Generalizing the Class Number Formula

Annette Huber

The class number formula is one of the miracles of number theory.

What is the class number? The fundamental theorem of arithmetic asserts that every integer has a unique decomposition into prime factors. The same holds true in the rings $\mathbb{Z}[i]$ and $\mathbb{Z}[e^{2\pi i/3}]$. To their dismay, number theorists discovered in the 19th century that it fails in more general rings of integers like $\mathbb{Z}[e^{2\pi i/N}]$ (cf. [S]). The failure has a quantitive measure, the *class number.*

Definition: *Let $\mathcal{O}_F$ be the ring of integers of a number field F (i.e., a finite field extension of $\mathbb{Q}$). The* class group *is the multiplicative semi-group of non-zero ideals of $\mathcal{O}_F$ modulo principal ideals. The order of the class group is called* class number.

Hence class number 1 is equivalent to the fundamental theorem of arithmetic.

Arguably the first real theorem in an algebraic number theory course is the finiteness of the class number. It is very hard to determine the class group. Luckily, the class number can also be computed by a completely different transcendental method.

Class number formula:

$$\zeta_F(0)^* = -\frac{R_F \cdot h}{w}$$

where h is the class number, w the number of roots of unity in F, R_F the regulator and $\zeta_F(0)^*$ the leading coefficient of the Dedekind ζ-function of F at the point 0.

The Dedekind ζ-function

$$\zeta_F : \mathbb{C} \setminus \{1\} \to \mathbb{C}$$

is a holomorphic function with a simple pole at 1. In the simplest case $F = \mathbb{Q}$ it agrees with the Riemann ζ-function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \ .$$

The regulator is a transcendental number computed from the logarithmic embedding of invertible elements of $\mathcal{O}_F$ to a certain $\mathbb{R}^n$. For more details see e.g. [L] or [N].

From the geometric point of view, number fields are zero-dimensional algebraic varieties over $\mathbb{Q}$. For example the field $\mathbb{Q}(e^{2\pi i/N})$ can be viewed as the set of points in $\mathbb{C}$ defined by $X^N = 1$. So what about varieties of higher dimension?

Indeed, the class number formula has been generalized - conjecturally! - to all varieties over $\mathbb{Q}$ (and even motives). The Dedekind ζ-function is replaced by the L-function of the variety, again a meromorphic function on $\mathbb{C}$. The leading coefficients at integer points are related to arithmetic invariants of the variety. Many people have contributed to this process. The most famous case is elliptic curves and the value of their L-function at 1: the Conjecture of Birch and Swinnerton-Dyer. For the general case Bloch, Deligne and Beilinson have to be mentioned. The final form (so far) is the *Tamagawa number conjecture* by Bloch and Kato in 1990 (see [BK], [F]).

Mentioning of the BSD Conjecture (which is a special case) should warn that these conjectures are extremely deep. The body of known cases is disappointingly modest ([K]).

In my talk I want to explain the statement of the Tamagawa number conjecture in a couple of cases in addition to the class number formula. I also want to discuss the elements of the proof in the case of Dirichlet characters, see [HK].

References

[BK] S. Bloch,K. Kato, L-functions and Tamagawa numbers of motives. The Grothendieck Festschrift, Vol. I, 333–400, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990.

[HK] A. Huber, G. Kings, Bloch-Kato conjecture and main conjecture of Iwasawa theory for Dirichlet characters. Duke Math. J. 119 (2003), no. 3, 393–464.

[F] J.-M. Fontaine, Valeurs spéciales des fonctions L des motifs, Séminaire Bourbaki, Vol. 1991/92. Astérisque No. 206 (1992), Exp. No. 751, 4, 205–249.

[K] G. Kings, The Bloch-Kato conjecture on special values of L-functions. A survey of known results. Les XXIIèmes Journées Arithmetiques (Lille, 2001). J. Théor. Nombres Bordeaux 15 (2003), no. 1, 179–198.

[L] S. Lang, Algebraic Number theory, Second edition. Graduate Texts in Mathematics, 110. Springer-Verlag, New York, 1994.

[N] J. Neukirch, Algebraic Number Theory, Translated from the 1992 German original and with a note by Norbert Schappacher. With a foreword by G. Harder. Grundlehren der Mathematischen Wissenschaften, 322. Springer-Verlag, Berlin, 1999.

[S] S. Singh, Fermat's enigma. The epic quest to solve the world's greatest mathematical problem. With a foreword by John Lynch. Walker and Company, New York, 1997.