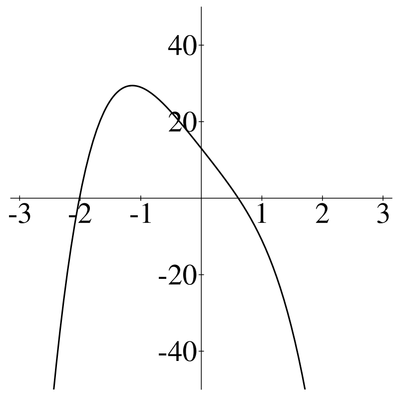
APPENDIX

Solutions for Section A

1. The graph is



(a) The range appears to be $y \leq 30$.
(b) The function has two zeros.

2. (a) The root is between 0.3 and 0.4, at about 0.35.
(b) The root is between 1.5 and 1.6, at about 1.55.
(c) The root is between -1.8 and -1.9, at about -1.85.

3. The root occurs at about -1.05

4. The root is between -1.7 and -1.8, at about -1.75.

5. The largest root is at about 2.5.

6. There is one root at $x = -1$ and another at about $x = 1.35$.

7. There is one real root at about $x = -1.1$.

8. The root occurs at about 0.9, since the function changes sign between 0.8 and 1.

9. Using a graphing calculator, we see that when x is around 0.45, the graphs intersect.

10. The root occurs between 0.6 and 0.7, at about 0.65.

11. The root occurs between 1.2 and 1.4, at about 1.3.

12. Zoom in on graph: $t = \pm 0.824$. [Note: t must be in radians; one must zoom in two or three times.]

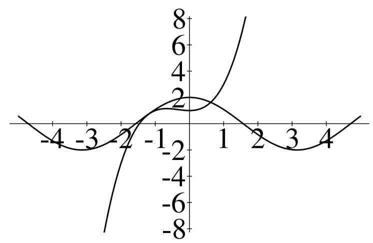
13. (a) Only one real zero, at about $x = -1.15$.
(b) Three real zeros: at $x = 1$, and at about $x = 1.41$ and $x = -1.41$.

14. First, notice that $f(3) \approx 0.5 > 0$ and that $f(4) \approx -0.25 < 0$.
1st iteration: $f(3.5) > 0$, so a zero is between 3.5 and 4.
2nd iteration: $f(3.75) < 0$, so a zero is between 3.5 and 3.75.
3rd iteration: $f(3.625) < 0$, so a zero is between 3.5 and 3.625.
4th iteration: $f(3.588) < 0$, so a zero is between 3.5 and 3.588.
5th iteration: $f(3.545) > 0$, so a zero is between 3.545 and 3.588.
6th iteration: $f(3.578) < 0$, so a zero is between 3.567 and 3.578.
7th iteration: $f(3.572) > 0$, so a zero is between 3.572 and 3.578.
8th iteration: $f(3.575) > 0$, so a zero is between 3.575 and 3.578.

Thus we know that, rounded to two places, the value of the zero must be 3.58. We know that this is the largest zero of $f(x)$ since $f(x)$ approaches -1 for larger values of x.

15. (a) Let $F(x) = \sin x - 2^{-x}$. Then $F(x) = 0$ will have a root where $f(x)$ and $g(x)$ cross. The first positive value of x for which the functions intersect is $x \approx 0.7$.
(b) The functions intersect for $x \approx 0.4$.

16. The graph is



We find one zero at about 0.6. It looks like there might be another one at about -1.2, but zoom in close... closer... closer, and you'll see that though the graphs are very close together, they do not touch, and so there is no zero near -1.2. Thus the zero at about 0.6 is the only one. (How do you know there are no other zeros off the screen?)

17. (a) Since f is continuous, there must be one zero between $\theta = 1.4$ and $\theta = 1.6$, and another between $\theta = 1.6$ and $\theta = 1.8$. These are the only clear cases. We might also want to investigate the interval $0.6 \leq \theta \leq 0.8$ since $f(\theta)$ takes on values close to zero on at least part of this interval. Now, $\theta = 0.7$ is in this interval, and $f(0.7) = -0.01 < 0$, so f changes sign twice between $\theta = 0.6$ and $\theta = 0.8$ and hence has two zeros on this interval (assuming f is not *really* wiggly here, which it's not). There are a total of 4 zeros.
(b) As an example, we find the zero of f between $\theta = 0.6$ and $\theta = 0.7$. $f(0.65)$ is positive; $f(0.66)$ is negative. So this zero is contained in $[0.65, 0.66]$. The other zeros are contained in the intervals $[0.72, 0.73]$, $[1.43, 1.44]$, and $[1.7, 1.71]$.
(c) You've found all the zeros. A picture will confirm this; see Figure A.1.

