

Newton's Law of Motion

Purpose

The purpose of this Lab is to introduce you to differential equations. As an important application, we will study Newton's Law of Motion and help you understand why Newton's calculations caused a revolution in human thought.

I. Introduction to Differential Equations.

A differential equation is an equation which involves the first derivative or higher derivatives of an unknown function. Finding the unknown function is called "solving" the differential equation. We have seen some simple examples already. In the Laboratory entitled "Introduction to Euler's Method," we asked whether we can find a function $y(t)$ if we know its derivative. In that Lab, we showed how to find $y(t)$ approximately by using tangent lines. The following example shows that we may be able to find $y(t)$ by guesswork.

Example 1. Suppose that an unknown function $y(t)$ satisfies

$$y'(t) = t^2$$

By differentiating, it is easy to see that the function $t^3/3$ satisfies the differential equation. However, this is not the only function which satisfies $y'(t) = t^2$, since for any constant c , the function

$$y(t) = \frac{t^3}{3} + c$$

also satisfies the differential equation. Thus, the differential equation has infinitely many solutions, each one corresponding to a different value of c . Now, suppose we know the value of y at $t = 0$. For example, suppose

$$y(0) = 5$$

then there is only one function in our family of functions, namely $y(t) = t^3/3 + 5$, which satisfies both the differential equation and the "initial condition" $y(0) = 5$. Several different solutions of the differential equation corresponding to different initial conditions are shown in Figure 1.

In Example 1, $\frac{t^3}{3}$ is called an antiderivative of t^2 . Thus, the procedure that we used to solve $y'(t) = t^2$ was first to find an antiderivative of t^2 . Then we observed that for each choice of c , the function $\frac{t^3}{3} + c$ was also a solution. In

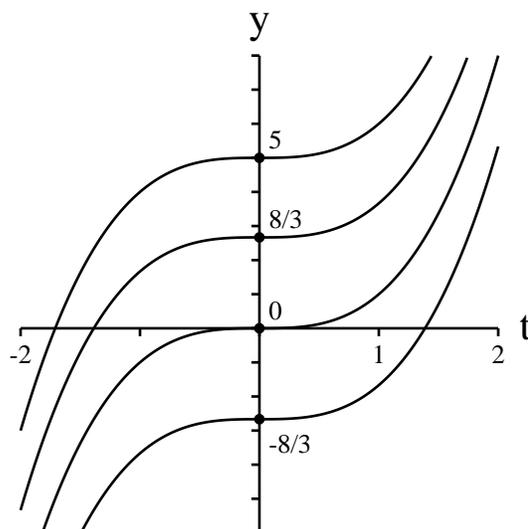


FIGURE 1. The solution curves to $y'(t) = t^2$ for the initial conditions $y(0) = 5$, $y(0) = \frac{8}{3}$, $y(0) = 0$, and $y(0) = -\frac{8}{3}$ respectively

general, a good strategy for solving the differential equation $y'(t) = f(t)$ is to find an **antiderivative** of f , that is, a function, $F(t)$, which satisfies $F'(t) = f(t)$. Then, for each choice of c , the function

$$y(t) = F(t) + c$$

will satisfy $y'(t) = f(t)$. (Can you explain why?) If $y(0)$ is given, we can solve the algebraic equation $y(0) = F(0) + c$ to find c . For instance, in the last example, $f(t) = t^2$, $F(t) = t^3/3$, and $c = 5$.

Example 2. Find the function $y(x)$ which satisfies

$$y'(x) = 3x^2 + 2, \quad \text{and} \quad y(0) = 4.$$

By trial and error we find that $x^3 + 2x$ is an antiderivative of $3x^2 + 2$. Thus, every function of the form

$$y(x) = x^3 + 2x + c$$

satisfies the differential equation. Since

$$4 = y(0) = 0^3 + 0 + c,$$

we see that $c = 4$. Thus, the solution of the differential equation which satisfies $y(0) = 4$ is

$$y(x) = x^3 + 2x + 4.$$

A differential equation together with an initial condition on the solution is often called an **initial-value problem**.

Practice Problems.

- (1) Find an antiderivative for each of the following functions:
 - (a) $f(x) = 2x + 3$.
 - (b) $f(x) = x^2 + x^7$.
 - (c) $f(x) = x^2 - 7x + 5$.
- (2) Solve the following initial-value problems:
 - (a) $y'(x) = 2x + 3, y(0) = 4$.
 - (b) $y'(x) = 2x + 3, y(0) = -2$.
 - (c) $y'(x) = x^2 + x^7, y(0) = 3$.
 - (d) $y'(x) = x^2 - 7x, y(0) = 1$.
- (3) Sometimes the initial condition is not given at $x = 0$ but at some other value of x . Show how to find the solution of the differential equation $y'(x) = 2x + 3$ which satisfies $y(2) = 5$.

