Series #4: Alternating Series

The series that you have studied so far in this course were series with only positive terms, except for some geometric series. We have learned that, if a series converges, then the individual terms much approach 0; more explicitly, if the series

$$\sum_{k=1}^{\infty} a_k \text{ converges, then } \lim_{k\to\infty} a_k = 0.$$

On the other hand, we have also seen that the terms, a_k, may go to 0, but the series, $\sum_{k=1}^{\infty} a_k$, could still diverge, such as happens with the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$, which diverges even though $\lim_{k\to\infty} \frac{1}{k} = 0$. We shall see that the result is very different if, instead of having all positive terms, the signs of the terms alternate.

Definition. A series $\sum_{k=1}^{\infty} a_k$ is said to be an <u>alternating series</u> if the signs of the terms alternate between positive and negative.

Examples. The following series are all examples of alternating series:

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

$$4 - \tfrac{4}{3} + \tfrac{4}{5} - \tfrac{4}{7} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^{k}4}{2k+1}$$

$$1 - 2 + \tfrac{2^2}{2!} - \tfrac{3^2}{3!} + \tfrac{2^4}{4!} - \dots = \sum_{k=0}^{\infty} \frac{(-2)^{k}}{k!}$$

$$1 - \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^3 + \dots = \sum_{k=0}^{\infty} \left(-\tfrac{1}{2}\right)^{k}$$

Because the last example in this list is a geometric series, we know that it converges and that its sum is $\frac{1}{1-(-\frac{1}{2})} = \frac{2}{3}$. Notice in the first two examples how the alternating signs can be represented in the shorthand notation. In fact if we are writing a mathematical statement about alternating series in general, we may want to exhibit the alternating signs in that manner. For example, if we stipulate that all the numbers $\{a_k\}_{k=1}^{\infty}$ are positive, then the expression $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ represents an alternating series.

Comparison of the harmonic series and the alternating harmonic series.

So that we can get an understanding of what effect the alternating signs have on a series, we will look closely at the partial sums of the harmonic series and the corresponding alternating harmonic series.

In long hand here are the two series we are comparing.

The harmonic series: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

and the alternating harmonic series: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

First, compare the partial sums of these two series.

Harmonic series

n	$S_n = \sum_{k=1}^{n} \frac{1}{k}$
1	1
2	1.5
3	1.83333
5	2.28333
10	2.92897
50	4.49921
100	5.18738
200	5.87803
400	6.56993
800	7.26245

Alternating harmonic series

n	$S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$
1	1
2	0.5
3	0.83333
5	0.78333
10	0.64563
50	0.68325
100	0.68817
200	0.69065
400	0.69190
800	0.69252

The partial sums of the harmonic series are increasing without bound—albeit they are increasing very slowly. In fact one reason this series is so interesting is that, even though we know that it diverges, it seems that it “almost converges.” The terms $\frac{1}{k}$ just weren't quite small enough to make the series converge. On the other hand, look at the partial sums of the *alternating* harmonic series. These partial sums are a strong indication that this series is converging.

Why should the alternating harmonic series converge?

Intuitively, what happens when we make the signs of a series alternate, as we did above in creating the alternating harmonic series from the harmonic series, is that we improve the chances of getting convergence: the alternating signs mean that we get some cancellation. The cancellation, combined with the fact that the individual terms are decreasing in size to zero, is enough to cause the series to converge. The degree to which the cancellation affects the series is evident when you note that the sum of the alternating harmonic series does not appear to be very large.

The argument for convergence.

We will look more closely at the alternating harmonic series, $\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$, and we will argue why it must converge. Consider the partial sums, one-by-one.

(1) We begin with $S_1 = 1$.

(2) To get S_2 we subtract $\frac{1}{2}$ from S_1. Note that the number we subtracted was smaller than the first term.

(3) To get S_3 we add $\frac{1}{3}$ to the previous partial sum. Note that the term we added was smaller than the last one we subtracted.

(4) To get S_4 we subtract $\frac{1}{4}$ from the previous partial sum. Note that the term we subtracted was smaller than the last one we added.