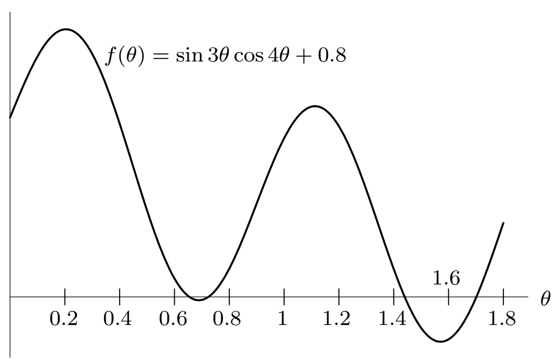


Figure A.1

18. (a) There appear to be two solutions: one on the interval from 1.13 to 1.14 and one on the interval from 1.08 to 1.09. From 1.13 to 1.14, $\frac{x^3}{\pi^3}$ increases from 0.0465 to 0.0478 while $(\sin 3x)(\cos 4x)$ decreases from 0.0470 to 0.0417, so they must cross in between. Similarly, going from 1.08 to 1.09, $\frac{x^3}{\pi^3}$ increases from 0.0406 to 0.0418 while $(\sin 3x)(\cos 4x)$ increases from 0.0376 to 0.0442. Thus the difference between the two changes sign over that interval, so their difference must be zero somewhere in between.
(b) Reasonable estimates are $x = 1.085$ and $x = 1.131$.

19. (a) The first ten results are:

n	0	1	2	3	4	5	6	7	8
1	3.14159	5.05050	5.50129	5.56393	5.57186	5.57285	5.57297	5.57299	5.57299

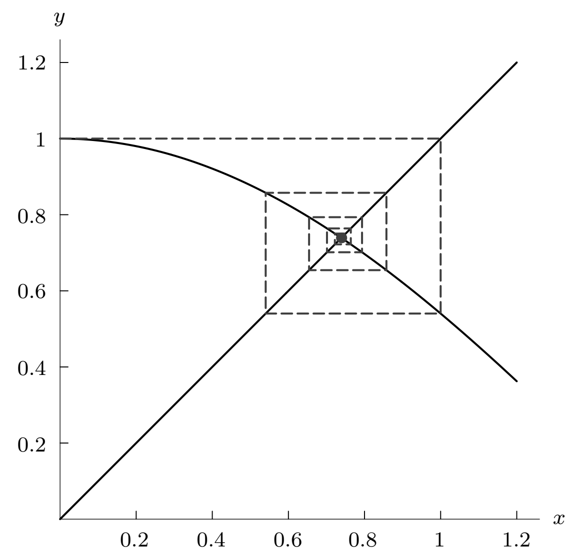
(b) The solution is $x \approx 5.573$. We started with an initial guess of 1, and kept repeating the given procedure until our values converged to a limit at around 5.573. For each number on the table, the procedure was in essence asking the question "Does this number equal 4 times the arctangent of itself?" and then correcting the number by repeating the question for 4 times the arctangent of the number.

(c) P_0 represents our initial guess of $x = 1$ (on the line $y = x$). P_1 is 4 times the arctangent of 1. If we now use take this value for P_1 and slide it horizontally back to the line $y = x$, we can now use this as a new guess, and call it P_2. P_3, of course, represents 4 times the arctangent of P_2, and so on. Another way to make sense of this diagram is to consider the function $F(x) = 4\arctan x - x$. On the diagram, this difference is represented by the vertical lines connecting P_0 and P_1, P_2 and P_3 and so on. Notice how these lines (and hence the difference between $\arctan x$ and x) get smaller as we approach the intersection point, where $F(x) = 0$.

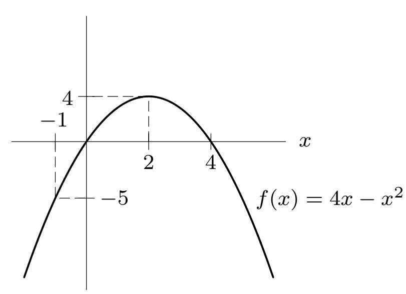
(d) For an initial guess of $x = 10$, the procedure gives a decreasing sequence which converges (more quickly) to the same value of about 5.573. Graphically, our initial guess of P_0 will lie to the right of the intersection on the line $y = x$. The iteration procedure gives us a sequence of $P_1, P_2, \ldots$ that zigzags to the left, toward the intersection point. For an initial guess of $x = -10$, the procedure gives an increasing sequence converging to the other intersection point of these two curves at $x \approx -5.573$. Graphically, we get a sequence which is a reflection through the origin of the sequence we got for an initial guess of $x = 10$. This is so because both $y = x$ and $y = \arctan x$ are odd functions.

