Present Value and Future Value

Purpose

We shall see that the ideas we used in the geometry and physics applications can also be applied to economic problems. In particular subdividing intervals, setting up Riemann sums, and computing limits helps us to answer some questions involving present value or future value of income.

Preview

Would you rather receive \$100 now, or 21 monthly payments of \$5 each? We will see how to answer this question based upon financial information. Then, using calculus, we will build upon this idea to develop the means for answering a question such as, what is the value of a stream of money being generated by a business over a long period of time.

Background: Compound Interest (discrete and continuous)

Many people consider the computation of the balance in a savings account as an addition process. For example, if a bank is paying 5% interest, we might view the computation as following: Balance $=$ Initial Deposit $+$ 5% of the Initial Deposit. But this can also be viewed as a multiplication: Balance $=$ (Initial Deposit) $\times$ (1.05). You will need these ideas in answering the questions below.

The following examples show where compound interest formulas come from. In all of them we will assume we have \$500 to deposit in an interest-bearing savings account.

1. Our hometown bank offers a savings account that pays 5% simple annual interest. (This means they add interest to the account at the end of each year.) Compute the balance in the account after ...

   (a) one year. (b) two years. (c) five years. (d) t years.

2. A new bank just opened in the mall that pays 5% interest on deposits, compounded monthly; i.e., $\frac{5}{12}$% of the initial deposit is added to the account after one month, and $\frac{5}{12}$% of the one-month balance is added to the account at the end of two months, and so on. Compute the balance in this account after ...

   (a) one year. (b) two years. (c) five years. (d) t years.

3. There's a bank in the supermarket that pays 5% interest, compounded daily. Compute the balance in this account after . . .

(a) one year. (b) two years. (c) five years. (d) t years.

Observation: Note that the balance after a given period of time goes up as we increase the number of times the account is compounded per year. That raises the question, just how large could the balance become if we raise the number of times it is compounded higher and higher? In the next two steps we answer that question.

4. Consider an account which pays 5% interest compounded n times a year. Construct a formula for $B_n(t)$, the balance in the account after t years. Your answers to parts (d) in the preceding problems should help.

5. Internetbank.com pays 5% interest, compounded continuously. (Loosely speaking, this means they compound interest "infinitely often.") To make sense of this idea, we use the formula for $B_n(t)$ from step 4, and we let $B(t)$ represent the balance in this continuously compounded account at any time t. We interpret $B(t)$ as follows:

$$B(t) = \lim_{n\to\infty} B_n(t).$$

Use the fact[1] that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ to show that $B(t) = \$500e^{0.05\,t}$.

6. Suppose you have access to a bank which pays 3% interest, compounded continuously.

(a) If we deposit \$500 now, how much would be in the account ten years from now? (Answer: \$674.93)

(b) How much would we have to deposit in the account today to have a balance of \$1000 at the end of ten years? (Answer: \$740.82)

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[1]In case you don't remember (or haven't seen) this result, it can be shown as follows: use L'Hôpital's Rule to show that $\lim_{n\to\infty} \frac{ln(1+\frac{x}{n})}{1/n} = x$. Then, because $\frac{ln(1+\frac{x}{n})}{1/n} = ln\big((1+\frac{x}{n})^n\big)$, we know that

$\lim_{n\to\infty} ln\big((1+\frac{x}{n})^n\big) = x$, and from this we get $\lim_{n\to\infty}(1+\frac{x}{n})^n = \lim_{n\to\infty} exp[\, ln\big((1+\frac{x}{n})^n\big)\,] = e^x$.

Present Value and Future Value. The last problem illustrates the notion of *present value* (PV) and *future value* (FV). In part (a) \$674.93 is called the *future value* (at the end of 10 years) of \$500 we have today. In part (b) \$740.82 is the *present value* of \$1000 to be received ten years from now.

7. We can use the concepts of PV and FV in the context of discrete compounding, also. For example, assume you can deposit money in an account which pays 3% interest compounded monthly. Suppose we have a contract that specifies we are to receive \$1000 at the end of ten years. What is the PV of this contract?

Summary and Generalization: You should now understand all parts of the general results. If we let r denote the simple annual interest rate (as a decimal), n the number of compounding periods per year, t the number of years, and B_0 the initial deposit, then

$$B_n(t) = B_0\left(1 + \tfrac{r}{n}\right)^{nt} \text{ and } \quad B(t) = \lim_{n\to\infty} B_0\left(1 + \tfrac{r}{n}\right)^{nt} = B_0 e^{rt}.$$

In terms of present value and future value, we can express these results as follows:

$$\text{FV} = \text{PV}\left(1 + \tfrac{r}{n}\right)^{nt} \quad \text{and} \quad \text{FV} = \lim_{n\to\infty} \text{PV}\left(1 + \tfrac{r}{n}\right)^{nt} = \text{PV}e^{rt}.$$

or

$$\text{PV} = \text{FV}\left(1 + \tfrac{r}{n}\right)^{-nt} \quad \text{and} \quad \text{PV} = \text{FV}e^{-rt}.$$

Part 1: Future Value and Present Value of a Sequence of Payments

In the background section we dealt with a one-time investment of money. We now consider what happens with there is a sequence of payments. The essential idea is that to compute the present value of a sequence of payments, you have to compute the sum of the present values of each individual payment.

