CHAPTER ONE

Solutions for Section 1.1

Exercises

1. $f(35)$ means the value of P corresponding to $t = 35$. Since t represents the number of years since 1950, we see that $f(35)$ means the population of the city in 1985. So, in 1985, the city's population was 12 million.

2. Since $T = f(P)$, we see that $f(200)$ is the value of T when $P = 200$; that is, the thickness of pelican eggs when the concentration of PCBs is 200 ppm.

3. (a) When the car is 5 years old, it is worth $\$6000$.

(b) Since the value of the car decreases as the car gets older, this is a decreasing function. A possible graph is in Figure 1.1:

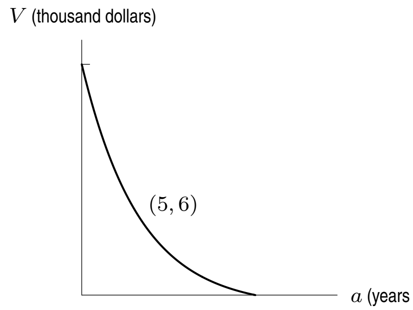


Figure 1.1

(c) The vertical intercept is the value of V when $a = 0$, or the value of the car when it is new. The horizontal intercept is the value of a when $V = 0$, or the age of the car when it is worth nothing.

4. The slope is $(1-0)/(1-0) = 1$. So the equation of the line is $y = x$.

5. The slope is $(3-2)/(2-0) = 1/2$. So the equation of the line is $y = (1/2)x + 2$.

6. Using the points $(-2, 1)$ and $(2, 3)$, we have

$$\text{Slope } = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2}.$$

Now we know that $y = (1/2)x + b$. Using the point $(-2, 1)$, we have $1 = -2/2 + b$, which yields $b = 2$. Thus, the equation of the line is $y = (1/2)x + 2$.

7. Rewriting the equation as

$$y = -\frac{12}{7}x + \frac{2}{7}$$

shows that the line has slope $-12/7$ and vertical intercept $2/7$.

8. Rewriting the equation of the line as

$$-y = \frac{-2}{4}x - 2$$
$$y = \frac{1}{2}x + 2,$$

we see the line has slope $1/2$ and vertical intercept 2.

9. Rewriting the equation of the line as

$$y = \frac{12}{6}x - \frac{4}{6}$$
$$y = 2x - \frac{2}{3},$$

we see that the line has slope 2 and vertical intercept $-2/3$.

10. (a) is (V), because slope is positive, vertical intercept is negative
(b) is (IV), because slope is negative, vertical intercept is positive
(c) is (I), because slope is 0, vertical intercept is positive
(d) is (VI), because slope and vertical intercept are both negative
(e) is (II), because slope and vertical intercept are both positive
(f) is (III), because slope is positive, vertical intercept is 0

11. (a) is (V), because slope is negative, vertical intercept is 0
(b) is (VI), because slope and vertical intercept are both positive
(c) is (I), because slope is negative, vertical intercept is positive
(d) is (IV), because slope is positive, vertical intercept is negative
(e) is (III), because slope and vertical intercept are both negative
(f) is (II), because slope is positive, vertical intercept is 0

12. $y = 5x - 3$. Since the slope of this line is 5, we want a line with slope $-\frac{1}{5}$ passing through the point $(2, 1)$. The equation is $(y - 1) = -\frac{1}{5}(x - 2)$, or $y = -\frac{1}{5}x + \frac{7}{5}$.

13. The line $y + 4x = 7$ has slope -4. Therefore the parallel line has slope -4 and equation $y - 5 = -4(x - 1)$ or $y = -4x + 9$. The perpendicular line has slope $\frac{-1}{(-4)} = \frac{1}{4}$ and equation $y - 5 = \frac{1}{4}(x - 1)$ or $y = 0.25x + 4.75$.

14. The line parallel to $y = mx + c$ also has slope m, so its equation is

$$y = m(x - a) + b.$$

The line perpendicular to $y = mx + c$ has slope $-1/m$, so its equation will be

$$y = -\frac{1}{m}(x - a) + b.$$

15. Since the function goes from $x = 0$ to $x = 5$ and between $y = 0$ and $y = 4$, the domain is $0 \le x \le 5$ and the range is $0 \le y \le 4$.

16. Since the function goes from $x = -2$ to $x = 2$ and from $y = -2$ to $y = 2$, the domain is $-2 \le x \le 2$ and the range is $-2 \le y \le 2$.