20. Starting with $x = 0$, and repeatedly taking the cosine, we get the numbers below. Continuing until the first three decimal places remain fixed under iteration, we have this list and diagram:

x	$\cos x$
0	0.735069
1	0.7401473
0.5403023	0.7356047
0.8575532	0.7414251
0.6542898	0.7375069
0.7934804	0.7401473
0.7013688	0.7383692
0.7639597	0.7395672
0.7221024	0.7387603
0.7504178	0.7393039
0.7314043	0.7389378
0.7442374	etc.

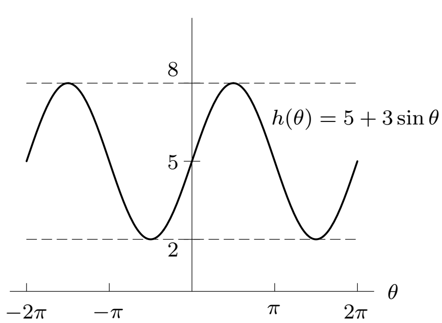


21.



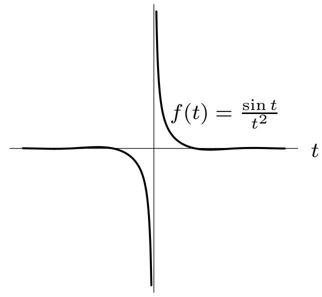
Bounded and $-5 \le f(x) \le 4$.

22.



Bounded and $2 \le h(\theta) \le 8$.

23.



Not bounded because $f(t)$ goes to infinity as t goes to 0.

Solutions for Section B

1. $2e^{i\pi/2}$
2. $5e^{i\pi}$
3. $\sqrt{2}e^{i\pi/4}$
4. $5e^{i4.069}$
5. $0e^{i\theta}$, for any θ.
6. $e^{3\pi i/2}$
7. $\sqrt{10}e^{i\theta}$, where $\theta = \arctan(-3) \approx -1.249 + \pi = 1.893$ is an angle in the second quadrant.
8. $13e^{i\theta}$, where $\theta = \arctan(-\frac{12}{5}) \approx -1.176$ is an angle in the fourth quadrant.
9. $-3 - 4i$
10. $-11 + 29i$
11. $-5 + 12i$
12. $1 + 3i$
13. $\frac{1}{4} - \frac{9i}{8}$
14. $3 - 6i$
15. $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
16. $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$ is one solution.
17. $5^3(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = -125i$
18. $\sqrt[4]{10}\cos\frac{\pi}{8} + i\sqrt[4]{10}\sin\frac{\pi}{8}$ is one solution.
19. One value of $\sqrt{i}$ is $\sqrt{e^{i\frac{\pi}{2}}} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$
20. One value of $\sqrt{-i}$ is $\sqrt{e^{i\frac{3\pi}{2}}} = (e^{i\frac{3\pi}{2}})^{\frac{1}{2}} = e^{i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$
21. One value of $\sqrt[3]{i}$ is $\sqrt[3]{e^{i\frac{\pi}{2}}} = (e^{i\frac{\pi}{2}})^{\frac{1}{3}} = e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$
22. One value of $\sqrt{7i}$ is $\sqrt{7e^{i\frac{\pi}{2}}} = (7e^{i\frac{\pi}{2}})^{\frac{1}{2}} = \sqrt{7}e^{i\frac{\pi}{4}} = \sqrt{7}\cos\frac{\pi}{4} + i\sqrt{7}\sin\frac{\pi}{4} = \frac{\sqrt{14}}{2} + i\frac{\sqrt{14}}{2}$
23. $(1+i)^{100} = (\sqrt{2}e^{i\frac{\pi}{4}})^{100} = (2^{\frac{1}{2}})^{100}(e^{i\frac{\pi}{4}})^{100} = 2^{50}\cdot e^{i\cdot 25\pi} = 2^{50}\cos 25\pi + i2^{50}\sin 25\pi = -2^{50}$
24. One value of $(1+i)^{2/3}$ is $(\sqrt{2}e^{i\frac{\pi}{4}})^{2/3} = (2^{\frac{1}{2}}e^{i\frac{\pi}{4}})^{\frac{2}{3}} = \sqrt[3]{2}e^{i\frac{\pi}{6}} = \sqrt[3]{2}\cos\frac{\pi}{6} + i\sqrt[3]{2}\sin\frac{\pi}{6} = \sqrt[3]{2}\cdot\frac{\sqrt{3}}{2} + i\sqrt[3]{2}\cdot\frac{1}{2}$
25. One value of $(-4+4i)^{2/3}$ is $[\sqrt{32}e^{(i3\pi/4)}]^{(2/3)} = (\sqrt{32})^{2/3}e^{(i\pi/2)} = 2^{5/3}\cos\frac{\pi}{2} + i2^{5/3}\sin\frac{\pi}{2} = 2i\sqrt[3]{4}$
26. One value of $(\sqrt{3}+i)^{1/2}$ is $(2e^{i\frac{\pi}{6}})^{1/2} = \sqrt{2}e^{i\frac{\pi}{12}} = \sqrt{2}\cos\frac{\pi}{12} + i\sqrt{2}\sin\frac{\pi}{12} \approx 1.366 + 0.366i$
27. One value of $(\sqrt{3}+i)^{-1/2}$ is $(2e^{i\frac{\pi}{6}})^{-1/2} = \frac{1}{\sqrt{2}}e^{i(-\frac{\pi}{12})} = \frac{1}{\sqrt{2}}\cos(-\frac{\pi}{12}) + i\frac{1}{\sqrt{2}}\sin(-\frac{\pi}{12}) \approx 0.683 - 0.183i$
28. Since $\sqrt{5} + 2i = 3e^{i\theta}$, where $\theta = \arctan\frac{2}{\sqrt{5}} \approx 0.730$, one value of $(\sqrt{5}+2i)^{\sqrt{2}}$ is $(3e^{i\theta})^{\sqrt{2}} = 3^{\sqrt{2}}e^{i\sqrt{2}\theta} = 3^{\sqrt{2}}\cos\sqrt{2}\theta + i3^{\sqrt{2}}\sin\sqrt{2}\theta \approx 3^{\sqrt{2}}(0.513) + i3^{\sqrt{2}}(0.859) \approx 2.426 + 4.062i$

