DUKE MATH MEET 2009: INDIVIDUAL ROUND

In the Individual Round there are five sub-rounds of two problems each to be solved individually. Like all rounds in the Duke Math Meet, only pencil and paper are allowed. At the start of each of the five sub-rounds, each student will turn over the sheet with the two questions, but will not be allowed to pick up their pencils. The moderator will then read the two questions aloud. When the moderator is finished reading the timer will begin and students may begin working. Students will have 10 minutes for each pair of problems and will receive 5-minute, 1-minute, and 15-second warnings for each pair of problems. When the 10 minutes are up, students must put down their pencils. Each correct answer from each student will add 1 point to their teams score.

Individual Round Problem 1 and 2

1. Let $p > 5$ be a prime. It is known that the average of all of the prime numbers that are at least 5 and at most p is 12. Find p.
2. The numbers $1, 2, \ldots, n$ are written down in random order. What is the probability that $n-1$ and n are written next to each other? (Give your answer in term of n.)

Individual Round Problem 3 and 4

3. The Duke Blue Devils are playing a basketball game at home against the UNC Tar Heels. The Tar Heels score N points and the Blue Devils score M points, where $1 < M, N < 100$. The first digit of N is a and the second digit of N is b. It is known that $N = a + b^2$. The first digit of M is b and the second digit of M is a. By how many points do the Blue Devils win?
4. Let $P(x)$ be a polynomial with integer coefficients. It is known that $P(x)$ gives a remainder of 1 upon polynomial division by $x + 1$ and a remainder of 2 upon polynomial division by $x + 2$. Find the remainder when $P(x)$ is divided by $(x + 1)(x + 2)$.

Individual Round Problem 5 and 6

5. Dracula starts at the point (0,9) in the plane. Dracula has to pick up buckets of blood from three rivers, in the following order: the Red River, which is the line $y = 10$; the Maroon River, which is the line $y = 0$; and the Slightly Crimson River, which is the line $x = 10$. After visiting all three rivers, Dracula must then bring the buckets of blood to a castle located at (8,5). What is the shortest distance that Dracula can walk to accomplish this goal?
6. Thirteen hungry zombies are sitting at a circular table at a restaurant. They have five identical plates of zombie food. Each plate is either in front of a zombie or between two zombies. If a plate is in front of a zombie, that zombie and both of its neighbors can reach the plate. If a plate is between two zombies, only those two zombies may reach it. In how many ways can we arrange the plates of food around the circle so that each zombie can reach exactly one plate of food? (All zombies are distinct.)

Individual Round Problem 7 and 8

7. Let $R_I, R_{II}, R_{III}, R_{IV}$ be areas of the elliptical region

$$\frac{(x-10)^2}{10} + \frac{(y-31)^2}{31} \leq 2009$$

that lie in the first, second, third, and fourth quadrants, respectively. Find $R_I - R_{II} + R_{III} - R_{IV}$.

8. Let r_1, r_2, r_3 be the three (not necessarily distinct) solutions to the equation $x^3 + 4x^2 - ax + 1 = 0$. If a can be any real number, find the minimum possible value of

$$\left(r_1 + \frac{1}{r_1}\right)^2 + \left(r_2 + \frac{1}{r_2}\right)^2 + \left(r_3 + \frac{1}{r_3}\right)^2.$$

Individual Round Problem 9 and 10

9. Let n be a positive integer. There exist positive integers $1 = a_1 < a_2 < \ldots < a_n = 2009$ such that the average of any $n - 1$ of elements of $\{a_1, a_2, \ldots, a_n\}$ is a positive integer. Find the maximum possible value of n.

10. Let $A(0) = (2, 7, 8)$ be an ordered triple. For each n, construct $A(n)$ from $A(n-1)$ by replacing the kth position in $A(n-1)$ by the average (arithmetic mean) of all entries in $A(n-1)$, where $k \equiv n \pmod 3$ and $1 \leq k \leq 3$. For example, $A(1) = \left(\frac{17}{3}, 7, 8\right)$ and $A(2) = \left(\frac{17}{3}, \frac{62}{9}, 8\right)$. It is known that all entries converge to the same number N. Find the value of N.