17. Since x goes from 1 to 5 and y goes from 1 to 6, the domain is $1 \le x \le 5$ and the range is $1 \le y \le 6$.

18. The domain is all numbers. The range is all numbers ≥ 2, since $x^2 \ge 0$ for all x.

19. The domain is all x-values, as the denominator is never zero. The range is $0 < y \le \dfrac{1}{2}$.

20. The value of $f(t)$ is real provided $t^2 - 16 \ge 0$ or $t^2 \ge 16$. This occurs when either $t \ge 4$, or $t \le -4$. Solving $f(t) = 3$, we have

$$\begin{aligned} \sqrt{t^2 - 16} &= 3 \\ t^2 - 16 &= 9 \\ t^2 &= 25 \end{aligned}$$

so

$$t = \pm 5.$$

21. Factoring gives

$$g(x) = \frac{(2 - x)(2 + x)}{x(x + 1)}.$$

The values of x which make $g(x)$ undefined are $x = 0$ and $x = -1$, when the denominator is 0. So the domain is all $x \ne 0, -1$. Solving $g(x) = 0$ means one of the numerator's factors is 0, so $x = \pm 2$.

22. For some constant k, we have $S = kh^2$.

23. We know that E is proportional to v^3, so $E = kv^3$, for some constant k.

24. We know that N is proportional to $1/l^2$, so

$$N = \frac{k}{l^2}, \quad \text{for some constant } k.$$

Problems

25. (a) The flat tire corresponds to a part of the graph where the velocity is zero. This could be (II) or (III). Since the velocity in (II) is higher for the later part of the interval, corresponding to speeding up, the answer is (II).
(b) This is (I), as the graph shows a positive velocity followed by a zero velocity.
(c) In (IV), the velocity is positive, zero (while the package is being dropped off) and then negative (the drive home).
(d) Graph (III) could represent a drive to the country, stopping to have lunch (velocity zero), and continuing on at a slower speed to look at the scenery. Other stories are possible.

26. See Figure 1.2.

27. See Figure 1.3.

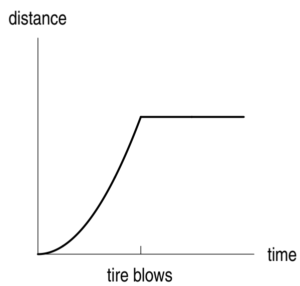


Figure 1.2

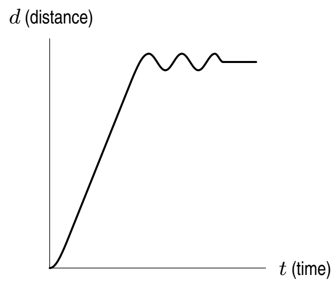


Figure 1.3

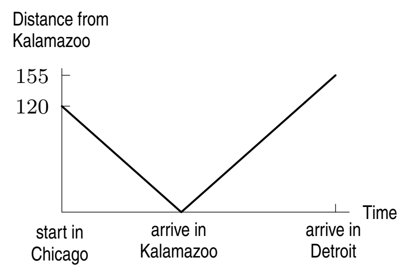


Figure 1.4

28. See Figure 1.4.

29. (a) We find the slope m and intercept b in the linear equation $C = b + mw$. To find the slope m, we use

$$m = \frac{\Delta C}{\Delta w} = \frac{48 - 32}{180 - 100} = 0.2.$$

We substitute to find b:

$$\begin{aligned} C &= b + mw \\ 32 &= b + (0.2)(100) \\ b &= 12. \end{aligned}$$

The linear formula is $C = 12 + 0.2w$.
(b) The slope is 0.2 dollars per kilogram. Each additional kilogram of waste costs 20 cents.
(c) The intercept is 12 dollars. The flat monthly fee to subscribe to the waste collection service is \$12. This is the amount charged even if there is no waste.

30. (a) Charge per cubic foot = $\dfrac{\Delta \$}{\Delta \text{ cu. ft.}} = \dfrac{105 - 90}{1600 - 1000} = \$0.025/\text{cu. ft.}$
Alternatively, if we let c = cost, w = cubic feet of water, b = fixed charge, and m = cost/cubic feet, we obtain $c = b + mw$. Substituting the information given in the problem, we have

$$\begin{aligned} 90 &= b + 1000m \\ 105 &= b + 1600m. \end{aligned}$$

Subtracting the first equation from the second yields $15 = 600m$, so $m = 0.025$.
(b) $c = b + 0.025w$, so $90 = b + 0.025(1000)$, which yields $b = 65$. Thus the equation is $c = 65 + 0.025w$.
(c) We need to solve the equation $130 = 65 + 0.025w$, which yields $w = 2600$.