29. We have

$$\begin{aligned}
i^{-1} &= \frac{1}{i} = \frac{1}{i} \cdot \frac{i}{i} = -i, \\
i^{-2} &= \frac{1}{i^2} = -1, \\
i^{-3} &= \frac{1}{i^3} = \frac{1}{-i} \cdot \frac{i}{i} = i, \\
i^{-4} &= \frac{1}{i^4} = 1.
\end{aligned}$$

The pattern is

$$i^n = \begin{cases} -i & n = -1, -5, -9, \cdots \\ -1 & n = -2, -6, -10, \cdots \\ i & n = -3, -7, -11, \cdots \\ 1 & n = -4, -8, -12, \cdots. \end{cases}$$

Since 36 is a multiple of 4, we know $i^{-36} = 1$.
Since $41 = 4 \cdot 10 + 1$, we know $i^{-41} = -i$.

30. Substituting $A_1 = 2 - A_2$ into the second equation gives

$$(1 - i)(2 - A_2) + (1 + i)A_2 = 0$$

so

$$\begin{aligned}
2iA_2 &= -2(1 - i) \\
A_2 &= \frac{-(1 - i)}{i} = \frac{-i(1 - i)}{i^2} = i(1 - i) = 1 + i
\end{aligned}$$

Therefore $A_1 = 2 - (1 + i) = 1 - i$.

31. Substituting $A_2 = 2 - A_1$ into the second equation gives

$$\begin{aligned}
(i - 1)A_1 + (1 + i)(2 - A_1) &= 0 \\
iA_1 - A_1 - A_1 - iA_1 + 2 + 2i &= 0 \\
-2A_1 &= -2 - 2i \\
A_1 &= 1 + i
\end{aligned}$$

Substituting, we have

$$A_2 = 2 - A_1 = 2 - (1 + i) = 1 - i.$$

32. (a) To divide complex numbers, multiply top and bottom by the conjugate of $1 + 2i$, that is, $1 - 2i$:

$$\frac{3 - 4i}{1 + 2i} = \frac{3 - 4i}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i} = \frac{3 - 4i - 6i + 8i^2}{1^2 + 2^2} = \frac{-5 - 10i}{5} = -1 - 2i.$$

Thus, $a = -1$ and $b = -2$.

(b) Multiplying $(1 + 2i)(a + bi)$ should give $3 - 4i$, as the following calculation shows:

$$(1 + 2i)(a + bi) = (1 + 2i)(-1 - 2i) = -1 - 2i - 2i - 4i^2 = 3 - 4i.$$

33. To confirm that $z = \dfrac{a + bi}{c + di}$, we calculate the product

$$\begin{aligned}
z(c + di) &= \left(\frac{ac + bd}{c^2 + d^2} = \frac{bc - ad}{c^2 + d^2} i\right)(c + di) \\
&= \frac{ac^2 + bcd - bcd + ad^2 + (bc^2 - acd + acd + bd^2)i}{c^2 + d^2} \\
&= \frac{a(c^2 + d^2) + b(c^2 + d^2)i}{c^2 + d^2} = a + bi.
\end{aligned}$$

34. (a)

$$z_1z_2 = (-3-i\sqrt{3})(-1+i\sqrt{3}) = 3+(\sqrt{3})^2+i(\sqrt{3}-3\sqrt{3}) = 6-i2\sqrt{3}.$$
$$\frac{z_1}{z_2} = \frac{-3-i\sqrt{3}}{-1+i\sqrt{3}}\cdot\frac{-1-i\sqrt{3}}{-1-i\sqrt{3}} = \frac{3-(\sqrt{3})^2+i(\sqrt{3}+3\sqrt{3})}{(-1)^2+(\sqrt{3})^2} = \frac{i\cdot 4\sqrt{3}}{4} = i\sqrt{3}.$$

