DUKE MATH MEET 2008: POWER ROUND: JAPANESE THEOREMS ON CYCLIC POLYGONS

In the Power Round the entire team of six students will have 60 minutes to answer this series of proof-based questions. The team members may collaborate freely, but like all rounds in the Duke Math Meet only pencil and paper can be used. After 60 minutes the team will submit all solutions. Solutions to different questions must be on different sheets of paper. Cross out anything you do not want graded. Teams will be given 30-minute, 5-minute, and 1-minute warnings. **Teams may use results of previous problems to solve later problems, even if the team has not submitted solutions to those previous problems.** The number of points earned for each problem varies, but the total is 16 points. This power round is divided into three somewhat independent parts.

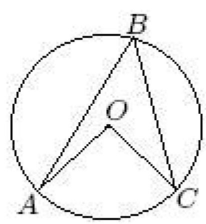
1. Basics (5 Points Total)

We will call a polygon "**cyclic**" if there is a single circle that passes through all of its vertices. (WARNING: Some people use the word "concyclic.") All triangles are cyclic, though not all quadrilaterals are. In the second and third sections you will prove two advanced theorems about cyclic quadrilaterals and cyclic polygons. In this section, though, you will prove two basic theorems that will be helpful in some of the later sections.

1A. (1 Point.) Prove the following statement:

A, B, and C are on a circle whose center is O, as in the diagram below. If B and O are on the same side of $\overleftrightarrow{AC}$ (or if one of these two points is on $\overleftrightarrow{AC}$), then $m\angle AOC = 2 \cdot (m\angle ABC)$.

(Hint: Draw $\overline{OB}$.)



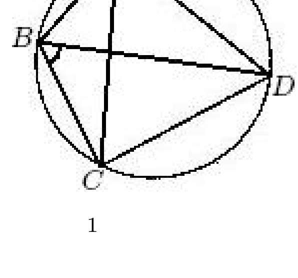
1B. (1 Point.) Prove the following statement:

A, B, and C are on a circle whose center is O. If B and O are on opposite sides of $\overleftrightarrow{AC}$ (or if one of these two points is on $\overleftrightarrow{AC}$), then $m\angle AOC = 360° - 2 \cdot (m\angle ABC)$.

(Hint: Draw $\overline{OB}$.)

The following are perhaps the two most useful basic results about cyclic quadrilaterals:

(i) (**Equal Angles Theorem**) The quadrilateral $ABCD$ is cyclic if and only if $m\angle CAD = m\angle CBD$, as in the diagram below. (You can use a different ordering of the points and get, for example, that $ABCD$ is cyclic if and only if $m\angle ABD = m\angle ACD$.)



(ii) (**Supplementary Angles Theorem**) The quadrilateral $ABCD$ is cyclic if and only if

$$(m\angle ABC) + (m\angle ADC) = 180°.$$

(Of course, you can use a different ordering of the points and get, for example, that $ABCD$ is cyclic if and only if $(m\angle BAD) + (m\angle BCD) = 180°$.)

You will prove these results in 1C, 1D, and 1E.

1C. (0.5 Points for Each Statement = 1 Point Total.) Prove that if the quadrilateral $ABCD$ is cyclic, then $m\angle CAD = m\angle CBD$ and $(m\angle ABC) + (m\angle ADC) = 180°$. (Hint: Use 1A and 1B.)

1D. (1 Point.) Prove that if in the quadrilateral $ABCD$ we have that $m\angle CAD = m\angle CBD$, then $ABCD$ is cyclic.