The process continues. We alternate adding and subtracting, always by a smaller and smaller amount. The partial sums have the following relationship to each other:

$$S_2 < S_4 < S_6 < \dots \quad \dots < S_5 < S_3 < S_1$$

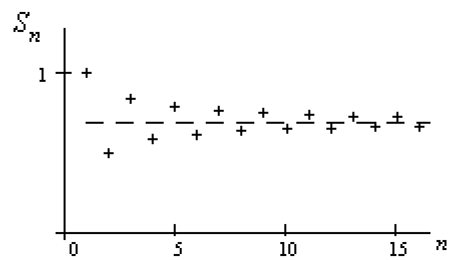
In particular the odd partial sums are decreasing and the even partial sums are increasing. If the partial sums are actually approaching a limit, then that limit must lie in the interval between these odd and even partial sums. If we can argue that this interval between the odd and even partial sums must be shrinking to length 0, then there must be a limiting value of the sequence of partial sums!

Let S_n and S_{n+1} be any two consecutive partial sums. How far apart are these numbers? Well, to get S_{n+1} from S_n you simply add the $(n+1)^{st}$ term: $\frac{(-1)^{n+1+1}}{n+1}$. It follows that the *size* of the difference between S_n and S_{n+1} is

$$\left|S_{n+1} - S_n\right| = \frac{1}{n+1}$$

Thus, as $n \to \infty$, the gap between the odd and even partial sums approaches 0. We have shown that the sequence of partial sums for the alternating harmonic series must converge!

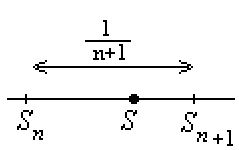
The illustration to the right shows graphically, for the alternating harmonic series, the relationship between the odd partial sums, the even partial sums, and the sum of the entire series. The number, n, denoting which partial sum is being computed, is on the horizontal axis. The sum of the series is represented by the horizontal asymptote, the odd partial sums by the points plotted above the asymptote, and the even partial sums by the points plotted below the asymptote.



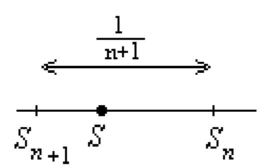
How big is the error when we approximate the sum of the alternating harmonic series with a partial sum?

We get a useful bonus from our convergence argument above. Even though we can't express the sum of series above exactly, we can approximate the sum as accurately as we please. Let's think of plotting the values the partial sums and the actual sum on a number line, and let's consider the distances between these sums. The actual sum of the series must lie between S_n and S_{n+1} for any choice of n (in other words, the sum of the series is between any two consecutive partial sums). Because the distance between S_n and S_{n+1} is $\frac{1}{n+1}$, the distance from the actual sum to S_n must be less than $\frac{1}{n+1}$. The situation is pictured below:

n even



n odd



For example, suppose we approximate $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ with $\sum_{k=1}^{500} \frac{(-1)^{k+1}}{k} = .692148180\ldots$.

We can conclude from our argument above that the error is less than $\frac{1}{501}$; i.e.,

$$\text{error} = \left| S - S_{500} \right| < \frac{1}{501} \; .$$

Alternating series in general.

Now that we have considered in detail what happens with the alternating harmonic series, we shall see what we can say about alternating series in general. We will use the following notation:

> In this context we use the symbols $a_1, a_2, a_3, \ldots$ to represent positive numbers. By assuming that the a_k are all positive, we know that when we write $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ we are representing an alternating series (i.e., we don't have to worry that some negative signs could be "hidden" in the terms a_k).

We are ready to address the question: under what conditions can we be certain that $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ will converge? Look at the argument that we used for the alternating harmonic series. What properties of that series were crucial in making the argument for convergence? Of course, there were the alternating signs—and we have those in the general alternating series, too. The only other property of the alternating harmonic series that we used was the fact that the terms $\frac{1}{k}$ were decreasing in size and approaching 0. (Could a sequence of terms decrease in size without approaching 0? Of course. And that's not good enough here because of the n^{th}-term test.) Those are the only properties we need: if we know that the general terms a_k are *strictly* decreasing to 0 (i.e., besides "decreasing to 0," no two can be equal), then the same argument we used for the alternating harmonic series will work for the general case.

Don't worry. We will not repeat the argument. Here is the general result:

Alternating Series Theorem.

It the terms, $a_1, a_2, a_3, \ldots,$ are all positive and strictly decreasing to 0, then the series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k$$

must converge. Furthermore, if we let S represent the sum of the series and S_n the n^{th} partial sum, then

$$\left|S - S_n\right| < a_{n+1}.$$

Example 1.