(b) We find (r_1,θ_1) corresponding to $z_1 = -3-i\sqrt{3}$:

$$r_1 = \sqrt{(-3)^2+(\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3};$$
$$\tan\theta_1 = \frac{-\sqrt{3}}{-3} = \frac{\sqrt{3}}{3}, \text{ so } \theta_1 = \frac{7\pi}{6}.$$

(See Figure B.2.) Thus,

$$-3-i\sqrt{3} = r_1e^{i\theta_1} = 2\sqrt{3}\,e^{i\frac{7\pi}{6}}.$$

We find (r_2,θ_2) corresponding to $z_2 = -1+i\sqrt{3}$:

$$r_2 = \sqrt{(-1)^2+(\sqrt{3})^2} = 2;$$
$$\tan\theta_2 = \frac{\sqrt{3}}{-1} = -\sqrt{3}, \text{ so } \theta_2 = \frac{2\pi}{3}.$$

(See Figure B.3.) Thus,

$$-1+i\sqrt{3} = r_2e^{i\theta_2} = 2e^{i\frac{2\pi}{3}}.$$

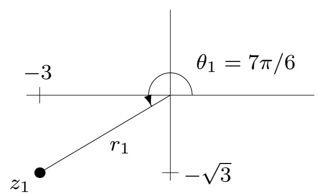


Figure B.2

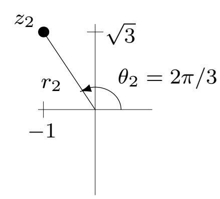


Figure B.3

We now calculate z_1z_2 and $\dfrac{z_1}{z_2}$.

$$\begin{aligned} z_1z_2 &= \left(2\sqrt{3}e^{i\frac{7\pi}{6}}\right)\left(2e^{i\frac{2\pi}{3}}\right) = 4\sqrt{3}e^{i(\frac{7\pi}{6}+\frac{2\pi}{3})} = 4\sqrt{3}e^{i\frac{11\pi}{6}} \\ &= 4\sqrt{3}\left[\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right] = 4\sqrt{3}\left[\frac{\sqrt{3}}{2}-i\frac{1}{2}\right] = 6-i2\sqrt{3}. \\ \frac{z_1}{z_2} &= \frac{2\sqrt{3}e^{i\frac{7\pi}{6}}}{2e^{i\frac{2\pi}{3}}} = \sqrt{3}e^{i(\frac{7\pi}{6}-\frac{2\pi}{3})} = \sqrt{3}e^{i\frac{\pi}{2}} \\ &= \sqrt{3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) = i\sqrt{3}. \end{aligned}$$

These agrees with the values found in (a).

35. First we calculate

$$z_1z_2 = (a_1+b_1i)(a_2+b_2i) = a_1a_2-b_1b_2+i(a_1b_2+a_2b_1).$$

Thus, $\overline{z_1z_2} = a_1a_2-b_1b_2-i(a_1b_2+a_2b_1)$.

Since $\bar{z}_1 = a_1-b_1i$ and $\bar{z}_2 = a_2-b_2i$,

$$\bar{z}_1\bar{z}_2 = (a_1-b_1i)(a_2-b_2i) = a_1a_2-b_1b_2-i(a_1b_2+a_2b_1).$$

Thus, $\overline{z_1z_2} = \bar{z}_1\bar{z}_2$.

36. If the roots are complex numbers, we must have $(2b)^2-4c<0$ so $b^2-c<0$. Then the roots are

$$\begin{aligned} x = \frac{-2b\pm\sqrt{(2b)^2-4c}}{2} &= -b\pm\sqrt{b^2-c} \\ &= -b\pm\sqrt{-1(c-b^2)} \\ &= -b\pm i\sqrt{c-b^2}. \end{aligned}$$

Thus, $p=-b$ and $q=\sqrt{c-b^2}$.

37. True, since $\sqrt{a}$ is real for all $a \geq 0$.

38. True, since $(x - iy)(x + iy) = x^2 + y^2$ is real.

39. False, since $(1 + i)^2 = 2i$ is not real.

40. False. Let $f(x) = x$. Then $f(i) = i$ but $f(\bar{i}) = \bar{i} = -i$.

41. True. We can write any nonzero complex number z as $re^{i\beta}$, where r and β are real numbers with $r > 0$. Since $r > 0$, we can write $r = e^c$ for some real number c. Therefore, $z = re^{i\beta} = e^c e^{i\beta} = e^{c+i\beta} = e^w$ where $w = c + i\beta$ is a complex number.