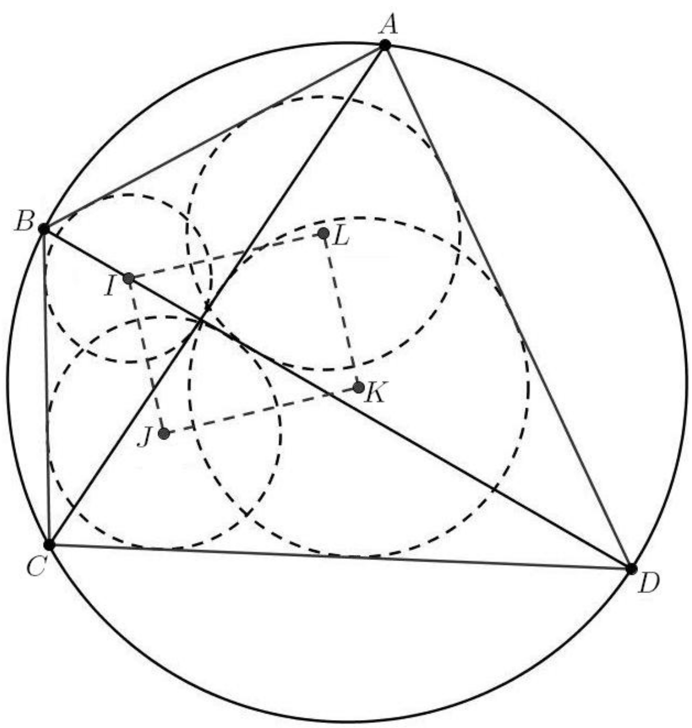
(Hint: Here is an outline of one way to prove this. If we can show that A is on the circle that passes through B, C, and D, then we will be done. Draw the circle that passes through B, C, and D. Let E be the point other than D where $\overleftrightarrow{AD}$ and the circle intersect. If we can show that $E = A$, then we will be done. In order to show this, draw $\overline{CE}$ and then use the result from 1C to prove that $m\angle CED = m\angle CAD$. Why does this show that $A = E$?)

1E. (1 Point.) Prove that if in the quadrilateral $ABCD$ we have that $(m\angle ABC) + (m\angle ADC) = 180°$, then $ABCD$ is cyclic.

(Hint: A proof similar to the one for 1D works here.)

2. The Japanese Theorem on Concyclic Quadrilaterals and The Japanese Theorem on Cyclic Polygons (6 Points Total)

The **Japanese Theorem on Cyclic Quadrilaterals** says the following: (See the diagram below.) Let $ABCD$ be a cyclic quadrilateral, and let I, J, K, and L be the centers of the circles inscribed within ΔABC, ΔBCD, ΔACD, and ΔABD, respectively. Then $IJKL$ is a rectangle.



2A. (0.5 Points Each = 2 Points Total.) Perform the details in each of the starred steps below to turn the outline into a complete proof of the Japanese Theorem on Cyclic Quadrilaterals. *You do NOT need to do any of the unstarred steps!* Of course, you are free to use their results in later steps.

(i) (*) State and prove a simple algebraic relation between $m\angle CAD$ and $m\angle CKD$ and prove that it is correct. (Hint: Focus on ΔACD, and draw in $\overline{AK}$, $\overline{CK}$, and $\overline{DK}$.)

(ii) (*) State, but do not prove, a similar algebraic relation between $m\angle CBD$ and $m\angle CJD$. Then use this relation and the one from (i), plus the Equal Angles Theorem to prove that $CJKD$ is cyclic.

(iii) Repeat steps (i) and (ii) to show that $DKLA$ is cyclic.

(iv) (*) State and prove a simple algebraic relation between $m\angle JKD$ and $m\angle BCD$ and prove that it is correct. (Hint: First use the Supplementary Angles Theorem. Then focus on ΔBCD.)

