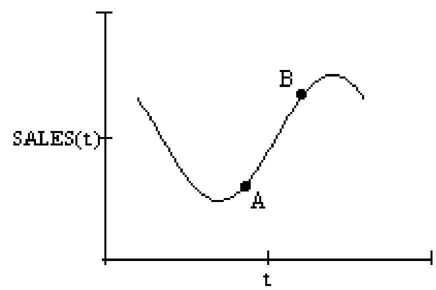
First and Second Derivatives and Roots

Purpose

The purpose of this lab is for you to acquire an understanding of the relationships among a function, it's first and second derivatives, and the roots of these functions. You will see how to use technology to enhance this understanding, and you will apply this understanding to critical analyses of non-mathematical statements.

Preview

Suppose the curve at the right gives the daily sales totals for a new item in your department store. It would certainly be of value to be able to identify the high and low points. But consider the points A and B on the curve. The rates of increase of sales at those two points are about the same, but it is obvious that at point A you would want to restock the item, and at B it might be preferable to let your supply on hand dwindle.



This type of marketing decision would be easy if we had a crystal ball that could show us the entire curve— in particular, the future curve. But, of course, we only have a “local” view. We shall see in this lab how we can use the local behavior of the graph to analyze a situation such as that above.

Part I: Zooming, Roots, and Extrema

1. Graph the function $f(t) = -2t^4 + 11t^3 + 15t^2 - 70t - 10$ on your calculator, using the calculator's default graphing ranges. Zoom out a couple of times and then zoom in on interesting parts of the curve until you are confident that you have both the “big picture” and a clear idea of the details of the graph.

2. Now have your calculator redraw the graph with a horizontal range of $[-3\,,6]$ and a vertical range of $[-100\,,200]$. On the attached graph paper carefully draw the graph that your calculator has produced.

3. Use your calculator to approximate the roots of the function. This operation will involve tracing along the curve and using the calculator's root finder. On your hand-drawn graph indicate the location, accurate to two decimal places, of the roots of f.

4. The function, f, makes several turns. At the points where the graph makes a turn, the y-coordinate of f is higher or lower than the surrounding points. Such a point on the graph of a function is called a *local extremum*, or more specifically, a *local maximum* or a *local minimum.* Use your calculator to find the locations of the extrema of f. (Your calculator may have a special key for this purpose. Without such a key, you could zoom in on the targeted point to find its coordinates.) Indicate on your hand-drawn graph the coordinates, accurate to two decimal places, of the extrema of this curve.

Part II: The First Derivative

1. Our next step is to superimpose the graph of f' on the graph of f. Your calculator may be able to do that automatically, or you may need to enter the expression for $f'(t)$ into the graphing menu and then graph the new expression. Either way, when you're done, your calculator's screen should show the graphs of both f and f'.

2. On the same graph where you drew the graph of f, draw the graph of f'. Be careful with the placement of the axis intersections and the turning points. You may want to draw f' with a different color pencil so your eye can easily follow one curve or the other.

3. Just as you have done with f, use your calculator to find the roots and extrema of f'. On the hand-drawn graph indicate the locations of these points accurate to two decimal places.

4. Describe the relationship between the roots of f' and the graph of f? Use the concept of tangent lines to explain your observation.

5. Consider the function $h(t) = f(t) + 70 = -2t^4 + 11t^3 + 15t^2 - 70t + 60$. Graph h along with its derivative. (Why does $h' = f'$?) Based on your observations of the graphs of h and its derivative, do you need to modify your answer to the previous question?

6. Describe the relationship between the sign of f' and the graph of f. Explain your observation in the context of rates of change and values of f.

7. What is the relationship between the locations of the roots of f and the locations of the roots of f'? In general if a differentiable function g has roots located at r_1 and r_2, what can you conclude about the behavior of g' between r_1 and r_2? Consider again the graphs of h and h' to test your conjecture. If there is a inconsistency, modify your conjecture so that it is true for both the functions f and h.

8. If it is possible, draw a graph of a differentiable function whose domain is $(-\infty, \infty)$ and such that f' has three distinct zeroes and f has one zero. Also, if it is possible, draw a graph of a differentiable function whose domain is $(-\infty, \infty)$ and such that f has three distinct zeroes and f' has one zero.

9. Use your conclusion from step 7 and your observations from step 8 to make a general statement about the relationship between the number of roots of a function g and the number of roots of g'; for example, if g' has exactly k roots, how many roots could g have?

