Individual Round

1. Ana, Bob, Cho, Dan, and Eve want to use a microwave. In order to be fair, they choose a random order to heat their food in (all orders have equal probability). Ana's food needs 5 minutes to cook, Bob's food needs 7 minutes, Cho's needs 1 minute, Dan's needs 12 minutes, and Eve's needs 5 minutes. What is the expected number of minutes Bob has to wait for his food to be done?

2. ABC is an equilateral triangle. H lies in the interior of ABC, and points X, Y, Z lie on sides AB, BC, CA, respectively, such that $HX \perp AB, HY \perp BC, HZ \perp CA$. Furthermore, $HX = 2, HY = 3, HZ = 4$. Find the area of triangle ABC.

3. Amy, Ben, and Chime play a dice game. They each take turns rolling a die such that the first person to roll one of his favorite numbers wins. Amy's favorite number is 1, Ben's favorite numbers are 2 and 3, and Chime's are 4, 5, and 6. Amy rolls first, Ben rolls second, and Chime rolls third. If no one has won after Chime's turn, they repeat the sequence until someone has won. What's the probability that Chime wins the game?

4. A point P is chosen randomly in the interior of a square $ABCD$. What is the probability that the angle $\angle APB$ is obtuse?

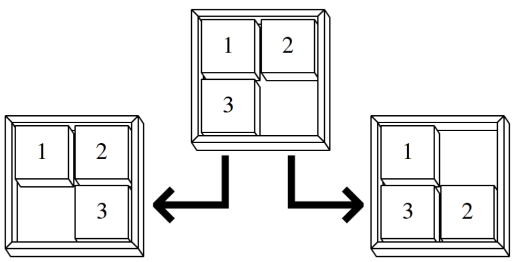
5. Let $ABCD$ be the quadrilateral with vertices $A = (3, 9), B = (1, 1), C = (5, 3)$, and $D = (a, b)$, all of which lie in the first quadrant. Let M be the midpoint of AB, N the midpoint of BC, O the midpoint of CD, and P the midpoint of AD. If $MNOP$ is a square, find (a, b).

6. Let M be the number of positive perfect cubes that divide 60^{60}. What is the prime factorization of M?

7. Given that x, y, and z are complex numbers with $|x| = |y| = |z| = 1$, $x + y + z = 1$ and $xyz = 1$, find $|(x + 2)(y + 2)(z + 2)|$.

8. If $f(x)$ is a polynomial of degree 2008 such that $f(m) = \frac{1}{m}$ for $m = 1, 2, \ldots, 2009$, find $f(2010)$.

9. A drunkard is randomly walking through a city when he stumbles upon a 2×2 sliding tile puzzle. The puzzle consists of a 2×2 grid filled with a blank square, as well as 3 square tiles, labeled 1, 2, and 3. During each turn you may fill the empty square by sliding one of the adjacent tiles into it. The following image shows the puzzle's correct state, as well as two possible moves you can make:



Assuming that the puzzle is initially in an incorrect (but solvable) state, and that the drunkard will make completely random moves to try and solve it, how many moves is he expected to make before he restores the puzzle to its correct state?

10. How many polynomials $p(x)$ exist such that the coefficients of $p(x)$ are a rearrangement of $\{0, 1, 2, \cdots, \deg p(x)\}$ and all of the roots of $p(x)$ are rational? (Note that the leading coefficient of $p(x)$ must be nonzero.)

Relay Round

Round 1

1. The sum of two prime numbers is 25. What is their product?
2. If D, U, K, and E are distinct positive integers and $D \cdot U \cdot K \cdot E = 24 \cdot \text{TNYWR}$, what is the maximum value of $D + U + K + E$?
3. Let n be TNYWR. There are n girls and n boys at a high school dance. The first boy dances with 1 randomly chosen girl, the second boy dances with 2 randomly chosen girls, and so on, with the last boy dancing with all n girls. If Andrew and Joanna are at the dance, what is the probability that they danced together?

Round 2

1. What is the 2011^{th} digit from the right of $\frac{2010^{2010}}{3}$?
2. Let k be TNYWR. What is the remainder when $1^3 + 2^3 + 3^3 + \cdots + k^3$ is divided by 7?
3. Let $r = \text{TNYWR} + 1$. A circular sector of angle measure $144°$ is removed from a circle of radius r, [diagram] and the straight sides of the resulting figure are joined to form the top of a cone. What is the volume of the cone?

Team Round

1. Find the smallest positive integer N such that $N!$ is a multiple of 10^{2010}.
2. An equilateral triangle T is externally tangent to three mutually tangent unit circles, as shown in the diagram. Find the area of T.
3. The polynomial $p(x) = x^3 + ax^2 + bx + c$ has the property that the average of its roots, the product of its roots, and the sum of its coefficients are all equal. If $p(0) = 2$, find b.
4. A regular pentagon $P = A_1A_2A_3A_4A_5$ and a square $S = B_1B_2B_3B_4$ are both inscribed in the unit circle. For a given pentagon P and square S, let $f(P, S)$ be the minimum length of the minor arcs A_iB_j, for $1 \leq i \leq 5$ and $1 \leq j \leq 4$. Find the maximum of $f(P, S)$ over all pairs of shapes.