(v) (*) State, but do not prove, a similar algebraic relation between $m\angle LKD$ and $m\angle BAD$. Then use this relation and the one from (iv), plus the Supplementary Angles Theorem to show that

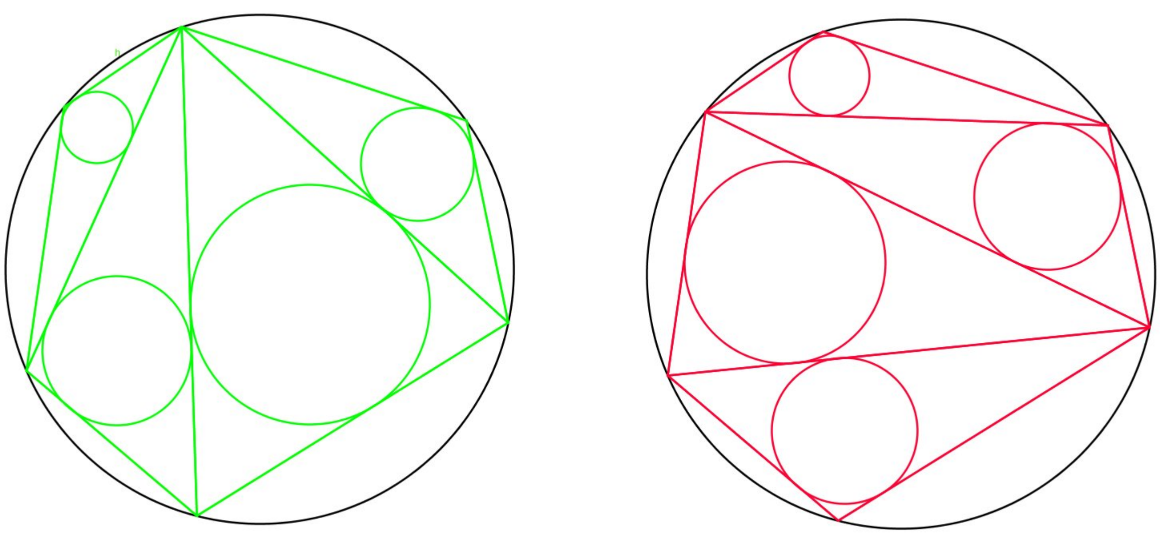
$$(m\angle LKD) + (m\angle JKD) = 270^\circ.$$

(vi) The previous step shows that $\angle JKL$ is right.

(vii) Repeat steps (i) to (vi) three more times to show that $\angle IJK$, $\angle KLI$, and $\angle LIJ$ are right and therefore $IJKL$ is a rectangle.

A **triangulation** of a polygon is a way of breaking a polygon and its interior into non-overlapping triangles such that every vertex of every triangle is a vertex of the original polygon. The diagram below gives two different triangulations of the same polygon.

The **Japanese Theorem on Cyclic Polygons** says the following: (See the diagram below.) For any cyclic polygon $\mathcal{P}$, the sum of the radii of the inscribed circles in any triangulation of $\mathcal{P}$ does not depend on the specific triangulation chosen. So, for example, the sum of the radii of the circles in the diagram below and to the left equals the sum of the radii of the circles in the diagram below and to the right.



We will prove this theorem using induction on the number of sides of $\mathcal{P}$.

2B. (2 Points.) In the diagram on the previous page, let ℓ_1 and ℓ_3 be the lines that are parallel to $\overline{AC}$ and pass through I and K. Let ℓ_2 and ℓ_4 be the lines that are parallel to $\overline{BD}$ and pass through J and L. Prove that the four lines ℓ_1, ℓ_2, ℓ_3, and ℓ_4 form a rhombus.

2C. (1 Point.) Explain why the result in 2B proves the Japanese Theorem on Cyclic Polygons in the case when $\mathcal{P}$ has four sides.

2D. (1 Point.) Now assume that for some integer $n \geq 4$, the result of The Japanese Theorem on Cyclic Polygons is true if $\mathcal{P}$ has n sides. Prove that the result of The Japanese Theorem on Cyclic Polygons is true if $\mathcal{P}$ has $n+1$ sides. With this step the induction, and therefore the proof of The Japanese Theorem on Cyclic Quadrilaterals, is complete.

(Hint: The result of 2C is helpful.)