31. (a) Given the two points $(0, 32)$ and $(100, 212)$, and assuming the graph in Figure 1.5 is a line,

$$\text{Slope} = \frac{212 - 32}{100} = \frac{180}{100} = 1.8.$$

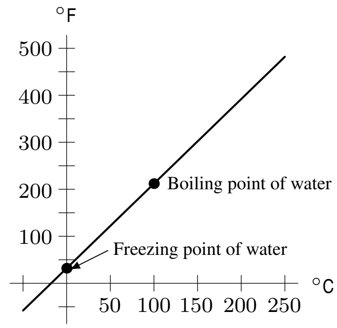


Figure 1.5

(b) The °F-intercept is $(0, 32)$, so

$$°\mathrm{F} = 1.8(°\mathrm{C}) + 32.$$

(c) If the temperature is 20°Celsius, then

$$°\mathrm{F} = 1.8(20) + 32 = 68°\mathrm{F}.$$

(d) If °F = °C, then

$$\begin{aligned} °\mathrm{C} &= 1.8°\mathrm{C} + 32 \\ -32 &= 0.8°\mathrm{C} \\ °\mathrm{C} &= -40° = °\mathrm{F}. \end{aligned}$$

32. (a) $f(30) = 10$ means that the value of f at $t = 30$ was 10. In other words, the temperature at time $t = 30$ minutes was 10°C. So, 30 minutes after the object was placed outside, it had cooled to 10 °C.

(b) The intercept a measures the value of $f(t)$ when $t = 0$. In other words, when the object was initially put outside, it had a temperature of a°C. The intercept b measures the value of t when $f(t) = 0$. In other words, at time b the object's temperature is 0 °C.

33. We are looking for a linear function $y = f(x)$ that, given a time x in years, gives a value y in dollars for the value of the refrigerator. We know that when $x = 0$, that is, when the refrigerator is new, $y = 950$, and when $x = 7$, the refrigerator is worthless, so $y = 0$. Thus $(0, 950)$ and $(7, 0)$ are on the line that we are looking for. The slope is then given by

$$m = \frac{950}{-7}$$

It is negative, indicating that the value decreases as time passes. Having found the slope, we can take the point $(7, 0)$ and use the point-slope formula:

$$y - y_1 = m(x - x_1).$$

So,

$$\begin{aligned} y - 0 &= -\frac{950}{7}(x - 7) \\ y &= -\frac{950}{7}x + 950. \end{aligned}$$

34. (a) This could be a linear function because w increases by 5 as h increases by 1.

(b) We find the slope m and the intercept b in the linear equation $w = b + mh$. We first find the slope m using the first two points in the table. Since we want w to be a function of h, we take

$$m = \frac{\Delta w}{\Delta h} = \frac{171 - 166}{69 - 68} = 5.$$

Substituting the first point and the slope $m = 5$ into the linear equation $w = b + mh$, we have $166 = b + (5)(68)$, so $b = -174$. The linear function is

$$w = 5h - 174.$$

The slope, $m = 5$, is in units of pounds per inch.

(c) We find the slope and intercept in the linear function $h = b + mw$ using $m = \Delta h/\Delta w$ to obtain the linear function

$$h = 0.2w + 34.8.$$

Alternatively, we could solve the linear equation found in part (b) for h. The slope, $m = 0.2$, has units inches per pound.

35. (a) We have the following functions.

(i) Since a change in p of \$5 results in a decrease in q of 2, the slope of $q = D(p)$ is $-2/5$ items per dollar. So

$$q = b - \frac{2}{5}p.$$

Now we know that when $p = 550$ we have $q = 100$, so

$$\begin{aligned}100 &= b - \frac{2}{5} \cdot 550\\ 100 &= b - 220\\ b &= 320.\end{aligned}$$

Thus a formula is

$$q = 320 - \frac{2}{5}p.$$

(ii) We can solve $q = 320 - \frac{2}{5}p$ for p in terms of q:

$$\begin{aligned}5q &= 1600 - 2p\\ 2p &= 1600 - 5q\\ p &= 800 - \frac{5}{2}q.\end{aligned}$$

The slope of this function is $-5/2$ dollars per item, as we would expect.

(b) A graph of $p = 800 - \dfrac{5}{2}q$ is given in Figure 1.6.