The **order** of a differential equation is the highest derivative that occurs in the equation. In Examples 1 and 2 we solved first order differential equations. In the following example we shall solve a second order differential equation.

Example 3. We wish to solve the differential equation

$$s''(t) = t^2$$

with initial conditions

$$s(0) = 2, \quad \text{and} \quad s'(0) = 4.$$

Because solving the differential equation means finding the function $s(t)$ and because $s''(t)$ is the *second* derivative of $s(t)$, we realize that we will have to antidifferentiate twice. An antiderivative for t^2 is $\frac{t^3}{3}$, thus, as we saw earlier, all functions of the form $\frac{t^3}{3} + b$ are antiderivatives, where b is a constant. That is, if

$$(1) \quad s'(t) = t^3/3 + b.$$

then $s''(t) = t^2$. To see that this is true, simply differentiate both sides of (1). Now, a family of antiderivatives for $t^3/3 + b$ is $t^4/12 + bt + c$ for any constant c . This suggests that if we define $s(t)$ by the formula

$$s(t) = \frac{1}{12}t^4 + bt + c.$$

then $s(t)$ should satisfy the differential equation $s''(t) = t^2$. You can check this by explicit differentiation.

We shall now use the initial conditions to determine b and c . Substituting $t = 0$ into $s(t) = \frac{1}{12}t^4 + bt + c$, we find

$$2 = s(0) = 0 + 0 + c$$

Therefore, $c = 2$. Substituting $t = 0$ into $s'(t) = t^3/3 + b$, we find

$$4 = s'(0) = 0 + b$$

Therefore, $b = 4$. Thus, the function

$$s(t) = t^4/12 + 4t + 2$$

satisfies both the differential equation and the initial conditions. You can check this directly.

From this example we can see a strategy for solving differential equations of the form

$$s''(t) = f(t)$$

If we can find a function G which satisfies $G''(t) = f(t)$, then the function

$$s(t) = G(t) + bt + c$$

satisfies the differential equation for any choice of the constants b and c . For instance, in Example 3, $G(t) = \frac{1}{12}t^4$. To see why this is true, differentiate $s(t)$ twice. We say that this second order differential equation has a “two-parameter” family of solutions, since for any choice of the “parameters” b and c we get a solution of the differential equation $s''(t) = f(t)$. Typically, b and c are determined from initial conditions as they were in Example 3.

Practice Problems.

- (1) Solve the following initial-value problems.
 - (a) $y''(t) = 3t + 2$, $y(0) = 4$, $y'(0) = 1$.
 - (b) $y''(t) = t^2 - 5$, $y(1) = -2$, $y'(1) = 2$.
- (2) Write short clear descriptions of the following terms:
 - (a) differential equation.
 - (b) antiderivative.
 - (c) initial-value problem.
 - (d) order of a differential equation.

Plug in! In this course you will learn several different methods for solving initial-value problems. Whatever the method, you can always check that the “solution” really is a solution by plugging it into the differential equation and checking whether the differential equation is satisfied. Similarly, you can always check that

the “solution” satisfies the initial conditions by evaluating the solution at the initial time.

Naming functions and variables. In Example 1 the name of the unknown function was “ y ” and the name of the independent variable was “ t ”. We were trying to find the unknown function $y(t)$. In Example 2 the name of the independent variable was “ x ”. In Example 3, the name of the unknown function was “ s ” and the name of the independent variable was “ t ”. We were trying to find the unknown function $s(t)$. What is going on here? The first (and most crucial) point is that the ideas and methods of calculus do not depend on the names which we give to functions and variables. If you know how to differentiate the function $f(x) = 3x^2 + 5$, then you also know how to differentiate the function $g(t) = 3t^2 + 5$. Nevertheless, names are very important. Standardized notation is efficient because one doesn't need to ask what the symbols mean and it makes the comparison of different solution techniques easy. Therefore, when we discuss the theory of how to solve different types of differential equations, we shall usually call the unknown function $y(t)$. That is, t is the name of the independent variable and y is the name of the unknown function. On the other hand, in applications it is natural to give special names to functions and variables which help us remember their meaning. In physics, we might be interested in $h(t)$, the height of a projectile as a function of time. In ecology, we might be interested in $C(d)$, the concentration of a pollutant as a function of the distance from the source. In public policy, we might be interested in $U(i)$, the number of unemployment claims as a function of interest rate. Thus, in applying calculus and differential equations to real world problems, we shall use a wide variety of names for functions and variables.