Consider the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$. To put this series into the notation of the theorem we would take $a_k = \frac{1}{\sqrt{k}}$. The terms $\frac{1}{\sqrt{k}}$ are obviously strictly decreasing to 0 and the series is alternating; therefore, the series must converge. Now suppose we use S_{100} to approximate the sum. What is the approximation and what can we say about the error?

$$S_{100} = \sum_{k=1}^{100} \frac{(-1)^{k+1}}{\sqrt{k}} = .5550236\ldots \text{ (from a calculator)}$$

And we know that the error must be less than $\frac{1}{\sqrt{101}}$; i.e.,

$$\textit{error} = \left|S - S_{100}\right| < a_{101} = \frac{1}{\sqrt{101}} .$$

Example 2.

Suppose you want to approximate the sum of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ with an error less than .001. How many terms must you include in the partial sum?

Solution: If we use S_n to approximate this sum, then the error will be less than $\frac{1}{\sqrt{n+1}}$. Because we want the error to be less than .001, we set up the inequality

$$\frac{1}{\sqrt{n+1}} \leq .001$$

and solve it for n:

$$\begin{aligned} 1 &\leq (.001)\sqrt{n+1} \\ \frac{1}{.001} &\leq \sqrt{n+1} \\ 1000 &\leq \sqrt{n+1} \\ (1000)^2 &\leq (n+1) \\ (1000)^2 - 1 &\leq n \end{aligned}$$

Therefore, if we compute a partial sum with at least $(1000)^2 - 1$ terms, then our approximation will have an error of less than .001 .

Absolute convergence.

We have seen that changing the signs of the terms in a series from all positive to alternating may change the answer to the convergence question. What happens if we take a series with alternating signs and make all the terms positive? We have only to look at examples we've already considered to see what could happen. The possibilities are illustrated by the cases we reiterate below:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \text{ converges, but } \sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges.}$$

$$\sum_{k=0}^{\infty} \left(-\tfrac{1}{2}\right)^k \text{ converges to } \tfrac{2}{3}\text{, and } \sum_{k=0}^{\infty} \left| \left(-\tfrac{1}{2}\right)^k \right| = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k \text{ converges to 2.}$$

It has become clear that, because of cancellation, mixing the signs of the terms gives the series a better chance of converging than if all the terms were positive. In the second case above we say that $\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k$ converges absolutely, because even if we take the absolute value of each term to create the new series $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$, this new series still converges. That result tells us something: the convergence of $\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k$ did not depend upon positive/negative cancellation of terms; rather, the terms $\left(-\frac{1}{2}\right)^k$ become so small as $k \to \infty$ that the series would converge even if there had been no cancellation.

Contrast the case that we just described with that of the alternating harmonic series, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$. We know from the Alternating Series Theorem that this series converges. But the fact that $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=0}^{\infty} \frac{1}{k}$ diverges, tells us that the convergence of the alternating harmonic series did, indeed, depend upon the positive/negative cancellation of terms. This contrast in the manner of convergence motivates us to make the following definition.

Definition. Suppose a series $\sum_{k=1}^{\infty} a_k$ converges. (The terms $\{a_k\}$ can have any sign.) If the series $\sum_{k=1}^{\infty} |a_k|$ also converges, then we say that $\sum_{k=1}^{\infty} a_k$ <u>converges absolutely</u>. On the other hand, if the series $\sum_{k=1}^{\infty} |a_k|$ diverges, then we say that $\sum_{k=1}^{\infty} a_k$ <u>converges conditionally</u>.

Example 3.

(1) $\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k$ converges absolutely.

(2) The alternating harmonic series converges conditionally.

(3) Suppose the geometric series $\sum_{k=0}^{\infty} r^k$ converges. By the Geometric Series Theorem we know that $|r| < 1$; thus, by the same theorem, $\sum_{k=0}^{\infty}|r^k|$ also converges. Therefore, every geometric series which converges, converges absolutely.

(4) $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k}}$ converges conditionally. We know the convergence is conditional because the Alternating Series Theorem implies that $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k}}$ converges, and we know from the Integral Test that $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ diverges.

(5) $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}$ converges absolutely.

Implications of absolute convergence.