5. Let a, b, c be three three-digit perfect squares that together contain each nonzero digit exactly once. Find the value of $a + b + c$.

6. There is a big circle P of radius 2. Two smaller circles Q and R are drawn tangent to the big circle P and tangent to each other at the center of the big circle P. A fourth circle S is drawn externally tangent to the smaller circles Q and R and internally tangent to the big circle P. Finally, a tiny fifth circle T is drawn externally tangent to the 3 smaller circles Q, R, S. What is the radius of the tiny circle T?

7. Let $P(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)(\cdots)$. This infinite product converges when $|x| < 1$. Find $P(\frac{1}{2010})$.

8. $P(x)$ is a polynomial of degree four with integer coefficients that satisfies $P(0) = 1$ and $P(\sqrt{2}+\sqrt{3}) = 0$. Find $P(5)$.

9. Find all positive integers $n \geq 3$ such that both roots of the equation

$$(n-2)x^2 + (2n^2 - 13n + 38)x + 12n - 12 = 0$$

are integers.

10. Let a, b, c, d, e, f be positive integers (not necessarily distinct) such that

$$a^4 + b^4 + c^4 + d^4 + e^4 = f^4.$$

Find the largest positive integer n such that n is guaranteed to divide at least one of a, b, c, d, e, f.

Tiebreaker Round

1. Let the series a_n be defined as $a_1 = 1$ and $a_n = \sum_{i=1}^{n-1} a_i a_{n-i}$ for all positive integers n. Evaluate $\sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i a_i$.

2. a, b, c, and d are distinct real numbers such that

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{d} = d + \frac{1}{a} = x.$$

Find $|x|$.

3. Find all ordered tuples (w, x, y, z) of complex numbers satisfying

$$\begin{aligned} x + y + z + xy + yz + zx + xyz &= -w \\ y + z + w + yz + zw + wy + yzw &= -x \\ z + w + x + zw + wx + xz + zwx &= -y \\ w + x + y + wx + xy + yw + wxy &= -z. \end{aligned}$$

Devil Round

1. Find all x such that $(\ln(x^4))^2 = (\ln(x))^6$.

2. On a piece of paper, Alan has written a number N between 0 and 2010, inclusive. Yiwen attempts to guess it in the following manner: she can send Alan a positive number M, which Alan will attempt to subtract from his own number, which we will call N. If M is less than or equal N, then he will erase N and replace it with $N - M$. Otherwise, Alan will tell Yiwen that $M > N$. What is the minimum number of attempts that Yiwen must make in order to determine uniquely what number Alan started with?

3. How many positive integers between 1 and 50 have at least 4 distinct positive integer divisors? (Remember that both 1 and n are divisors of n.)

4. Let F_n denote the n^{th} Fibonacci number, with $F_0 = 0$ and $F_1 = 1$. Find the last digit of

$$\sum_{i=0}^{97!+4} F_i.$$

5. Find all prime numbers p such that $2p + 1$ is a perfect cube.

6. What is the maximum number of knights that can be placed on a 9×9 chessboard such that no two knights attack each other?

7. S is a set of 9 consecutive positive integers such that the sum of the squares of the 5 smallest integers in the set is the sum of the squares of the remaining 4. What is the sum of all 9 integers?

8. In the following infinite array, each row is an arithmetic sequence, and each column is a geometric sequence. Find the sum of the infinite sequence of entries along the main diagonal.

$$\begin{array}{ccccc} 1 & ? & \frac{1}{2} & ? & \cdots \\ \frac{1}{2} & ? & ? & ? & \cdots \\ ? & ? & ? & ? & \cdots \\ ? & ? & ? & ? & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

9. Let $x > y > 0$ be real numbers. Find the minimum value of $\frac{x}{y} + \frac{4x}{x-y}$.

10. A regular pentagon $P = A_1A_2A_3A_4A_5$ and a square $S = B_1B_2B_3B_4$ are both inscribed in the unit circle. For a given pentagon P and square S, let $f(P, S)$ be the minimum length of the minor arcs A_iB_j, for $1 \le i \le 5$ and $1 \le j \le 4$. Find the maximum of $f(P, S)$ over all pairs of shapes.

11. Find the sum of the largest and smallest prime factors of $9^4 + 3^4 + 1$.

12. A transmitter is sending a message consisting of 4 binary digits (either ones or zeros) to a receiver. Unfortunately, the transmitter makes errors: for each digit in the message, the probability that the transmitter sends the correct digit to the receiver is only 80%. (Errors are independent across all digits.) To avoid errors, the receiver only accepts a message if the sum of the first three digits equals the last digit modulo 2. If the receiver accepts a message, what is the probability that the message was correct?

13. Find the integer N such that

$$\prod_{i=0}^{8} \sec\left(\frac{\pi}{9}2^i\right) = N.$$