# Fourier Analysis of a Musical Sound

**Purpose**

In this lab you will learn why the same note played on different musical instruments sounds different. The key to this understanding is the use of Fourier series to analyze musical sounds.

**Preview**

This lab is divided into four parts. Part one deals with the physical nature of a sound wave and with its perception. Part two deals with the distinguishing characteristics of musical sounds. In parts three and four you will use discrete Fourier analysis to find the harmonic structure of a musical sound.

**Part 1**

*The Physical Origin of Sound*. Sounds are variations in pressure which propagate through the air (or other media such as water). We will illustrate how such variations arise by thinking about the motion of a plucked guitar string. As the string moves up (Figure 1), it pushes air molecules in front of it, thereby making the air more dense above the string and less dense below the string. Higher density means higher air pressure, so the air pressure will be a little higher than average above the string and a little lower below. As the string comes back down (Figure 2), it compresses the air molecules beneath it that have rushed in to fill the low pressure area (pictured in Figure 1) and creates a new low pressure area above the string.

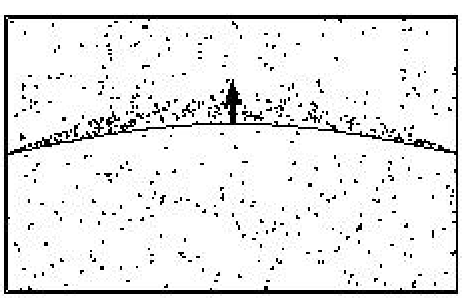


Figure 1

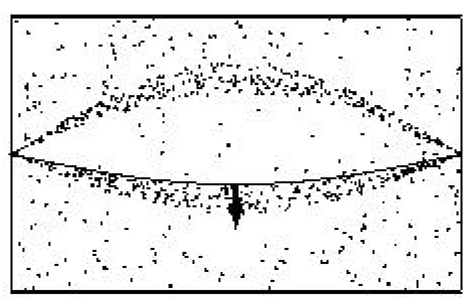


Figure 2

As the string continues to vibrate, alternate high and low pressure areas propagate away from the string (Figure 3).

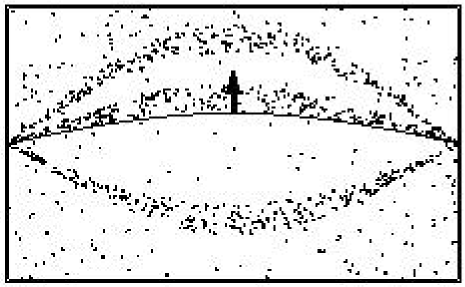


Figure 3

If one measured the pressure, $P,$ as a function of time, $t$, at some point near the string, one would expect $P(t)$ to be a periodic function (since the guitar string is vibrating periodically) which oscillates above and below normal pressure. Thus, we could write the function as

$$P(t) = a_0 + V(t)$$

where $V(t)$ is the deviation from the normal pressure $a_0$. Sounds produced by other means (e.g., speech or thunderclaps) would usually produce very complicated functions, $P(t)$ and $V(t)$, which would not necessarily be periodic.

*The Perception of Sound*. When the variations in pressure arrive at the ear, they propagate down the ear canal and set the tympanic membrane (the ear drum) in motion. This motion causes the three bones of the middle ear to oscillate. The vibrations of the third bone cause fluid movement in the cochlea (inner ear), and this fluid movement causes regions of a membrane (called the basilar membrane) to vibrate. Neurons (nerve cells) all along the basilar membrane send electrical signals to the brain in proportion to the local movement of the membrane. Different pressure deviation functions, $V(t)$, at the tympanic membrane cause different patterns of vibration on the basilar membrane, and therefore different patterns of electrical signals to the brain. Thus, different sounds (that is, pressure functions) are perceived differently.

**Part 2: Musical Sounds**

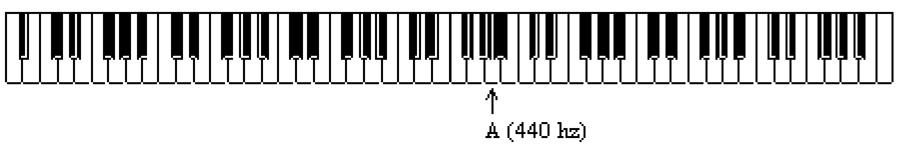
Most musical instruments produce sounds that are periodic in time. That is, there is a number, $p$, called the **period** of the sound so that $V(t+p) = V(t)$ for all $t$. Put differently, after $p$ units of time the pressure function repeats itself. For typical musical sounds, $p$ is a very small fraction of a second. The number of repetitions per second, $f$, is called the **frequency** of the sound, and you can see that

$$f = \frac{1}{p}$$

One repetition per second is called a **hertz**, named after Heinrich Hertz (1857-1894), the physicist who first broadcast and received radio waves. Human beings can hear sounds at frequencies ranging from approximately 20 hertz to 18,000 hertz.

