DUKE MATH MEET 2009: POWER ROUND

In the Power Round the entire team of six students will have 60 minutes to answer this series of proof-based questions. The team members may collaborate freely, but as with all rounds in the Duke Math Meet, only pencil/pen and paper may be used (no calculators of any kind). After 60 minutes the team will submit all solutions. Solutions to different-numbered questions must be on different sheets of paper. Cross out anything you do not want graded. Teams will be given 30-minute, 5-minute, and 1-minute warnings. **Teams may use results of previous problems to solve later problems, even if the team has not submitted solutions to those previous problems.** The number of points earned for each problem varies, but the total is 20 points. To receive full points, solutions to problems must be clear mathematical proofs.

1. (1 Point.) Consider all possible colorings of the faces of a cube with colors white and black. (A coloring is an assignment of each face with a color.) We say that two colorings are *rotationally equivalent* if one can be rotated to form the other, and otherwise we call the colorings *rotationally distinct*. How many rotationally distinct colorings are there?

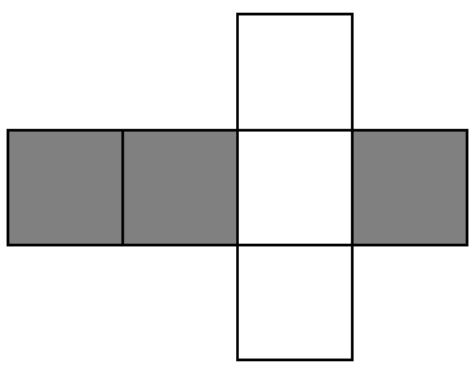
2. Such a question becomes much harder if there are, say, 10 colors instead of 2 (you can no longer get the answer by simple counting). For this, we'll need to know more about the symmetries of a cube.

   (a) (0.5 Points.) Pick a face of the cube and call it A. Consider rotations of the cube into itself. How many such rotations take face A to itself? (The "rotation" that does not move the cube at all **always** counts as a rotation.)

   (b) (0.5 Points.) Pick another face and call it B. Count the number of rotations of the cube into itself that move A to B. Explain why this answer is the same as the answer to part (a) by giving a way to match up the rotations.

   (c) (1 Point.) Use this to count the total number of rotations of a cube into itself.

3. (a) (1 Point.) Consider the following coloring of a cube. (Wrap the following diagram into a cube.) Count the number of rotations of the cube into itself that do not change the color of any face—rotations that take black faces to black faces, and white faces to white faces.



   (b) (1.5 Points.) Using the answers from (c) and (d), count the number of colorings (assignments of each face with some color) that are rotationally equivalent to the above cube. *i.e.*, count the number of colorings that can be formed by rotating the above cube into itself.

4. Let N be the number of rotationally distinct colorings of a cube with 10 colors. For each rotation r of the cube into itself, call the number of colorings (not necessarily rotationally distinct) c that rotation r leaves unchanged $\text{Fix}(r)$. Now consider all ordered pairs (r, c) in which rotation r leaves coloring c unchanged. (In counting these ordered pairs a coloring simply means an assignment of colors to faces; we do not require the colorings to be rotationally distinct.)

(a) (1.5 Points.) Count the number of all possible ordered pairs (r, c), expressing the answer using the function $\text{Fix}(r)$.

5. (a) (2.5 Points.) Count the number of ordered pairs (r, c) using another method, and express the answer in terms of N. (You'll need the ideas from problem 3 as well as some additional cleverness.)

   (b) (1 Point.) Prove that N is the average of $\text{Fix}(r)$ over all rotations r.

6. We can classify rotations of the cube into itself into equivalent types, where two rotations are of the same type if they do the same thing except possibly to different faces and in possibly different orientations. For instance, all $90°$ rotations around a face are of the same type; all $180°$ rotations around a face are of the same type, etc.

   (a) (1.5 Points.) Completely classify all possible rotations into equivalent types, and count the number of rotations of each type. In other words, complete the following table.

Type of rotation	**Number of rotations of that type**
The "do-nothing" rotation	1
$90°$ rotations about a face	?
...	...

   (b) (1 Points.) Explain why if rotation r_1 and rotation r_2 are of the same type, then $\text{Fix}(r_1) = \text{Fix}(r_2)$.

   (c) (1.5 Point.) Compute $\text{Fix}(r)$ for each type of rotation. Recall the definition of $\text{Fix}(r)$ in problem 4—consider colorings with 10 colors.

7. (2 Points.) Use the classification of rotations of a cube and the formula from 5(b) to find N: the number of rotationally distinct colorings of a cube using 10 colors.

Now apply the same method to compute the following:

8. (1.5 Points.) The number of rotationally distinct colorings of a cube using n colors. Express your answer in terms of n.

9. (2 Points.) The number of rotationally distinct colorings of the tetrahedron using n colors. Express your answer in terms of n.