We have seen that absolute convergence tells us something about the sizes of the terms being added: if a series converges absolutely, then the terms, a_k, become so small as $k\to\infty$, that positive/negative cancellation is not necessary to ensure that $\sum_{k=1}^{\infty} a_k$ converge. In other words, if $\sum_{k=1}^{\infty} a_k$ converges absolutely, then we could change the signs of any or all terms in any way we please and still get a series that would converge (although we would be changing the sum to which it converges). This observation can be said in a slightly different—but equivalent—way in the following theorem.

Absolute Convergence Theorem.
If the series $\sum_{k=1}^{\infty} \left|a_k\right|$ converges, then the series $\sum_{k=1}^{\infty} a_k$ must also converge.

It is common to think of this theorem as stating that "Absolute convergence implies convergence." Of course, you might ask, "So what? If a series converges, we know we can approximate the sum with a partial sum. If it diverges, then forget looking for an approximation. Why should we care about this distinction between absolute and conditional convergence?" The next example illustrates how we can use this theorem, and the comments that follow it hint at why mathematicians have made a distinction between absolute and conditional convergence.

Example 4.

Does the series $\sum_{k=1}^{\infty} \frac{sin(k)}{k^2}$ converge? We know that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. Because $0 \le \left|\frac{sin(k)}{k^2}\right| \le \frac{1}{k^2}$ for all $k \ge 1$, the series $\sum_{k=1}^{\infty} \left|\frac{sin(k)}{k^2}\right|$ converges by the Comparison Test. Thus, $\sum_{k=1}^{\infty} \frac{sin(k)}{k^2}$ must converge by the Absolute Convergence Theorem.

Why did we use absolute convergence? Note that the series $\sum_{k=1}^{\infty} \frac{sin(k)}{k^2}$ looks very similar to a p-series, but we can't use the p series result, because of the terms, $sin(k)$, which change in value and change in sign. Furthermore, the direct comparison, $\sum_{k=1}^{\infty} \frac{sin(k)}{k^2} \le \sum_{k=1}^{\infty} \frac{1}{k^2}$ does not help, even though $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, because $\sum_{k=1}^{\infty} \frac{sin(k)}{k^2}$ has negative terms; i.e., the sequence of partial sums for $\sum_{k=1}^{\infty} \frac{sin(k)}{k^2}$ is not an increasing sequence, so being bounded is not enough to ensure convergence.

Some comments on theory.

We are accustomed to the associative and commutative properties of addition. But it turns out that neither of these properties hold for conditionally convergent series. For example, the series $(5 - 5^{\frac{1}{2}}) + (5^{\frac{1}{2}} - 5^{\frac{1}{3}}) + (5^{\frac{1}{3}} - 5^{\frac{1}{4}}) + (5^{\frac{1}{4}} - 5^{\frac{1}{5}}) + \ldots$ converges to $5 - 1 = 4$, whereas the series $5 - (5^{\frac{1}{2}} - 5^{\frac{1}{2}}) - (5^{\frac{1}{3}} - 5^{\frac{1}{3}}) - (5^{\frac{1}{4}} - 5^{\frac{1}{4}}) - \ldots$ converges to 5. [You can check the partial sums—just pay close attention to where the parentheses are.] In fact a remarkable result of advanced calculus is that if a series is conditionally convergent, then we can rearrange the terms to make it add up to any number we please! But there is some good news awaiting you in advanced calculus: if a series is absolutely convergent, then the associative and commutative laws of (infinite) addition hold true. In particular, all rearrangements and regroupings of an absolutely convergent series have the same sum.

We mention these matters, not for you to try to prove them (that's hard) or to be overly concerned about them, but to give you another idea as to why mathematicians have made a distinction between absolute and conditionally convergent series.

The Ratio Test revisited.

In our earlier presentation of the Ratio Test we considered only series of positive terms. Now that we have the Absolute Convergence Theorem in hand, we can drop the "positive" part of the previous hypothesis.

The Generalized Ratio Test

Suppose the series $\sum_{k=1}^{\infty} a_k$ has no zero terms. Consider the limit: $\mathrm{L} = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$.

If $\mathrm{L} < 1$, then the series $\sum_{k=1}^{\infty} a_k$ converges absolutely.