If the physical action that produces the sound becomes stronger (like plucking a string harder), then the pressure deviation function $V(t)$ will have larger values. These larger variations in pressure cause the vibrations of the three bones in the middle ear and the vibrations of the basilar membrane to be larger, which in turn cause the neurons sending signals to the brain to fire at a higher rate. This process is how we perceive the size of the pressure changes as the *loudness* of the sound.

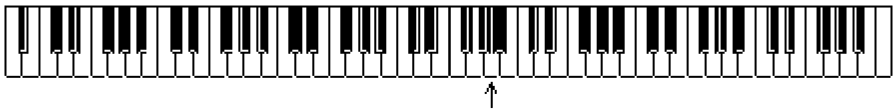
The *pitch* of a musical sound is defined as its frequency, and by tradition, certain frequencies are given names and called *notes*. The most widely accepted tuning system, in place since the mid-nineteenth century, uses the following method to assign note names to specific frequencies. The *A* note (above middle C) is assigned the frequency 440 hertz. This is the note you hear from the oboe when an orchestra is tuning up. The range of frequencies that either doubles or halves the frequency of a given note is divided up into 12 steps, and the frequencies at each step are given the note names *A, A#, B, C, C#, D, D#, E, F, F#, G, G#, A,* where the symbol # is read "sharp." These steps are chosen so that the ratios of frequencies of adjacent notes are equal[1].




If you start at any key on a piano and count 12 different keys (black and white) in either direction, the twelfth key will have the same note name as the one you started with, and it will have either half or twice the frequency of the original note. A note whose period is half another is said to be one "octave" above the other. When you sing the *do-re-mi* scale, the *do* at the beginning and the *do* at the end are one octave apart. A full-size piano has 88 keys. (The black keys are the sharps.) Thus, the

[1]In practice we choose to make slight deviations from this formula because of the ear's perception of the different frequencies.

frequency span of a piano is a little more than 7 octaves. The highest *A* note on the piano ($3^{rd}$ white key from the right end) has a frequency of 3520 hertz.




1. The first note on the left end of the piano keyboard is also an *A* note. Find its frequency.

2. Using the fact that the ratio of the frequencies between any two successive notes is constant, find the frequency of the *C* note above the middle *A*.

It is a common observation that the same note played on different instruments sounds different. The reason for this is that the perception of the sound depends not only on the pitch (that is, the frequency which is determined by the period) but also on the details of the pressure variation function $V(t)$ as $t$ goes through the period. Notice how different the functions $V(t)$ look for flutes, clarinets, bassoons, and trumpets (see page 508 of your textbook).

Let us suppose that our sound has period $p$. If the pressure deviation function has the simple form of a sine wave of period $p$,

$$V(t) = b_1 sin\,\left(\frac{2\pi}{p}\,t\right),$$

then the sound is called a **pure tone**. (Musical instruments do not give pure tones, but pure tones can be produced electronically.) To simplify the notation, we will use the symbol $\omega$ for the expression $\frac{2\pi}{p}$; *i.e.*,

$$\omega \equiv \frac{2\pi}{p}\ .$$

Notice that

$$V_1(t) = a_1 cos(\omega t) + b_1 sin(\omega t)$$

is also a (phase shifted) pure tone, since it can be rewritten as $A\,sin(\omega t - \phi)$ for appropriate $A$ and $\phi$ (page 585 of the textbook). Using the same $\omega$ as above, we see that the following function is another simple pressure deviation function with period $p$:

$$V_2(t) = a_2 cos(2\omega t) + b_2 sin(2\omega t)\ .$$

This function repeats a pure tone twice in the same period $p$; i.e. $V_2(t)$ is a pure tone of period $\frac{p}{2}$ and frequency $\frac{2}{p}$. (Note that a function can have many periods; e.g., $sin\, t$ can be view as having period $4\pi$ or $6\pi$, as well as the fundamental period $2\pi$.) In general, for any positive integer, $n$, the pressure deviation function

$$V_n(t) = a_n cos(n\omega t) + b_n sin(n\omega t)$$

repeats a pure tone $n$ times in the same period $p$; i.e., $V_n(t)$ has period $p$ (and period $\frac{p}{n}$) and frequency $\frac{n}{p}$.

