DUKE MATH MEET 2007: INDIVIDUAL ROUND

In the Individual Round there are four sub-rounds of two problems each to be solved individually. Like all rounds in the Duke Math Meet, only pencil and paper are allowed. At the start of each of the four sub-rounds, each student will turn over the sheet with the two questions, but will not be able to pick up their pencils. The moderator will then read the two questions aloud. When the moderator is finished reading the timer will begin. Students will have 10 minutes for each pair of problems and will receive 5-minute, 1-minute, and 15-second warnings for each pair of problems. When the 10 minutes are up, students must put down their pencils. Each correct answer from each student will add 1 point to their team's score.

Individual Round Problems 1 and 2

1. There are 32 balls in a box: 6 are blue, 8 are red, 4 are yellow, and 14 are brown. If I pull out three balls at once, what is the probability that none of them are brown?
2. Circles A and B are concentric, and the area of circle A is exactly 20% of the area of circle B. The circumference of circle B is 10. A square is inscribed in circle A. What is the area of that square?

Individual Round Problems 3 and 4

3. If $x^2 + y^2 = 1$ and $x, y \in \mathbb{R}$, let q be the largest possible value of $x + y$ and p be the smallest possible value of $x + y$. Compute pq.
4. Yizheng and Jennifer are playing a game of ping-pong. Ping-pong is played in a series of consecutive matches, where the winner of a match is given one point. In the scoring system that Yizheng and Jennifer use, if one person reaches 11 points before the other person can reach 10 points, then the person who reached 11 points wins. If instead the score ends up being tied 10-to-10, then the game will continue indefinitely until one person's score is two more than the other person's score, at which point the person with the higher score wins. The probability that Jennifer wins any one match is 70% and the score is currently at 9-to-9. What is the probability that Yizheng wins the game?

Individual Round Problems 5 and 6

5. The squares on an 8×8 chessboard are numbered left-to-right and then from top-to-bottom (so that the top-left square is #1, the top-right square is #8, and the bottom-right square is #64). 1 grain of wheat is placed on square #1, 2 grains on square #2, 4 grains on square #3, and so on, doubling each time until every square of the chessboard has some number of grains of wheat on it. What fraction of the grains of wheat on the chessboard are on the rightmost column?
6. Let f be any function that has the following property: For all real numbers x other than 0 and 1,
$$f\left(1 - \frac{1}{x}\right) + 2f\left(\frac{1}{1-x}\right) + 3f(x) = x^2.$$
Compute $f(2)$.

Individual Round Problems 7 and 8

7. Find all solutions of:
$$\left(x^2 + 7x + 6\right)^2 + 7\left(x^2 + 7x + 6\right) + 6 = x.$$
8. Let ΔABC be a triangle where $AB = 25$ and $AC = 29$. C_1 is a circle that has AB as a diameter and C_2 is a circle that has BC as a diameter. D is a point on C_1 so that $BD = 15$ and $CD = 21$. C_1 and C_2 clearly intersect at B; let E be the other point where C_1 and C_2 intersect. Find all possible values of ED.