Devil Round

June 9, 2006

1. [D1] Let a and b be complex numbers such that $a^3 + b^3 = -17$ and $a + b = 1$. What is the value of ab?

2. [D2] Let $AEFB$ be a right trapezoid, with $\measuredangle AEF = \measuredangle EAB = 90°$. The two diagonals EB and AF intersect at point D, and C is a point on AE such that $AE \perp DC$. If $AB = 8$ and $EF = 17$, what is the lenght of CD?

3. [D3] How many three-digit numbers abc (where each of a, b, and c represents a single digit, $a \neq 0$) are there such that the six-digit number $abcabc$ is divisible by 2, 3, 5, 7, 11, or 13?

4. [D4] Let S be the sum of all numbers of the form $\frac{1}{n}$ where n is a postive integer and $\frac{1}{n}$ terminates in base b, a positive integer. If S is $\frac{15}{8}$, what is the smallest such b?

5. [D5] Sysyphus is having an birthday party and he has a square cake that is to be cut into 25 square pieces. Zeus gets to make the first straight cut and messes up badly. What is the largest number of pieces Zeus can ruin (cut across)? Diagram?

6. [D6] Given $(9x^2 - y^2)(9x^2 + 6xy + y^2) = 16$ and $3x + y = 2$. Find x^y.

7. [D7] What is the prime factorization of the smallest integer N such that $\frac{N}{2}$ is a perfect square, $\frac{N}{3}$ is a perfect cube, $\frac{N}{5}$ is a perfect fifth power?

8. [D8] What is the maximum number of pieces that an spherical watermelon can be divided into with four straight planar cuts?

9. [D9] How many ordered triples of integers (x, y, z) are there such that $0 \leq x, y, z \leq 100$ and

$$(x-y)^2+(y-z)^2+(z-x)^2 \geq (x+y-2z)+(y+z-2x)^2+(z+x-2y)^2.$$

10. [D10] Find all real solutions to $(2^x-4)^2+(4^x-2)^3=(4^x+2^x-6)^3$.

11. [D11] Let f be a function that takes integers to integers that also has the following property: $f(x) = \begin{cases} x-5 & \text{if } x \geq 50 \\ f(f(x+12)) & \text{if } x < 50 \end{cases}$. Evaluate $f(2)+f(39)+f(58)$.

12. [D12] If two real numbers are chosen at random (i.e. uniform distribution) from the interval $[0, 1]$, what is the probability that theit difference will be less than $\frac{3}{5}$?

13. [D13] Let a, b, and c be positive integers, not all even, such that $a < b$, $b = c - 2$, and $a^2 + b^2 = c^2$. What is the smallest possible value for c?

14. [D14] Let $ABCD$ be a quadrilateral whose diagonals intersect at O. If $BO = 8$, $OD = 8$, $AO = 16$, $OC = 4$, and $AB = 16$, then find AD.

15. [D15] Let P_0 be a regular icosahedron with an edge length of 17 units. For each nonnegative integer n, recursively construct P_{n+1} from P_n by performing the following procedure on each face of P_n: glue a regular tetrahedron to that face such that three of the vertices of the tetrahedron are the midpoints of the three adjacent edges of the face, and the last vertex extends outside of P_n. Express the number of square units in the surface area of P_{17} in the form

$$\frac{u^v \cdot w\sqrt{x}}{y^z},$$

where u, v, w, x, y, and z are integers, all greater than or equal to 2, that satisfy the following conditions: the only perfect square that evenly divides x is 1, the GCD of u and y is 1, and neither u nor y divides w. ANSWERS WRITTEN IN ANY OTHER FORM WILL NOT BE CONSIDERED CORRECT!