It was the spectacular discovery of Joseph Fourier that *every* function, $P(t)$, of period $p$ can be written

$$P(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n cos(n\omega t) + b_n sin(n\omega t)\right) .$$

In our musical context, $a_0$ is the normal pressure, and the sum is $V(t)$, the pressure deviation. Notice that this result shows that *any* pressure deviation function $V(t)$ of period $p$ can be written as a pure tone of frequency $\frac{1}{p}$ (corresponding to $n = 1$ in the sum) plus a sum of pure tones whose frequencies are integral multiples of this basic frequency (the terms with $n \geq 2$ in the sum). In music, these pure tones which are integer multiples of the basic frequency are called **overtones** or **harmonics**. Different musical instruments have different combinations of these overtones in their pressure deviation functions; hence, they sound different.

**Part 3: Calculations with Fourier Series**

As in your text, we will simplify the notation by assuming that $\omega = 1$; that is, we assume that the period of the sound is $2\pi$. From the point of view of Fourier analysis, it is the values of the *a*'s and *b*'s which determine the “timbre” (or “quality”) of a musical sound. Their values represent the relative strength of the individual harmonics. For example, the vibrating reed in a clarinet produces relatively weak even harmonics—that is, $a_2,\ b_2,\ a_4, b_4,$ etc., are relatively small.

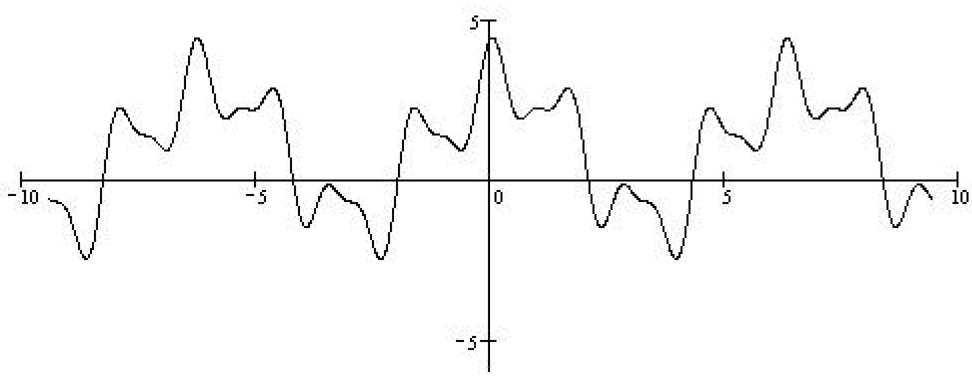
1. Use your calculator to graph the Fourier polynomial with $a_0 = 0,\ a_1 = .5,$ $b_1 = 1,\ \ a_2 = b_2 = 0,\ \ a_3 = .3,\ \ b_3 = .4,\ \ a_4 = .03,\ \ b_4 = .09,\ a_5 = .5$ and $b_5 = .16.$ Make sure your calculator is in radian mode, and use horizontal and vertical graphing ranges [0,12] and [$-1.7$, 1.7] respectively. Notice the relative strength of the harmonics, and compare your graph to the graph of the sound of the clarinet pictured in the textbook on page 464.

2. Use your calculator to graph the Fourier polynomial with $a_0 = -.25,\ a_1 = .32,$ $b_1 = .04,\ a_2 = .6,\ b_2 = .2,\ a_3 = b_3 = 1,\ a_4 = .2,\ b_4 = .3,\ a_5 = .2,$ and $b_5 = .1.$ Change the horizontal and vertical ranges to [0, 24] and [−2.5, 2.5] respectively. Which of the graphs in the section on Fourier series in the textbook does it resemble?

**Part 4: Constructing the Fourier Series from Sound**

As you saw in Part 3, it is a rather easy task to construct the pressure function when we are given the amplitudes of the individual harmonics. On the other hand the reverse problem of coming up with the harmonic structure of a musical sound, when we know only experimental values of its pressure function, is a more challenging and more interesting problem.

