Duke Math Meet
Team Round

November 11th, 2006

1. What is the smallest positive integer x such that $\frac{1}{x} < \sqrt{12011} - \sqrt{12006}$?

2. Two soccer players run a drill on a 100 foot by 300 foot rectangular soccer field. The two players start on two different corners of the rectangle separated by 100 feet, then run parallel along the long edges of the field, passing a soccer ball back and forth between them. Assume that the ball travels at a constant speed of 50 feet per second, both players run at a constant speed of 30 feet per second, and the players lead each other perfectly and pass the ball as soon as they receive it, how far has the ball travelled by the time it reaches the other end of the field?

3. A trapezoid $ABCD$ has AB and CD both perpendicular to AD and $BC = AB + AD$. If $AB = 26$, what is $\frac{CD^2}{AD+CD}$?

4. A hydrophobic, hungry, and lazy mouse is at $(0,0)$, a piece of cheese at $(26,26)$, and a circular lake of radius $5\sqrt{2}$ is centered at $(13,13)$. What is the length of the shortest path that the mouse can take to reach the cheese that also does not also pass through the lake?

5. Let a, b, and c be real numbers such that $a+b+c=0$ and $a^2+b^2+c^2=3$. If $a^5+b^5+c^5 \neq 0$, compute $\frac{\left(a^3+b^3+c^3\right)\left(a^4+b^4+c^4\right)}{a^5+b^5+c^5}$.

6. Let S be the number of points with integer coordinates that lie on the line segment with endpoints $\left(2^{2^2}, 4^{4^4}\right)$ and $\left(4^{4^4}, 0\right)$. Compute $\log_2(S-1)$

7. For a positive integer n let $f(n)$ be the sum of the digits of n. Calculate
$$f(f(f(2^{2006}))).$$

8. If a_1, a_2, a_3, a_4 are roots of $x^4 - 2006x^3 + 11x + 11 = 0$, find $|a_1^3 + a_2^3 + a_3^3 + a_4^3|$.

9. A triangle ABC has M and N on sides BC and AC, respectively, such that AM and BN intersect at P and the areas of triangles ANP, APB, and PMB are 5, 10, and 8 respectively. If R and S are the midpoints of MC and NC, respectively, compute the area of triangle CRS.

10. Jack's calculator has a strange button labelled "PS." If Jack's calculator is displaying the positive integer n, pressing PS will cause the calculator to divide n by the largest power of 2 that evenly divides n, and then adding 1 to the result and displaying that number. If Jack randomly chooses an integer k between 1 and 1023, inclusive, and enters it on his calculator, then presses the PS button twice, what is the probability that the number that is displayed is a power of 2?