If $\mathrm{L} > 1$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

The hypothesis that the series must have no zero terms is really no restriction at all. If there are some zeros in the series, we just leave them out.

The first conclusion follows from combining the earlier version of the Ratio Test and the Theorem on Absolute Convergence. The second conclusion follows from the proof of the Ratio Test, wherein we show that if $\mathrm{L} > 1$, the terms of the series (with or without absolute values) fail the n^{th}-term test.

Example 5.

Consider the series $\sum_{k=0}^{\infty} \frac{(-2)^k}{k!}$. If we apply the Generalized Ratio Test to the series we get $\mathrm{L} = \lim_{k\to\infty} \frac{\left|\frac{(-2)^{k+1}}{(k+1)!}\right|}{\left|\frac{(-2)^k}{k!}\right|} = 0$. (You should check this result.) It follows that the series converges absolutely.

Example 6.

For which values of x does the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^{2k}}{k}$ converge?

Solution: Using the Generalized Ratio Test we get

$$\mathrm{L} = \lim_{k\to\infty} \frac{\left|\frac{(-1)^{k+1+1}x^{2(k+1)}}{k+1}\right|}{\left|\frac{(-1)^{k+1}x^{2k}}{k}\right|} = \lim_{k\to\infty} \frac{x^2 k}{k+1} = x^2 .$$

We can set $x^2 < 1$, and solve for x to determine which values of x make L less than 1, and hence make the series converge absolutely. We find that $-1 < x < 1$. Furthermore, we see that if $x > 1$ or $x < -1$, then $\mathrm{L} > 1$ and the series must diverge. We have answered the question of convergence for all values of x except -1 and 1. We can treat those as special cases. If $x = \pm 1$, then the series becomes the alternating harmonic series, which we know converges conditionally.

Exercises.

For problems 1-6 determine if the series converges. If it does converge, either compute the sum (if you can), or approximate the sum with S_{50} and find a bound on the error.

1. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\pi}{6k!}$

2. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{k+1}$

3. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3}$

4. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}3^k}{7^k}$

5. $\sum_{k=1}^{\infty} \left((-1)^{k+1}(.001 + \frac{1}{k})\right)$

6. $\sum_{k=2}^{\infty} \frac{(-1)^k}{1-k^2}$

In problems 6-10 the series all converge. Either compute the sum precisely, or determine which partial sum you must use to approximate the sum with an error less than .01 and compute the approximation.

7. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^4}$

8. $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k}$

9. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$

10. $\sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{4}\right)^{2k+1}}{2k+1}$

11. Many Taylor Series are alternating; thus, we could use the Alternating Series Theorem to determine an error bound when we use a partial sum.

(a) For values of x in the interval $[-1,1]$, how many terms of the Taylor series (with base point 0) for $sin(x)$ would we need to ensure that the error is less than 0.1 ?

(b) The approximation $sin(x) \approx x$ is commonly used for small values of x. For what interval can we be sure that the error is less than 0.05 ?

12. When we're considering the question of convergence of a series, we can ignore a finite number of terms at the beginning. (Of course, we can't ignore any terms when we're computing the sum.) For example, you would have to use such a strategy with the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}k}{k^2+100}$. Explain carefully how you know this series must converge, and determine how large n must be for the partial sum, S_n, to approximate the sum with an error less than .01 . [Hint: Watch the conditions of the Alternating Series Theorem. And you can use derivatives to show when a function is decreasing.]

For each series below determine all values of x for which the series converges.

13. $\sum_{k=1}^{\infty} \frac{x^k}{k!}$

14. $\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k}$

15. $\sum_{k=1}^{\infty} \frac{(-1)^k 2^k x^k}{5^k}$

16. For which values of x will the first 500 terms of the Taylor Series (about 0) for $ln(x+1)$ approximate $ln(x+1)$ with an error less than 0.1 ?

Answers to selected problems

1. Converges. $S_{50} = 0.330977\ldots$ with an error less than $\frac{\pi}{6(51!)}$.
2. Diverges by the n^{th} Term Test.
4. Converges to the sum 0.3 exactly.
5. Diverges by the n^{th} Term Test.
7. Use $S_3 = -0.9498\ldots$
8. The exact sum is $\frac{2}{3}$.

11(b). $-0.669 < x < 0.669$

13. Converges for all real numbers.
14. Converges for $-1 < x \leq 1$.