42. False, since $(1 + 2i)^2 = -3 + 4i$.

43.
$$\begin{aligned}
1 = e^0 = e^{i(\theta-\theta)} &= e^{i\theta}e^{i(-\theta)} \\
&= (\cos\theta + i\sin\theta)(\cos(-\theta) + i\sin(-\theta)) \\
&= (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) \\
&= \cos^2\theta + \sin^2\theta
\end{aligned}$$

44. Using Euler's formula, we have:

$$e^{i(2\theta)} = \cos 2\theta + i\sin 2\theta$$

On the other hand,

$$e^{i(2\theta)} = \left(e^{i\theta}\right)^2 = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$$

Equating imaginary parts, we find

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

45. Using Euler's formula, we have:

$$e^{i(2\theta)} = \cos 2\theta + i\sin 2\theta$$

On the other hand,

$$e^{i(2\theta)} = \left(e^{i\theta}\right)^2 = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$$

Equating real parts, we find

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

46. Differentiating Euler's formula gives

$$\frac{d}{d\theta}(e^{i\theta}) = ie^{i\theta} = i(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta$$

Since in addition $\dfrac{d}{d\theta}(e^{i\theta}) = \dfrac{d}{d\theta}(\cos\theta + i\sin\theta) = \dfrac{d}{d\theta}(\cos\theta) + i\dfrac{d}{d\theta}(\sin\theta)$, by equating imaginary parts, we conclude that $\dfrac{d}{d\theta}\sin\theta = \cos\theta$.

47. Differentiating Euler's formula twice gives

$$\frac{d^2}{d\theta^2}(e^{i\theta}) = \frac{d^2}{d\theta^2}(\cos\theta + i\sin\theta) = \frac{d^2}{d\theta^2}(\cos\theta) + i\frac{d^2}{d\theta^2}(\sin\theta).$$

But

$$\frac{d^2}{d\theta^2}(e^{i\theta}) = i^2e^{i\theta} = -e^{i\theta} = -\cos\theta - i\sin\theta.$$

Equating real parts, we find

$$\frac{d^2}{d\theta^2}(\cos\theta) = -\cos\theta.$$

48. Replacing θ by $-x$ in the formula for $\sin\theta$:

$$\sin(-x) = \frac{1}{2i}\left(e^{-ix} - e^{ix}\right) = -\frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = -\sin x.$$

49. Replacing θ by $(x+y)$ in the formula for $\sin\theta$:

$$\begin{aligned}
\sin(x+y) &= \frac{1}{2i}\left(e^{i(x+y)} - e^{-i(x+y)}\right) = \frac{1}{2i}\left(e^{ix}e^{iy} - e^{-ix}e^{-iy}\right) \\
&= \frac{1}{2i}\left((\cos x + i\sin x)(\cos y + i\sin y) - (\cos(-x) + i\sin(-x))(\cos(-y) + i\sin(-y))\right) \\
&= \frac{1}{2i}\left((\cos x + i\sin x)(\cos y + i\sin y) - (\cos x - i\sin x)(\cos y - i\sin y)\right) \\
&= \sin x\cos y + \cos x\sin y.
\end{aligned}$$

50. Since x_1, y_1, x_2, y_2 are each functions of the variable t, differentiating the sum gives

$$\begin{aligned}
(z_1 + z_2)' &= (x_1 + iy_1 + x_2 + iy_2)' = (x_1 + x_2 + i(y_1 + y_2))' \\
&= (x_1 + x_2)' + i(y_1 + y_2)' \\
&= \left(x_1' + x_2'\right) + i\left(y_1' + y_2'\right) \\
&= (x_1 + iy_1)' + (x_2 + iy_2)' \\
&= z_1' + z_2'.
\end{aligned}$$

Differentiating the product gives

$$\begin{aligned}
(z_1 z_2)' &= ((x_1 + iy_1)(x_2 + iy_2))' = (x_1x_2 - y_1y_2 + i(y_1x_2 + x_1y_2))' \\
&= (x_1x_2 - y_1y_2)' + i(y_1x_2 + x_1y_2)' \\
&= \left(x_1'x_2 + x_1x_2' - y_1'y_2 - y_1y_2'\right) + i\left(y_1'x_2 + y_1x_2' + x_1'y_2 + x_1y_2'\right) \\
&= [x_1'x_2 - y_1'y_2 + i(x_1'y_2 + y_1'x_2)] + [x_1x_2' - y_1y_2' + i(y_1x_2' + x_1y_2')] \\
&= \left(x_1' + iy_1'\right)(x_2 + iy_2) + (x_1 + iy_1)\left(x_2' + iy_2'\right) \\
&= z_1'z_2 + z_1z_2'.
\end{aligned}$$

Solutions for Section C

1. (a) $f'(x) = 3x^2 + 6x + 3 = 3(x+1)^2$. Thus $f'(x) > 0$ everywhere except at $x = -1$, so it is increasing everywhere except perhaps at $x = -1$. The function is in fact increasing at $x = -1$ since $f(x) > f(-1)$ for $x > -1$, and $f(x) < f(-1)$ for $x < -1$.