10. You have just made observations that mathematicians have recorded as theorems. Complete the statement of each of the following theorems:

Maximum/Minimum Value Theorem: If the function f has a local minimum or a local maximum at $x = c$, and if $f'(c)$ exists, then $f'(c) =$ ____________ .

Rolle's Theorem: If $f(r_1) = 0$ and $f(r_2) = 0$ and if f is differentiable on the interval $[r_1, r_2]$, then f' ... .

Part III: Concavity and Inflection Points

1. Refer again to the graph shown in the preview. If you look closely at a small region on the curve around each of the points A and B, you will see that although the rates of increase of sales at these two points are the same, there is a difference in the shape of the curve at the two points. Can you describe the difference?

2. The terms that we use to distinguish the characteristics you observed above are “concave up” and “concave down.” Is the curve concave up or concave down at A? ... at B? State either a definition or a characterization of the phrases “concave up” and “concave down” sufficient to enable someone who knows no calculus to use them correctly in describing features of a graph.

3. You will notice that this graph changes concavity between points A and B. The point at which this change occurs is called an *inflection point*, which we will designate by “IP.” On the diagram in the preview indicate the location of the IP, and note that the curve “flexes” at that point.

4. Where on the sales curve in the preview is the tangent line the steepest? Describe the trend of the slopes of tangent lines just before and just after the point that you've chosen.

Part IV: Derivatives and Inflection Points

1. We will now turn our attention back to the function, f, which you graphed, along with its derivative function, in Part II. On the curve representing f locate the IPs as best you can from simple observation, and label them. (There are two.)

2. Describe the relationship between the location of the IPs of f and the graph of f'? Explain why this relationship should be true.

3. Use your calculator to locate the extrema of f', accurate to two decimal places. Explain how to use this information to find the IPs of f . On your hand-drawn graph of f indicate, accurate to two decimal places, the coordinates of the IPs.

4. Now have your calculator superimpose the graph of f'' on the graphs of f and f'. Then carefully draw by hand the curve representing f'' on the same graph paper where you have already drawn f and f''. (As before, it's best to use a different color.) Try to place the roots and turning point as accurately as you can.

5. Lightly sketch vertical lines through each of the IPs of f. Describe the relationship between the IPs of f and the graph of f''. Describe the relationship between the concavity of f and the sign of f'' .

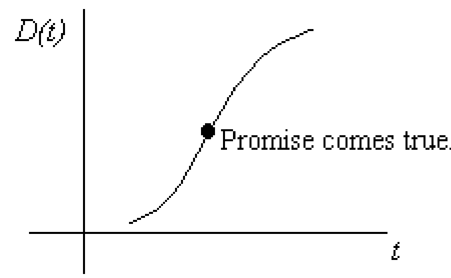
6. Use your calculator to approximate the roots of f'' accurate to two decimal places. Against which previous results can you check these answers?

Critical Analysis

Consider the following statement: *If I am elected to office, I promise to decrease the annual national deficit.*

We will use calculus to analyze what is being said here. The statement is actually about the national debt over time, because the annual deficit is simply the rate at which the debt is growing, where we measure time in years. Let $D(t)$ represent the size of the national debt at year t. The fact that there is an annual deficit implies that $D'(t) > 0$ for the current year. If the politician speaking these words can keep this promise, then there will soon be an economic situation where $D''(t) < 0$.

This curve could represent what is happening: the debt has been rising at a (presumably) increasing rate. Our politician promises to bring about the inflection in the curve at the right. It is clear, of course, that there was no promise to make $D'(t) < 0$.



Exercises

In each of the following exercises there is a statement and a definition of a function. Based upon what is given in the statement, make a sketch of the function and its derivatives. Explain why your graphs look the way they do.

1. *The Dow Jones average is continuing to increase, but at a decreasing rate.* Let $A(t)$ represent the Dow Jones average on day t.

2. *There is new data that suggests that the annual rate of growth of the world population peaked last year.* Let $P(t)$ be the number of people alive in year t.

Report

Your report should include the following:

1. A copy of the completed “Graphing Page.”

2. Complete statements of the *Maximum/Minimum Value Theorem* and *Rolle's Theorem.* In each case include a sketch to illustrate the theorem.

3. Your graphs in step 8.

4. The answer to step 9 in Part 2, along with a sketch to illustrate your argument.

5. A mathematical analysis of each of the statements in the Exercises.

6. Finally, look back at the preview graph of sales over time. Explain in terms of the first and second derivatives how you could know when to restock and when to cut back on orders. Explain how you could use sales data from electronic cash registers to get the information that you would need to make the restocking decision.

Graphing Page