Notes on Simpson's Rule

Background

The idea of Simpson's Rule is to approximate a definite integral $\int_a^b f(x)\,dx$ as follows:

1. Subdivide the interval [a,b] into n subintervals. Make sure n is even.
2. Corresponding to the usual $x_0, x_1, x_2, \ldots, x_n$ notation for the endpoints of the subintervals of $[a, b]$ (i.e., $x_k = a + k\,\Delta x$, for $k = 0, 1, 2, \ldots$), we introduce y_0 for $f(x_0)$, y_1 for $f(x_1)$, etc.
3. Construct a parabolic arc over each consecutive pair of subintervals. Use the Fundamental Theorem of Calculus to compute the area under each parabolic arc separately, then sum these $\frac{n}{2}$ areas to approximate $\int_a^b f(x)\,dx$.
4. The result of step (3), after simplification, is the following approximation, S_n, of $\int_a^b f(x)\,dx$ given by Simpson's Rule using n (where n is even) subintervals:
   $\mathrm{S}_n = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$

Note the 1-4-2-4-2-4…-2-4-1 pattern of the coefficients. In the special case of using only two subintervals, we have $\mathrm{S}_2 = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$.

Example: We'll use Simpson's Rule with n $=$ 4 to approximate ln2 $= \int_1^2 \frac{1}{x}dx$.

$\Delta x = \frac{2-1}{4} = \frac{1}{4}$. $x_0 = 1, x_1 = \frac{5}{4}, x_2 = \frac{6}{4}, x_3 = \frac{7}{4}, x_4 = \frac{8}{4} = 2.$

And because $f(x) = \frac{1}{x}$, we get: $y_0 = 1, y_1 = \frac{4}{5}, y_2 = \frac{4}{6}, y_3 = \frac{4}{7}, y_4 = \frac{1}{2}.$

Thus, $\mathrm{S}_4 = \frac{\frac{1}{4}}{3}\left(1 + 4\left(\frac{4}{5}\right) + 2\left(\frac{4}{6}\right) + 4\left(\frac{4}{7}\right) + \frac{1}{2}\right) \approx 0.69325.$

When we check ln2 on the calculator, we see that ln2 $\approx$ 0.69314; i.e., Simpson's Rule with only 4 subintervals was quite a good approximation in this case.

Note on the presentation in the textbook

In you look in the textbook you'll see that Simpson's Rule is given as a weighted average of the Midpoint Rule (MR) and the Trapezoid Rule (TR). In particular, if we divide $[a,b]$ into N subintervals, then

$$\mathrm{S_N} = \frac{\mathrm{2MR+TR}}{3} .$$

To reconcile the results of the formula I gave you in class with the results of the formula given in the textbook, you have to take n $=$ 2N. In other words, the amount of arithmetic required is about the same either way. I presented the $\mathrm{S_n}$ formula in class, because that's the one Simpson derived, and the associated argument seems more motivated to me.