(b) The original equation can have at most one root, since it can only pass through the x-axis once if it never decreases. It must have one root, since $f(0) = -6$ and $f(1) = 1$.

(c) The root is in the interval $[0, 1]$, since $f(0) < 0 < f(1)$.

(d) Let $x_0 = 1$.

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{12} = \frac{11}{12} \approx 0.917 \\
x_2 &= \frac{11}{12} - \frac{f\left(\frac{11}{12}\right)}{f'\left(\frac{11}{12}\right)} \approx 0.913 \\
x_3 &= 0.913 - \frac{f(0.913)}{f'(0.913)} \approx 0.913.
\end{aligned}$$

Since the digits repeat, they should be accurate. Thus $x \approx 0.913$.

2. Let $f(x) = x^3 - 50$. Then $f(\sqrt[3]{50}) = 0$, so we can use Newton's method to solve $f(x) = 0$ to obtain $x = \sqrt[3]{50}$. Since $f'(x) = 3x^2$, f' is always positive, and f is therefore increasing. Consequently, f has only one zero. Since $3^3 = 27 < 50 < 64 = 4^3$, let $x_0 = 3.5$. Then

$$\begin{aligned} x_0 &= 3.5 \\ x_1 &= 3.5 - \frac{f(3.5)}{f'(3.5)} \approx 3.694 \end{aligned}$$

Continuing, we find

$$\begin{aligned} x_2 &\approx 3.684 \\ x_3 &\approx 3.684. \end{aligned}$$

Since the digits repeat, x_3 should be correct, as can be confirmed by calculator.

3. Let $f(x) = x^4 - 100$. Then $f(\sqrt[4]{100}) = 0$, so we can use Newton's method to solve $f(x) = 0$ to obtain $x = \sqrt[4]{100}$. $f'(x) = 4x^3$. Since $3^4 = 81 < 100 < 256 = 4^4$, try 3.1 as an initial guess.

$$\begin{aligned} x_0 &= 3.1 \\ x_1 &= 3.1 - \frac{f(3.1)}{f'(3.1)} \approx 3.164 \\ x_2 &= 3.164 - \frac{f(3.164)}{f'(3.164)} \approx 3.162 \\ x_3 &= 3.162 - \frac{f(3.162)}{f'(3.162)} \approx 3.162 \end{aligned}$$

Thus $\sqrt[4]{100} \approx 3.162$.

4. Let $f(x) = x^3 - \frac{1}{10}$. Then $f(10^{-1/3}) = 0$, so we can use Newton's method to solve $f(x) = 0$ to obtain $x = 10^{-1/3}$. $f'(x) = 3x^2$. Since $\sqrt[3]{\frac{1}{27}} < \sqrt[3]{\frac{1}{10}} < \sqrt[3]{\frac{1}{8}}$, try $x_0 = \frac{1}{2}$. Then $x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)} \approx 0.467$. Continuing, we find $x_2 \approx 0.464. x_3 \approx 0.464$. Since $x_2 \approx x_3, 10^{-1/3} \approx 0.464$.

5. Let $f(x) = \sin x - 1 + x$; we want to find all zeros of f, because $f(x) = 0$ implies $\sin x = 1 - x$.
Graphing $\sin x$ and $1 - x$ in Figure C.4, we see that $f(x)$ has one solution at $x \approx \frac{1}{2}$.

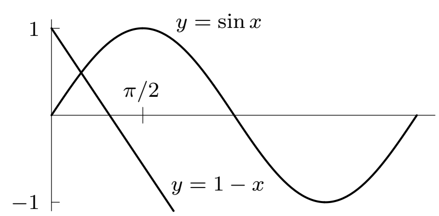


Figure C.4

Letting $x_0 = 0.5$, and using Newton's method, we have $f'(x) = \cos x + 1$, so that

$$x_1 = 0.5 - \frac{\sin(0.5) - 1 + 0.5}{\cos(0.5) + 1} \approx 0.511,$$

$$x_2 = 0.511 - \frac{\sin(0.511) - 1 + 0.511}{\cos(0.511) + 1} \approx 0.511.$$

Thus $\sin x = 1 - x$ has one solution at $x \approx 0.511$.

