A PROBLEM LIST FOR COMPACT HYPERKÄHLER MANIFOLDS

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Abstract. In his thesis, Caldararu described twisted Fourier-Mukai transforms for elliptic fibrations. In this talk I will describe how certain holomorphic symplectic manifolds can be deformed to integrable systems, i.e. fibrations by abelian varieties. These are higher dimensional analogues of elliptic K3 surfaces, and twisted Fourier-Mukai transforms once again arise.

1. Title of talk: Twisted Fourier-Mukai transforms for holomorphic symplectic manifolds

2. A problem list for compact hyperkähler manifolds

In the 50s Berger classified the possible holonomy groups of Riemannian manifolds. Examples of Riemannian manifolds with non-generic holonomy include Calabi-Yau, G_2, and Spin(7) manifolds; for these manifolds, we now know of hundreds of compact examples (for instance, see Joyce's book [11]). In contrast, hyperkähler manifolds also have non-generic holonomy, but we know of very few compact examples. Currently the list of compact irreducible hyperkähler manifolds is as follows:

- K3 surfaces, the only examples in dimension four,
- the Hilbert scheme $S^{[n]}$ of n points on a K3 surface S (the case $n = 2$ was discovered by Fujiki [2], the general case by Beauville [1]),
- the generalized Kummer varieties [1],
- the Mukai moduli spaces of sheaves on K3 and abelian surfaces [15],
- two recent examples due to O'Grady [17, 18].

The Mukai moduli spaces are deformations of Beauville's examples, whereas O'Grady's examples are genuinely new. Note that a compact hyperkähler manifold admits a holomorphic symplectic structure, and conversely, by Yau's theorem, a compact Kähler holomorphic symplectic manifold admits a hyperkähler metric, making the study of these kinds of manifolds equivalent. In all of the above examples, the existence of a hyperkähler metric is deduced in this way.

2.1. **Can we classify compact hyperkähler manifolds up to deformations?** This is really a fundamental problem which encomposes many others in the subject. Huybrechts [9] has proved a finiteness result for the number of deformation classes of holomorphic symplectic manifolds in each dimension (under some additional hypotheses). Moreover, in most dimensions we only know of the two examples due to Beauville. Given the strong rigidity properties of a hyperkähler structure, one might be led to suspect that there can't be many more.

A similar problem is the classification of positive quaternion-Kähler manifolds. Conjecturely they are all symmetric spaces, and this has been proved in dimension eight by Poon and Salamon [19] (see also LeBrun and Salamon [12]) and more recently in dimension twelve by Herrera and Herrera [5]. Moreover, their are links between these two kind of geometry, though not directly between *compact* examples of each kind.

2.2. **What kinds of bounds on the topological invariants of a compact hyperkähler manifold can we obtain?** Verbitsky [23] found that the cohomology ring admits an $\mathfrak{so}(5)$ action which generalizes the hard Lefschetz theorem, and places some restrictions on the Hodge diamonds which can occur. In [20] Salamon found a linear relation for the Betti numbers. This was later used by Guan [3] in dimension eight to prove some quite strong bounds on the Betti numbers: only a small number of possible Hodge diamonds are allowed. Guan's results generalize to provide bounds on at least some of the Betti numbers in dimension twelve. Hitchin and the author [6] found that in all dimensions the characteristic number given by $\hat{A}^{1/2}$ must be positive, which leads to inequalities for the Chern numbers.

2.3. **Can the analogies between compact hyperkähler manifolds and complex surfaces be exploited?** All of the known examples are built from either K3s or abelian surfaces (more generally, two-dimensional complex tori). On a compact hyperkähler manifold, the second integral cohomology can be made into a lattice by using the Beauville-Bogomolov quadratic form [1]. This provides another analogy with complex surfaces, whose middle cohomology has a quadratic form given by intersection pairing.

We might expect that the general hyperkähler manifold can be deformed to one that has a surface canonically associated to it. For example, the Hilbert schemes of K3 surfaces are codimension one in their moduli space: if X is a hyperkähler manifold in this moduli space, we can deform it to a Hilbert scheme then recover the K3 surface. In general, can we recover the lattice of a complex surface from the second integral cohomology of a hyperkähler manifold?

2.4. **Can we better understand O'Grady's examples?** Obviously we'd like to generalize the construction and produce more examples. If this is not possible, we'd like to know why we get these two somewhat anomolous examples, in (complex) dimensions six and ten. We also know very little about the topology of these examples: their Betti numbers (except for b_2 [17, 18]), Hodge diamonds, Chern numbers.

2.5. **Is there a global Torelli theorem for compact hyperkähler manifolds?** The local moduli space of compact hyperkähler manifolds has been well studied: a comprehensive account can be found in [8]. However, the naive generalization of the global Torelli theorem for K3 surfaces fails for several reasons (Namikawa [16] has constructed a counter-example), and the result needs to be reformulated in higher dimensions.