II. Newton's Law of Motion.

Newton's Law of Motion

$$F = ma$$

expresses a relationship among the pushes and pulls on an object (the force F), the mass of the object (m), and the acceleration of the object (a). We shall discuss Newton's Law of motion for several special cases in which the object moves along a line. If the function $s(t)$ gives the position of the object on the line at time t , then the rate of change of $s(t)$, namely $s'(t)$, is called the velocity, and the rate of change of $s'(t)$, namely $s''(t)$, is called the acceleration. Thus, Newton's Law of Motion gives us information about the second derivative of $s(t)$. The

mathematical problem is to use this information about the second derivative and information about the object at time $t = 0$ to calculate $s(t)$ itself. You will be asked to make several calculations and to reflect on the philosophical impact of Newton's achievements.

Falling Bodies. Suppose that a rock is dropped from a height of 200 feet. We shall calculate its position as a function of time as it falls to the ground. We will measure time t in seconds starting at the time of release and we will measure distance in feet and denote the height above the ground at time t by the function $h(t)$. Thus, in this case, the "line" referred to above is the vertical line determined by the initial position of the rock and the place where it hits the ground. The concepts of velocity and acceleration are already familiar:

$$\begin{aligned}h(t) &= \text{distance above the ground} \\h'(t) &= \text{rate of change of distance} = \text{velocity} \\h''(t) &= \text{rate of change of velocity} = \text{acceleration.}\end{aligned}$$

We are given the information that $h(0) = 200$.

We assume that the only force acting on the rock is the force due to gravity. For a rock this assumption is reasonable; for a feather it would not be reasonable since the resistance of the air would significantly affect the feather. Near the surface of the earth, the force of gravity is

$$F = -mg$$

where $g = 32$ feet per second per second. Thus, in this case, Newton's Law can be written

$$mh''(t) = -m(32).$$

The negative sign indicates that the force pushes in the direction that causes h to decrease. Notice that m occurs on both sides of the equation above, so dividing by m , we find

$$h''(t) = -32.$$

This is the differential equation satisfied by $h(t)$. Since this is a second order differential equation, we expect that we will need two initial conditions to determine the solution. We were told explicitly that

$$h(0) = 200$$

. The other condition is implicit in the description of the problem we are trying to solve. We were told that the rock "is dropped", which says that it was not thrown

down or thrown up, but released with initial velocity zero. That is,

$$h'(0) = 0.$$

The differential equation, along with the two initial conditions is just the type of second order initial-value problem that you learned to solve earlier in the Lab.

Problem 1. Solve the initial-value problem for the differential equation and show that $h(t) = -16t^2 + 200$.

Now that we have computed $h(t)$, the height of the rock at all times t (until it hits the earth), you can use $h(t)$ to calculate other quantities of interest. For example, the height after 2 seconds is $h(2) = (-16)2^2 + 200 = 136$ feet. The distance fallen in t seconds is:

$$h(0) - h(t) = 200 - (-16t^2 + 200) = 16t^2 \text{ feet.}$$

Problem 2. For what values of t will $h(t)$ actually represent the height of the rock above the ground?

Problem 3. Suppose that the rock discussed above is not dropped but instead is thrown upward (from a platform 200 feet above the ground) with an initial velocity of 100 feet per second.

- Let $h(t)$ be the distance above the ground as a function of time. Write down the differential equation and initial conditions that are to be satisfied by $h(t)$.
- Compute $h(t)$.
- Compute the height above ground at times $t = 1, 2, \dots, 7$.
- Compute the time when the rock will hit the ground.
- Compute the greatest height above the ground. (Hint: this will happen at the time when the velocity is zero.)

Notice that the mass m appears on both sides of Newton's equation of motion for falling bodies and therefore cancels out. Thus, the motion is independent of the mass, a fact discovered by Galileo. This is strictly true only in a vacuum since the resistance of air exerts a force counter to any motion and the resistance depends on the size and surface properties of the body. The motion *does* depend on g , which is $32ft/(sec)^2$ on the surface of the earth. If we had asked the same questions on the surface of the moon, where $g = 5ft/(sec)^2$, the answers would have been different. For more practice problems on motion under the influence of gravity, see Section 6.5 of your text.

Motion on a Line. Suppose that an object of mass m moves along a line. The object could be a ping-ball or a subatomic particle. Think of the line as the x -axis and let $s(t)$ denote the position of the object on the axis at time t . The time t is measured in some time units (seconds, minutes, or hours) and s is measured in some distance units (feet or miles). If the object is pushed by a constant force F acting parallel to the line, then, according to Newton's Law of Motion,

$$s''(t) = F/m$$

so the acceleration is constant. Of course, the appropriate units for F depend on the units chosen for s and t . For simplicity, we let all the units remain unspecified. If $F > 0$ the force acts to push the object in the positive x direction. Similarly, if $s'(t) > 0$, then the object is moving in the positive x direction.