6. Let $f(x) = \cos x - x$. We want to find all zeros of f, because $f(x) = 0$ implies that $\cos x = x$. Since $f'(x) = -\sin x - 1$, f' is always negative (as $-\sin x$ never exceeds 1). This means f is always decreasing and consequently has at most 1 root. We now use Newton's method. Since $\cos 0 > 0$ and $\cos\frac{\pi}{2} < \frac{\pi}{2}$, $\cos x = x$ for $0 < x < \frac{\pi}{2}$. Thus, try $x_0 = \frac{\pi}{6}$.

$$\begin{aligned} x_1 &= \frac{\pi}{6} - \frac{\cos\frac{\pi}{6} - \frac{\pi}{6}}{-\sin\frac{\pi}{6} - 1} \approx 0.7519, \\ x_2 &\approx 0.7391, \\ x_3 &\approx 0.7390. \end{aligned}$$

$x_2 \approx x_3 \approx 0.739$. Thus $x \approx 0.739$ is the solution.

7. Let $f(x) = e^{-x} - \ln x$. Then $f'(x) = -e^{-x} - \frac{1}{x}$. We want to find all zeros of f, because $f(x) = 0$ implies that $e^{-x} = \ln x$. Since e^{-x} is always decreasing and $\ln x$ is always increasing, there must be only 1 solution. Since $e^{-1} > \ln 1 = 0$, and $e^{-e} < \ln e = 1$, then $e^{-x} = \ln x$ for some x, $1 < x < e$. Try $x_0 = 1$. We now use Newton's method.

$$\begin{aligned} x_1 &= 1 - \frac{e^{-1} - 0}{-e^{-1} - 1} \approx 1.2689, \\ x_2 &\approx 1.309, \\ x_3 &\approx 1.310. \end{aligned}$$

Thus $x \approx 1.310$ is the solution.

8. Let $f(x) = e^x \cos x - 1$. Then $f'(x) = -e^x \sin x + e^x \cos x$. Now we use Newton's method, guessing $x_0 = 1$ initially.

$$x_1 = 1 - \frac{f(1)}{f'(1)} \approx 1.5725$$

Continuing: $x_2 \approx 1.364$, $x_3 \approx 1.299$, $x_4 \approx 1.293$, $x_5 \approx 1.293$. Thus $x \approx 1.293$ is a solution. Looking at a graph of $f(x)$ suffices to convince us that there is only one solution.

9. Let $f(x) = \ln x - \frac{1}{x}$, so $f'(x) = \frac{1}{x} + \frac{1}{x^2}$.
Now use Newton's method with an initial guess of $x_0 = 2$.

$$\begin{aligned} x_1 &= 2 - \frac{\ln 2 - \frac{1}{2}}{\frac{1}{2} + \frac{1}{4}} \approx 1.7425, \\ x_2 &\approx 1.763, \\ x_3 &\approx 1.763. \end{aligned}$$

Thus $x \approx 1.763$ is a solution. Since $f'(x) > 0$ for positive x, f is increasing: it must be the only solution.

10. (a) One zero in the interval $0.6 < x < 0.7$.
(b) Three zeros in the intervals $-1.55 < x < -1.45$, $x = 0$, $1.45 < x < 1.55$.
(c) Two zeros in the intervals $0.1 < x < 0.2$, $3.5 < x < 3.6$.

11. $f'(x) = 3x^2 + 1$. Since f' is always positive, f is everywhere increasing. Thus f has only one zero. Since $f(0) < 0 < f(1)$, $0 < x_0 < 1$. Pick $x_0 = 0.68$.

$$\begin{aligned} x_0 &= 0.68, \\ x_1 &= 0.6823278, \\ x_2 &\approx 0.6823278. \end{aligned}$$

Thus $x \approx 0.682328$ (rounded up) is a root. Since $x_1 \approx x_2$, the digits should be correct.

12. Let $f(x) = x^2 - a$, so $f'(x) = 2x$.
Then by Newton's method, $x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n}$
For $a = 2$:
$x_0 = 1$, $x_1 = 1.5$, $x_2 \approx 1.416$, $x_3 \approx 1.414215$, $x_4 \approx 1.414213$ so $\sqrt{2} \approx 1.4142$.
For $a = 10$:
$x_0 = 5$, $x_1 = 3.5$, $x_2 \approx 3.17857$, $x_3 \approx 3.162319$, $x_4 \approx 3.162277$ so $\sqrt{10} \approx 3.1623$.
For $a = 1000$:
$x_0 = 500$, $x_1 = 251$, $x_2 \approx 127.49203$, $x_3 \approx 67.6678$, $x_4 \approx 41.2229$, $x_5 \approx 32.7406$, $x_6 \approx 31.6418$, $x_7 \approx 31.62278$, $x_8 \approx 31.62277$ so $\sqrt{1000} \approx 31.6228$.
For $a = \pi$:
$x_0 = \frac{\pi}{2}$, $x_1 \approx 1.7853$, $x_2 \approx 1.7725$ $x_3 \approx 1.77245$, $x_4 \approx 1.77245$ so $\sqrt{\pi} \approx 1.77245$.