Duke Math Meet 2012

Power Round Solutions

1. If v is a vertex of G, which has n edges, then v may have vertex at most $n-1$ - it can be connected to every other vertex of G, but no more.

2. (a) Every edge has two endpoints, so if the sum was odd some edge would have only one endpoint, a contradiction.

   (b) We have $S(G) = 2|E|$. Every edge has two endpoints, hence contributes a total of 2 to the sum of all degrees.

   (c) We have $S(G) \leq |V|^2 - |V|$, with equality if every pair of vertices is connected. Such a graph is called a complete graph.

3. (a) We can represent the people at the party as vertices, and put an edge between two people if they have shaken hands.

   (b) No. If somebody shakes nobody's hands, and somebody shakes everybody's hands, then those two people have both not shaken hands and shaken hands, a contradiction.

   (c) Every person may shake between 0 and $n-1$ hands. There are n distinct values in this range, but 0 and $n-1$ cannot occur simultaneously. Hence n people may be sorted by how many hands they have shaken; as there are only $n-1$ possible categories, some two people have shaken the same number of hands.

4. (a) The total number of possible edges that a graph with $|V|$ vertices may have is $(|V|^2 - |V|)/2$. Hence the condition given is that G has at most $2/3$ of all possible edges. If G has greater than $2/3$ of all possible edges, then by the pigeonhole principle some triangle has all 3 edges in E.

   (b) We repeat the analysis from the previous problem, but consider all subgraphs with 5 vertices instead. We can note by explicit construction that none of the four non-equivalent graphs on 5 vertices with 7 edges are triangle-free. Hence by the same analysis as above, if G has more than $6/10 = 3/5$ of the number of possible edges, some 5 vertices have 7 edges among them, and hence G is not triangle-free.

5. If G is disconnected, then it has one component with size at most $n/2$, so that the maximum possible degree of every vertex in this component is at most $n/2 - 1 < n/2$, a contradiction. Hence G is connected.

6. (a) No; there can be multiple longest paths.

   (b) Yes; P is in this case a Hamiltonian path. We can't at this point say that G contains a Hamiltonian cycle, though.

   (c) If $k = n$, then P is a Hamiltonian path; as (v_1, v_n) is an edge of G, P can be extended to a Hamiltonian cycle.

7. (a) Suppose first that (v_1, v_n) is an edge of G. Then one of $v_1, \cdots, v_n$ has a neighbor not in P; hence we may cyclically permute the elements of P and then append the neighbor not in P to get P' with $k+1$ vertices.

   Now if (v_1, v_n) is not an edge of G, then there are $n/2$ vertices among $v_1, \cdots, v_{n-1}$ preceding neighbors of v_1 and $n/2$ vertices among $v_1, \cdots, v_{n-1}$ that are neighbors of v_n. Hence $P' = (v_1, v_2, \cdots, v_j, v_n, v_{n-1}, v_{n-2}, \cdots, v_{j+1}, v_1)$ is a cycle. As one of the vertices of P' has a neighbor not in P', there exists a P'' with at least $k+1$ vertices.

   (b) If the longest path P is not a Hamiltonian path (i.e., $k < n$) then it can be extended to a longer path by the above, contradicting the assumption that it is the longest path. Hence P is a Hamiltonian path.

8. (a) There are $n/2$ vertices among $v_1, \cdots, v_{n-1}$ preceding neighbors of v_1 and $n/2$ vertices among $v_1, \cdots, v_{n-1}$ that are neighbors of v_n. Hence by the pigeonhole principle one of v_j $(1 \leq j \leq n-1)$ satisfies the desired properties.

   (b) In this case the path $P' = (v_1, v_2, \cdots, v_j, v_n, v_{n-1}, v_{n-2}, \cdots, v_{j+1}, v_1)$ is a Hamiltonian cycle.

9. If $\delta(G) \geq n/2$, then the conditions for Ore's theorem are clearly satsified; hence G contains a Hamiltonian cycle.

10. We modify the argument in 7(a) and 8(a) as follows: either v_1, v_n are adjacent, or we still have enough vertices since $\deg(v_1) + \deg(v_n) \geq n$.