Problem 4. Suppose that $F/m = 5$ and that the particle starts two units to the right of the origin (that is, $s(0) = 2$) with initial velocity $s'(0) = -10$. Describe the subsequent motion of the particle. Hint: first use the differential equation and the initial conditions to find $s(t)$. Then describe the motion using that fact that $s(t)$ is the position and $s'(t)$ is the velocity at each time t .

Problem 5. In elementary textbooks it is often stated that "Newton's first law says that objects will move with constant velocity if the force applied to them is zero." Explain why this follows automatically from Newton's Law of Motion.

Problem 6. A particle moves on a line with acceleration $s''(t) = -4t$. Suppose that $s(0) = 6$ and $s'(0) = 24$. Find the functions $s(t)$ and $s'(t)$. Graph the functions $s(t)$, $s'(t)$, and $s''(t)$ on the same page and describe the motion in words.

III. Calculus and Human History.

Isaac Newton, an Englishman, was born in 1642, the year of Galileo's death. He showed no particular scientific talents as a child or as a student at Cambridge, but the full flowering of his genius appeared when he left Cambridge (to avoid the plague) in 1666 and took up residence in the country. There, in the space of two years, Newton invented the differential and integral calculus, proved the binomial theorem, showed that white light is composed of many colors, and found the law of universal gravitation. Of course, Newton did not make these inventions in scientific isolation, for he knew Kepler's laws of planetary motion and the mechanics of Galileo and others.

Of these discoveries, only the decomposition of light was communicated to the Royal Society at the time. Newton put aside his dynamical calculations and did not return to them for many years, and concentrated his attention on optics and

chemistry. He returned to mechanics in 1679 when there was much general speculation about what sort of force is exerted by the sun on the planets. In 1684, the astronomer Edmond Halley went to consult Newton.

“Without mentioning either his own speculations or those of Hooke and Wren, he at once indicated the object of his visit by asking Newton what would be the curve described by the planets on the supposition that gravity diminished as the square of the distance. Newton immediately answered ‘*an ellipse.*’ Struck with joy and amazement, Halley asked how he knew it. ‘Why,’ replied he, ‘I have calculated it’; and being asked for his calculation, he could not find it, but promised to send it to him.” (from an account by John Conduitt, a contemporary of Newton and Halley.)

At Halley’s urging, Newton devoted his full efforts to writing a description of his discoveries. His work was communicated to the Royal Society in 1686 and published (at Halley’s expense) in 1687 as *Philosophiae Naturalis Principia Mathematica*. The German mathematician Gottfried Leibnitz (1646 – 1716) is credited with discovering calculus independently.

Calculation and Prediction. Newton’s scientific discoveries had an enormous impact on philosophy, religion, and literature, because he showed that the answers to certain mechanical questions could be calculated. That is, in fact, what you have done in several simple cases in problems 1 - 4. Suppose that we know the mass and the forces acting on a particle. Then, given the initial position and initial velocity, we can calculate where the particle will be in the future. It was natural for the thinkers of the 18th century to believe that all other fields, not only the sciences, but human social organization and the study of human nature itself, would soon have laws and methods of calculation which would enable one to calculate and predict. The French mathematician Laplace said:

“We may regard the present state of the universe as the effect of its past and the cause of the future. An intellect which at any given moment knew all of the forces that animate nature and the mutual positions of the beings that compose it, if this intellect were vast enough to submit its data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom: for such an intellect nothing could be uncertain; and the future, just like the past, would be present before its eyes.”

And the French philosopher Voltaire wrote:

“It would be very singular that all nature, all the planets, should obey eternal laws, and that there should be a little animal, five feet high, who, in contempt of these laws, could act as he pleased, solely according to his own caprice.”

This is not some ancient, dry, philosophical dispute. Indeed, differing views of human nature play a major role in current disputes over social policy and the “best” forms of political organization. Here is contemporary socio-biologist E. O. Wilson (from *On Human Nature*, 1978) :

“But, to the extent that the new naturalism is true, its pursuit seems certain to generate two great spiritual dilemmas. The first is that no species, ours included, possesses a purpose beyond the imperative created by its genetic history...I believe that the human mind is constructed in a way that locks it inside this fundamental human constraint and forces it to make choices with a purely biological instrument. If the brain evolved by natural selection, even the capacities to select particular esthetic judgments and religious beliefs must have arisen by the same mechanistic process. They are either direct adaptations to past environments in which the ancestral human population evolved or at most constructions thrown up secondarily by deeper, less visible activities that were once adaptive in this stricter, biological sense.

Essay Question. Do individuals have free will? In other words, are a person's decisions and innermost feelings both determined by his or her biological inheritance and environmental past? What does this have to do with mathematics? Write a short (two page) essay on these questions.

Report. For this Lab your report should be in two parts. Your answers to the six problems should be written up and turned in to your Lab Assistant. Your essay should be turned in to your classroom instructor.