Suppose that we measure the values of the pressure function of a musical instrument playing a particular note. (We again assume for simplicity that the period of this note is $2\pi$.) Below is a graph of the pressure function and a table containing some $y$ values, measured from the graph at thirteen different times.




| $t$ | $-\pi$ | $\frac{-5\pi}{6}$ | $\frac{-4\pi}{6}$ | $\frac{-3\pi}{6}$ | $\frac{-2\pi}{6}$ | $\frac{-\pi}{6}$ | 0 | $\frac{\pi}{6}$ | $\frac{2\pi}{6}$ | $\frac{3\pi}{6}$ | $\frac{4\pi}{6}$ | $\frac{5\pi}{6}$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(t)$ | $-0.6$ | $-1.5$ | $-1.133$ | 2.2 | 1.367 | 1.1 | 4.4 | 2 | 2.233 | 2.8 | $-0.267$ | $-0.6$ | $-0.6$ |

Your goal in this part is to find a Fourier polynomial that approximates the given pressure function—and hence the harmonic structure of the sound. We will compute the coefficients for the first four harmonics of $P(t)$. The table above shows values of $P(t)$ taken $\frac{\pi}{6}$ units apart.

1. Write down an integral to show how to compute $a_2$symbolically. Now write out the expression for the left-hand sum approximation of the integral, and compute the

value of this expression by hand. Your computed value should be about $-0.3$ . You now have one of the nine numbers we are seeking.

2. Explain why using the left-hand sum, or the right-hand sum, or the trapezoid rule to make the approximation in the last step would all produce the same result in this case.

3. Now use the advanced features of your calculator to do the same computation again; i.e., enter the values for $t$ and $P(t)$ in a convenient format, such as in "lists," and then use your calculator's list features to compute the left-hand sum without having to write down the value of each term in the sum. You should, of course, get the same value you computed in step 1.

4. Repeat the process you used in step 3 to compute the constant term and all the other coefficients through the fourth harmonic. (Thus, you will have computed 9 numbers.) After you compute the coefficients, graph your Fourier polynomial and compare it to the original graph.

5. In the last step you probably noticed a small descrepancy between your graph and the orginal graph. To improve our representation one might try computing more Fourier coefficients. Compute the coefficients for the fifth, sixth, and seventh harmonics, and then make a new graph. Does your representation match the original graph now? If not, try replacing the value you computed for $a_5$ with 0, and graph the Fourier representation again. Does it match the original function now?

6. You have just seen that our numerical approximation of the value of $a_5$ was not a good one. Explain why continued approximations of coefficients for higher harmonics will not be reliable based on these data.

7. In this last step you will explore the effects that the different harmonic vibrations have on the fundamental. For this exercise it is highly recommended that you use the following web site:

*http://webphysics.ph.msstate.edu/javamirror/ntnujava/sound/sound.html*

Although this exercise can be done with your calculator, it would be tedious to do so. On the other hand you will likely find the web site fun. It will allow you easily to produce the required graphs, and if you use a computer with speakers, you can actually listen to the sound which corresponds to the pressure function.

After your computer loads the above web page, you will see a graph and two sets of sliders. The blue sliders control the *b*'s and the green sliders the *a*'s in the Fourier polynomial. (There is one exception: the first blue slider, $b_0$, applies a vertical stretching factor to the function. Thus, you can ignore the first blue slider.) Use the mouse to move the sliders and watch the changes in the graph. The values of the coefficients can be seen in a window above the graph. If you right click on a slider, it will return to 0 or slide to 1. The play button (above the graph) will allow you to hear the sound.

8. After you've experimented for a while, return all sliders to 0. Consider the function $f(t) = sin(t) + \frac{1}{3}sin(3t) + b_5 sin(5t)$. As $b_5$ ranges from 1 to $-1$, describe the changes in the pressure function. You should include some graphs in your description. (Note that the sixth blue slider is $b_5$.)

9. Repeat the experiment in step 4 with the function $f(t) = cos(t) + a_{10}\, cos(10t)$.

## Report

Your report should include your responses to all the questions and sketches of all the graphs. Be sure to show all your calculations and clearly label all of your graphs.

**Joseph Fourier** (1768-1830) was a French mathematician who was also involved in politics and conducted research on antiquities. He invented Fourier series when trying to understand and predict the flow of heat in metal bars. His work was rejected by the French Academy in 1807 and was eventually published as *Théorie Analytic de la Chaleur* in 1822. Often mathematics invented for one purpose is extremely useful in other contexts, just as Fourier's work on heat flow led to an understanding of musical sounds. Fourier's calculations, ideas, and questions, were the foundation of the branch of mathematics now called harmonic analysis.