Suppose Publishers' Clear Glass House is advertising its “\$1 million” sweepstakes. But when you read the fine print, you discover that the \$1 million will be paid to the winner in 40 annual payments of \$25,000 each, starting on the day of the selection of the winner. Assume that you can deposit money in an account which pays 6% interest, compounded twice a year.

1. What is the present value (PV) of the second $25,000 payment? The third $25,000 payment? The n^{th} $25,000 payment? Now compute the PV of the $1 million grand prize. Suppose you win and then your cousin, Gil Bates, offers you $400,000 for the grand prize. Would you take the offer? Why (not)?

2. While you're pondering the last offer from Gil, he returns to you with another offer. He says his company, Bikers' Loft, will pay you and your heirs $20,000 a year, starting now and continuing forever, if you'll turn the grand prize over to him. Compute the present value of this offer. [Hint: you'll need the Geometric Series Theorem.] Is this a better offer? Why (not)?

3. Suppose you discover a Savings and Loan that pays 6% interest, compounded continuously. How much does this change the PV of Gil's offer of $20,000 a year forever? Explain this change intuitively. Does this alter your response to Gil's latest offer?

Part 2: Future Value and Present Value of an Income Stream

You have seen in the *Background* section that there is only a small difference between compounding several times a year and compounding continuously. You also saw that working with the expression for continuous compounding is somewhat easier than working with the discrete formula. Thus, even if we are faced with the usual case of discrete compounding, we may use an assumption of continuous compounding to simplify our computations, which are, after all, only estimates based on financial conditions which may change.

A similar situation occurs with income. For example, suppose the marketing giant, Wal-More, expects to earn a certain profit each year, but the profit on a given day depends upon the time of year. Instead of viewing this profit as 365 discrete values for each year, we could view it as in “income stream”; i.e., we use a continuous function, $p(t)$, to represent the rate at which Wal-More is making money at any time t. (We usually measure t in years, and $p(t)$ in dollars per year.) We shall see that computing PV and FV of such an income stream is an easy way to estimate the value of the company's future income.

Let $p(t)$ be the rate at which profit is “streaming” into Wal-More. We will compute the present value of this income stream for over a period of time. We assume that the current, continuously compounded interest rate is r.

1. First, we look at the next 10 years. We shall use the same strategy that we used in computing quantities such as volumes and arc length: divide this 10-year period into N subintervals, each of length Δt, with endpoints $\{t_0,\ t_1, \ldots,\ t_N\}$. We construct a Right Hand Sum to approximate the PV of this income stream:

(a) Explain why the expression $p(t_k)\,\Delta t$ would approximate the profit during the time from t_{k-1} to t_k.

(b) What expression would approximate the PV of the profit during the time from t_{k-1} to t_k ?

(c) Carefully describe the meaning of the expression $\sum_{k=1}^{N} e^{-rt_k}\, p(t_k)\,\Delta t$.

(d) As we let $N \to \infty$, the expression above takes on a value represented by a definite integral. Write down the integral, and explain what it means.

2. Suppose $p(t) = 50$ is a constant income stream (in units of \$1000 per year). Assume the current (continuously compounded) interest rate is 4%.

(a) Compute the PV of this income stream over the next 10 years.

(b) If we expect this income stream to last a long time, we could represent the situation mathematically by assuming it goes on forever. Write down an integral that would represent the PV of this stream under the assumption it will last forever. Does this integral converge? Explain intuitively how this result can make sense.

3. Suppose $p(t) = 1.5\,t$ (in units of \$1000 per year) models an increasing income stream, which we expect to last for a very long time.

(a) Assuming the current interest rate is 7% (compounded continuously), compute the PV of this income stream.

(b) Assuming the current interest rate is given by the constant r (compounded continuously), compute the PV of this income stream as a function of r. If this income stream were yours, under what conditions would you be willing to sell it for $200,000?

(c) Explain how your decision to sell or not will depend upon whether the current value of r is relatively high or low. Also explain why your answer makes sense intuitively.

Report

Your report should include your responses to all the problems and questions in parts 1 and 2 of this lab.