DUKE MATH MEET 2007: DEVIL ROUND

In the Devil Round, all teams will be broken apart and re-assembled randomly to form new teams of about 10 people each. There will be 5 sets of 3 problems each for a total of 15 problems. At the start of the Devil Round, one person from each team will run to the front of the room and grab the first set of problems and run back to solve them with their team. When the team wishes to submit the answers to a set, one person from the team will run to the front of the room to submit their answers and grab the next set of problems to take back to their team. There will be a combined total time of 10 minutes to solve all 15 problems; awards will be given to the fastest team that obtains the highest score.

Devil Round Problems 1, 2, and 3

1. If:
$$a^2 + b^2 + c^2 = 1000,$$
$$(a + b + c)^2 = 100, \text{ and}$$
$$ab + bc = 10,$$
what is ac?
2. If a and b are real numbers such that $a \neq 0$ and the numbers 1, $a + b$, and a are, in some order, the numbers 0, $\frac{b}{a}$, and b, what is $b - a$?
3. Of the first 120 natural numbers, how many are divisible by at least one of 3, 4, 5, 12, 15, 20, and 60?

Devil Round Problems 4, 5, and 6

4. For positive real numbers a, let p_a and q_a be the maximum and minimum values, respectively, of $\log_a(x)$ for $a \leq x \leq 2a$. If $p_a - q_a = \frac{1}{2}$, what is a?
5. Let ABC be an acute triangle and let a, b, and c be the sides opposite the vertices A, B, and C, respectively. If $a = 2b \sin A$, what is the measure of angle B?
6. How many ordered triples (x, y, z) of positive integers satisfy the equation
$$x^3 + 2y^3 + 4z^3 = 9?$$

Devil Round Problems 7, 8, and 9

7. Joe has invented a robot that travels along the sides of a regular octagon. The robot starts at a vertex of the octagon and every minute chooses one of two directions (clockwise or counterclockwise) with equal probability and moves to the next vertex in that direction. What is the probability that after 8 minutes the robot is directly opposite the vertex it started from?
8. Find the nonnegative integer n such that when
$$\left(x^2 - \frac{1}{x}\right)^n$$
is completely expanded the constant coefficient is 15.
9. For each positive integer k, let
$$f_k(x) = \frac{kx + 9}{x + 3}.$$
Compute
$$f_1 \circ f_2 \circ \cdots \circ f_{13}(2).$$

Devil Round Problems 10, 11, and 12

10. Exactly one of the following five integers cannot be written in the form

$$x^2 + y^2 + 5z^2,$$

where x, y, and z are integers. Which one is it?

$$2003, 2004, 2005, 2006, 2007.$$

11. Suppose that two circles C_1 and C_2 intersect at two distinct points M and N. Suppose that P is a point on the line MN that is outside of both C_1 and C_2. Let A and B be the two distinct points on C_1 such that AP and BP are each tangent to C_1 and B is inside C_2. Similarly, let D and E be the two distinct points on C_2 such that DP and EP are each tangent to C_2 and D is inside C_1. If $AB = \frac{5\sqrt{2}}{2}$, $AD = 2$, $BD = 2$, $EB = 1$, and $ED = \sqrt{2}$, find AE.
12. How many ordered pairs (x, y) of positive integers satisfy the following equation?

$$\sqrt{x} + \sqrt{y} = \sqrt{2007}.$$

Devil Round Problems 13, 14, and 15

13. The sides BC, CA, and CB of triangle ABC have midpoints K, L, and M, respectively. If

$$AB^2 + BC^2 + CA^2 = 200,$$

what is $AK^2 + BL^2 + CM^2$?
14. Let x and y be real numbers that satisfy:

$$x + \frac{4}{x} = y + \frac{4}{y} = \frac{20}{xy}.$$

Compute the maximum value of $|x - y|$.
15. 30 math meet teams receive different scores which are then shuffled around to lend an aura of mystery to the grading. What is the probability that no team receives their own score? Express your answer as a decimal accurate to the nearest hundredth.