DUKE MATH MEET 2007: RELAY ROUND

In the Relay Round each team of six students will divide into two groups of three students each. There are two sub-rounds in the Relay Round, each group of three students will work together. In each of the two sub-rounds, the three students in a group will receive different problems. The problems that the second and third students in each group will have the symbol "TNYWR" within the problem statement. This stands for "The Number You Will Receive." This is because the first student in the group is supposed to write down the answer to his or her problem and pass that answer to the second student, and similarly the second student is to pass the answer to his or her problem to the third students. The second student will need the first student's answer to completely solve the second question, and similarly the third student will need the second student's answer to completely solve the third question. The first and second students can only pass numbers to the second and third students; there are to be no symbols or other expressions. The one exception is that students can underline numbers to indicate, for example, whether their answer is a 6 or a 9.

The first and second students may pass as many answers as they wish; however, the third student can only submit answers after three minutes or six minutes, and only the third students's answer will be graded. If a group submits an answer after six minutes then their answer after three minutes, if they gave one, will be discarded. If a group obtains the correct answer after three minutes, they will earn 4 points for their team. If a group obtains the correct answer after six minutes, they will earn 2 points for their team. Therefore, a group should not, for example, submit the same answer after six minutes that they did after three minutes. The moderator will give 15-second warnings before the three minute deadline and the six minute deadline.

First Relay Round

1A. How many of the following eight integers are divisible by 36?

$$0, 3636, 33636, 33663, 64278, 72702, 90000, 91980.$$

1B. Let k be TNYWR. The following polynomial equation in x has exactly one real solution. What is it?

$$(x-k)^{10} = x^{10}.$$

1C. Let k be TNYWR. How many positive prime numbers are less than or equal to $18k$?

Second Relay Round

2A. The following equation has exactly one solution (x, y) where x and y are both positive integers. What is $x + y$?

$$\frac{x^3}{y} + \frac{y^3}{x} = 288.$$

2B. Let k be TNYWR. A lattice point is an ordered pair (x, y) such that x and y are both integers. How many lattice points are either inside of or on the boundary of the triangle whose vertices have coordinates $(0, 0)$, $(k, 0)$, and $(0, k)$?

2C. Let k be TNYWR. Find the minimum of the function

$$f(x) = x^2 + \frac{k}{x^2+1}$$

for $x \in \mathbb{R}$.