3. Carnot's Theorem and The Japanese Theorem on Cyclic Polygons (5 Points Total)

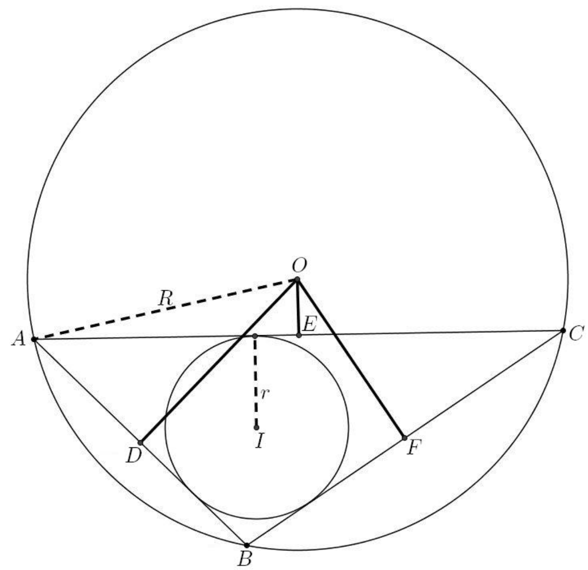
Carnot's Theorem states the following: (Refer to the diagram below.) Let ABC be a triangle, let O be the center of its circumscribed circle and R its radius, and let I be the center of its inscribed circle and r be its radius. Let D, E, and F be the feet of the perpendiculars from O to $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$, respectively. Then:

$$\|\overline{OD}\| + \|\overline{OE}\| + \|\overline{OF}\| = R + r,$$

where for $X \in \{D, E, F\}$, $\|\overline{OX}\|$ means;

(i) the length of $\overline{OX}$ if the interior of $\overline{OX}$ intersects the interior of ΔABC and
(ii) the negative of the length of $\overline{OX}$ if the interior of $\overline{OX}$ does not intersect the interior of ΔABC.

In the diagram below, $\|\overline{OE}\|$ is negative and both $\|\overline{OD}\|$ and $\|\overline{OF}\|$ are positive.



3A. (2 Points.) Prove Carnot's theorem in the case when O is outside or on the boundary of ΔABC.
(Hint: Draw the altitudes of ΔABC.)

3B. (1 Point.) Prove Carnot's theorem in the case when O is inside or on the boundary of ΔABC. (You will have proved Carnot's theorem twice when O is on the boundary of ΔABC; this is intentional and may be helpful later.)
(Hint: The proof will be similar to the one in 3A.)

3C. (1 Point.) Let $\mathcal{P}$ be the cyclic polygon $P_1P_2\cdots P_n$. Let O be the center of the circle that passes through all of the vertices of $\mathcal{P}$ and let R be the radius of the circle. For each i between 1 and n, let Q_i be the foot of the perpendicular from O to $\overleftrightarrow{P_iP_{i+1}}$, where P_{n+1} means "P_1." Prove that the sum of the radii of the inscribed circles in any triangulation is exactly equal to:

$$\|\overline{OQ_1}\| + \|\overline{OQ_2}\| + \cdots + \|\overline{OQ_n}\| - R(n-2),$$

where for each i between 1 and n, $\|OQ_i\|$ means:

(i) the length of $\overline{OQ_i}$ if the interior of $\overline{OQ_i}$ intersects the interior of $\mathcal{P}$ and
(ii) the negative of the length of $\overline{OQ_i}$ if the interior of $\overline{OQ_i}$ does not intersect the interior of $\mathcal{P}$.

(There was originally a typo in this section of the power round. The relevant expression was written as:

$$\|\overline{OQ_1}\| + \|\overline{OQ_2}\| + \cdots + \|\overline{OQ_n}\| + R(n-2).$$

All teams who pointed out that there was a typo received full credit, as did all teams who proved the corrected version.)

3D. (1 Point.) Use the result of 3C to give a second proof of the Japanese Theorem on Cyclic Polygons.