2.6. **Can we understand mirror symmetry for compact hyperkähler manifolds?** Verbitsky [24] proved that generic (Picard number one) compact hyperkähler manifolds are self-mirror, by exhibiting an isomorphism between the A-model and B-model VFAs (variation of Frobenius algebras). There is still much more to investigate: Kontsevich's homological mirror symmetry, mirror manifolds in the presence of B-fields. For K3 surfaces, most of this has already been extensively studied. In higher dimensions, the 'quantum' moduli space has been studied by Huybrechts [10]. In the non-compact case, Hausel and Thaddeus [4] have found pairs of mirror manifolds with non-trivial B-field.

The Strominger-Yau-Zaslow conjecture [22] should be of particular interest for compact hyperkähler manifolds. By the hyperkähler rotation trick, a special Lagrangian fibration can be made into a holomorphic Lagrangian fibration. Therefore, unlike in the case of Calabi-Yau threefolds fibred by Lagrangian tori, for hyperkähler manifolds we have at our disposal all of the machinery of complex geometry.

2.7. **Which compact hyperkähler manifolds are integrable systems?** Equivalently, which compact hyperkähler manifolds admit Lagrangian fibrations? Conjecturely, every hyperkähler manifold can be deformed to such a fibration (see [21]). In (complex) dimension four, these Lagrangian fibrations have been studied by Markushevich [13]. In arbitrary dimension, Matsushita [14] has shown that this is more-or-less the only kind of fibration that a compact hyperkähler manifold can admit. Hurtubise [7] has also studied the local structure of these fibrations.

References

[1] A. Beauville, *Variétés Kählériennes dont le 1ére classe de Chern est nulle*, Jour. Diff. Geom. **18** (1983), 755-782. 1, 1, 2

[2] A. Fujiki, *On primitively symplectic compact Kähler V-manifolds of dimension four*, in Classification of algebraic and analytic manifolds. Progr. Math. **39** (1983), 71-250. 1

[3] D. Guan, *On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four*, Math. Res. Lett. **8** (2001), no. 5-6, 663-669. 2

[4] T. Hausel and M. Thaddeus, *Examples of mirror partners arising from integrable systems*, C. R. Acad. Sci. Paris Sér. I Math., **333** (2001), No. 4, 313-318. 3

[5] H. Herrera and R. Herrera, *A result on the* $\widehat{A}$ *and elliptic genera on non-spin manifolds with circle actions*, C. R. Math. Acad. Sci. Paris **335** (2002), no. 4, 371-374. 2

[6] N. Hitchin and J. Sawon, *Curvature and characteristic numbers of hyperähler manifolds*, Duke Math. Jour. **106** (2001), No. 3, 599-615. 2

[7] J. Hurtubise, *Integrable systems and algebraic surfaces*, Duke Math. Jour. **83** (1996), No. 1, 19-50. 3

[8] D. Huybrechts, *Compact hyperkähler manifolds: basic results*, Invent. Math. **135** (1999), no. 1, 63-113. 3

[9] D. Huybrechts, *Finiteness results for hyperkähler manifolds*, preprint **math.AG/0109024**. 1

[10] D. Huybrechts, *Moduli spaces of hyperkähler manifolds and mirror symmetry*, Trieste Lectures, September 2002, preprint **math.AG/0210219**. 3

[11] D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press, 2000. 1

[12] C. LeBrun and S. Salamon, *Strong rigidity of positive quaternion-Kähler manifolds*, Invent. Math. **118** (1994), no. 1, 109-132. 2

[13] D. Markushevich, *Lagrangian families of Jacobians of genus 2 curves*, J. Math. Sci. **82** (1996), no. 1, 3268-3284. 3

[14] D. Matsushita, *On fibre space structures of a projective irreducible symplectic manifold*, Topology **38** (1999), No. 1, 79-83. Addendum, Topology **40** (2001), No. 2, 431-432. 3

[15] S. Mukai, *Symplectic structure of the moduli space of simple sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984), 101-116. 1

[16] Y. Namikawa, *Counter-example to global Torelli problem for irreducible symplectic manifolds*, Math. Ann. **324** (2002), no. 4, 841-845. 3

[17] K. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, J. Reine Angew. Math. **512** (1999), 49-117. 1, 2

[18] K. O'Grady, *A new six dimensional irreducible symplectic variety*, preprint **math.AG/0010187**. 1, 2

[19] Y. S. Poon and S. Salamon, *Quaternionic Kähler 8-manifolds with positive scalar curvature*, J. Differential Geom. **33** (1991), no. 2, 363-378. 2

[20] S. Salamon, *On the cohomology of Kähler and hyper-Kähler manifolds*, Topology **35** (1996), no. 1, 137-155. 2

[21] J. Sawon, *Abelian fibred holomorphic symplectic manifolds*, Proceedings of the Ninth Gökova Geometry-Topology Conference, Turkey, May/June 2002, to appear. 3

[22] A. Strominger, S-T. Yau, E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. **B 479** (1996), 243-259. 3

[23] M. Verbitsky, *Action of the Lie algebra of SO(5) on the cohomology of a hyper-Kähler manifold*, (English translation) Funct. Anal. Appl. **24** (1990), no. 3, 229-230. 2

[24] M. Verbitsky, *Mirror symmetry for hyperkähler manifolds*, in Mirror symmetry III, AMS/IP Stud. Adv. Math., **10**, Providence, RI, (1999), 115-156. 3

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