Duke Math Meet 2011: Individual Round

In the Individual Round there are five sub-rounds of two problems each to be solved individually. Like all rounds in the Duke Math Meet, only pencil and paper are allowed. At the start of each of the five sub-rounds, each student will turn over the sheet with the two questions, but will not be allowed to pick up their pencils. The moderator will then read the two questions aloud. When the moderator is finished reading the timer will begin and students may begin working. Students will have 10 minutes for each pair of problems and will receive 5-minute, 1-minute, and 15-second warnings for each pair of problems. When the 10 minutes are up, students must put down their pencils. Each correct answer from each student will add 1 point to their team's score.

Individual Round Problems 1 and 2

1. Elsie M. is fixing a watch with three gears. Gear A makes a full rotation every 5 minutes, gear B makes a full rotation every 8 minutes, and gear C makes a full rotation every 12 minutes. The gears continue spinning until all three gears are in their original positions at the same time. How many minutes will it take for the gears to stop spinning?

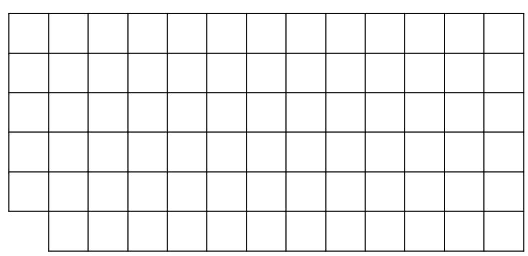
2. Optimus has to pick 10 distinct numbers from the set of positive integers $\{2, 3, 4, \ldots, 29, 30\}$. Denote the numbers he picks by $a_1, a_2, \ldots, a_{10}$. What is the least possible value of

$$d(a_1) + d(a_2) + \cdots + d(a_{10}),$$

where $d(n)$ denotes the number of positive integer divisors of n? For example, $d(33) = 4$ since 1, 3, 11, and 33 divide 33.

Individual Round Problems 3 and 4

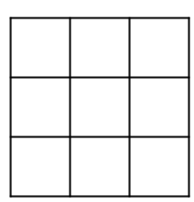
3. Michael is given a large supply of both 1×3 and 1×5 dominoes and is asked to arrange some of them to form a 6×13 rectangle with one corner square removed. What is the minimum number of 1×3 dominoes that Michael can use?



4. Andy, Ben, and Chime are playing a game. The probabilities that each player wins the game are, respectively, the roots a, b, and c of the polynomial $x^3 - x^2 + \frac{111}{400}x - \frac{9}{400} = 0$ with $a \leq b \leq c$. If they play the game twice, what is the probability of the same player winning twice?

Individual Round Problems 5 and 6

5. TongTong is doodling in class and draws a 3×3 grid. She then decides to color some (that is, at least one) of the squares blue, such that no two 1×1 squares that share an edge or a corner are both colored blue. In how many ways may TongTong color some of the squares blue? TongTong cannot rotate or reflect the board.



6. Given a positive integer n, we define $f(n)$ to be the smallest possible value of the expression

$$|\,\square\, 1 \,\square\, 2 \,\cdots\, \square\, n|,$$

where we may place a $+$ or a $-$ sign in each box. So, for example, $f(3) = 0$, since $|+1+2-3| = 0$. What is $f(1) + f(2) + \cdots + f(2011)$?

Individual Round Problems 7 and 8

7. The Duke Men's Basketball team plays 11 home games this season. For each game, the team has a $\frac{3}{4}$ probability of winning, except for the UNC game, which Duke has a $\frac{9}{10}$ probability of winning. What is the probability that Duke wins an odd number of home games this season?

8. What is the sum of all integers n such that $n^2 + 2n + 2$ divides $n^3 + 4n^2 + 4n - 14$?

Individual Round Problems 9 and 10

9. Let $\{a_n\}_{n=1}^{N}$ be a finite sequence of increasing positive real numbers with $a_1 < 1$ such that

$$a_{n+1} = a_n\sqrt{1 - a_1^2} + a_1\sqrt{1 - a_n^2}$$

and $a_{10} = 1/2$. What is a_{20}?

10. Three congruent circles are placed inside a unit square such that they do not overlap. What is the largest possible radius of one of these circles?