \in C_0^\infty(\Sigma) \]

\[ \iff H \equiv 0 \]
No compact minimal surfaces.

Follows by the maximum principle since $x_1, x_2, x_3$ are harmonic on $\Sigma$ and thus when $\Sigma$ is compact all constant.
Minimal surfaces as an area of mathematical research goes back to work of Euler and Lagrange ~ 1740
Conjecture (Calabi ~ 1965):

Prove that a complete minimal surface must be unbounded.
Conjecture II (Calabi ~ 1965):

A complete nonflat minimal surface has unbounded projection in every line.
The immersed versions of the conjectures are false.

Jorge-Xavier (1980):

There exists a non-flat immersed minimal surface in a slab, i.e., between two parallel planes.
Nadirashvili (1996):

There exists an immersed minimal disk in a ball.
Embedded = injective
Emmersion
Thm (C-M):

The Calabi–Yau Conjectures holds for Embedded Surfaces.
**Proper**

An immersed surface \( f : \mathbb{S} \to \mathbb{R}^3 \) is proper if each pre-image of a compact subset of \( \mathbb{R}^3 \) is compact, i.e., if for each compact subset \( C \subseteq \mathbb{R}^3 \), \( f^{-1}(C) \) is compact.
We will prove much more than the actual conjectures. In fact we will show a Chord-arc bound giving an effective version of properness.

Chord-arc:

Always: extrinsic distances \leq intrinsic distances

Will show:

intrinsic distances \leq extrinsic distances
Thm (C-M)

$\Sigma \subseteq \mathbb{R}^3$ embedded minimal disk
$B_{2R}(0)$ intrinsic ball in $\Sigma \setminus \Theta \Sigma$
$\sup_{B_{2R}} |A|^2 > \gamma_0^{-2} \quad (R > \rho_0)$
$B_{\rho_0}$

then for $x \in \partial B_{\rho_0}$

$C \cdot \text{dist}_{\Sigma} (x, 0) \leq |x| + \rho_0$

$|A|^2 = k_1^2 + k_2^2$

principal curvatures

$0 = H = k_1 + k_2$
Curvature normalization in this theorem is needed as can be seen from the helicoid.

**Helicoid:**

\[(s \cos t, s \sin t, t)\]

\[s, t \in \mathbb{R}\]
Cor: A complete embedded minimal surface with finite topology must be proper.

Hence, Calabi's first conjecture follows.
Cor There exists $\varepsilon > 0$ s.t. if

$$\Sigma \subseteq \{ x_3 > 0 \} \cap \mathbb{R}^3$$

is an embedded minimal disk with $\Delta_2 (x) \leq \varepsilon \setminus \mathcal{E}$ and $|x| < \varepsilon R$, then

$$\sup_{B_R(x)} |A|^2 \leq R^{-2}$$

(Intrinsic one-sided curvature est.)
Disk condition needed in this theorem as can be seen from the catenoid.

Catenoid rotationally symmetric minimal surface
Because of the logarithmic growth, a piece of a catenoid can be scaled in close to a plane with huge curvature.
Cor: The plane is the only complete embedded minimal surface with finite topology in a half-space.
Proof of the Chord-arc bound for embedded disks

Three main steps
We will need the following consequence of the one-sided curvature estimate for properly embedded minimal disks:

**Cor.** \( \exists \varepsilon > 0 \) s.t.:

If \( \Sigma_1, \Sigma_2 \subseteq B_{2R} \) are compact embedded minimal disks with \( \partial \Sigma_1, \partial \Sigma_2 \subseteq \partial B_{2R} \) and \( x_i \in B_{2R} \cap \Sigma_i \), then \( \sup_{B_{2R}(x_i)} |A_{\Sigma_i}| < R^{-2} \).
Step 1

Ag another consequence of the estimates for properly embedded disks:

We have good Chord-arc bounds which are independent of the particular disk.
Suppose now that $\Sigma$ is an embedded minimal disk not nec. proper.

Suppose also that in all balls of radius $R$ each component of the disk is compact.

We want to show that also in all balls of radius $5R$ each component of the disk is compact.
If not, we may assume that $\exists x, y$ in a ball of radius $5R$ s.t.

$$y \in \partial B_5(x)$$

for some hugely large $5R$. 

$\sigma$: curve from $x$ to $y$ contained in $B_{5R} \cap \Sigma$.
Denote by \( \mathfrak{f}_x \) the points on \( \mathcal{S} \) which are very far apart. Since there are very many of these points all cramped into \( B_{SR} \) at least two of them must be extrinsically close.
Start at these two points and use the cor. of the one-sided curv. est. for properly emb. disks. Namely, by that consequence we get a priori curv. est. on large intrinsic balls by iteration.
Once we have large intrinsic balls which are extrinsically close and with a priori curvature bounds, improvements kick in via stability and show that each of these balls are much flatter than expected and hence could not combine within the ball of radius $5R$, giving the desired contradiction.
Final step

Locating the first scale where it is not proper and then use the above to get a contradiction.

This show properness for disks.

Properness for disks is then used to show properness in the general case.