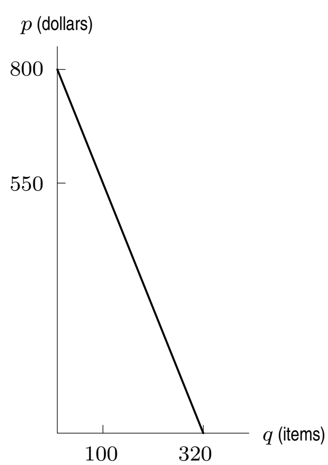


Figure 1.6

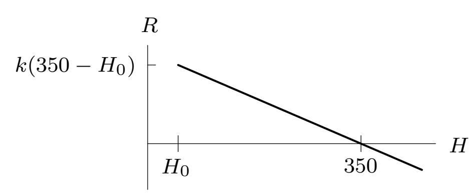


Figure 1.7

36. (a) $R = k(350 - H)$, where k is a positive constant.

If H is greater than $350°$, the rate is negative, indicating that a very hot yam will cool down toward the temperature of the oven.

(b) Letting H_0 equal the initial temperature of the yam, the graph of R against H looks like figure 1.7.

Note that by the temperature of the yam, we mean the average temperature of the yam, since the yam's surface will be hotter than its center.

37. Given $l - l_0 = al_0(t - t_0)$ with l_0, t_0 and a all constant,

(a) We have $l = al_0(t - t_0) + l_0 = al_0t - al_0t_0 + l_0$, which is a linear function of t with slope al_0 and y-intercept at $(0, -al_0t_0 + l_0)$.

(b) If $l_0 = 100$, $t_0 = 60°$F and $a = 10^{-5}$, then

$$\begin{aligned}l = 10^{-5}(100)t - 10^{-5}(100)(60) + 100 &= 10^{-3}t + 99.94\\ &= 0.001t + 99.94\end{aligned}$$

(c) If the slope is positive, (as in (b)), then as the temperature rises, the length of the metal increases: it expands. If the slope were negative, then the metal would contract as the temperature rises.

38. (a) Assembling the given information, we have

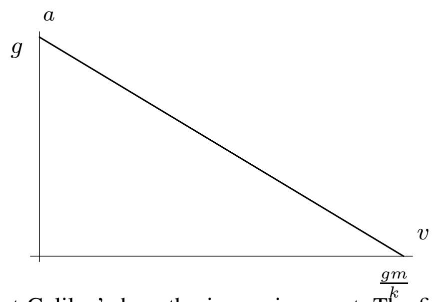
$$F = ma = F_g - F_r = mg - kv,$$

where k is the constant that relates velocity to air resistance (which depends on the shape of the object).

(b) Solving the above equation for a, we have

$$a = g - \frac{k}{m}v$$

(c)



39. Looking at the given data, it seems that Galileo's hypothesis was incorrect. The first table suggests that velocity is not a linear function of distance, since the increases in velocity for each foot of distance are themselves getting smaller. Moreover, the second table suggests that velocity is instead proportional to *time*, since for each second of time, the velocity increases by 32 ft/sec.

Solutions for Section 1.2

Exercises

1. Initial quantity $= 5$; growth rate $= 0.07 = 7\%$.

2. Initial quantity $= 7.7$; growth rate $= -0.08 = -8\%$ (decay).

3. Initial quantity $= 3.2$; growth rate $= 0.03 = 3\%$ (continuous).

4. Initial quantity $= 15$; growth rate $= -0.06 = -6\%$ (continuous decay).

5. (a) The function is linear with initial population of 1000 and slope of 50, so $P = 1000 + 50t$.
(b) This function is exponential with initial population of 1000 and growth rate of 5%, so $P = 1000(1.05)^t$.

6. (a) This is a linear function with slope -2 grams per day and intercept 30 grams. The function is $Q = 30 - 2t$, and the graph is shown in Figure 1.8.

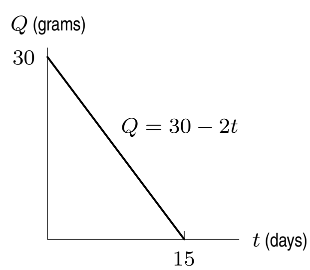


Figure 1.8

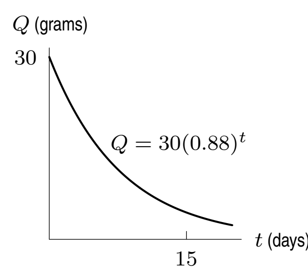


Figure 1.9

(b) Since the quantity is decreasing by a constant percent change, this is an exponential function with base $1 - 0.12 = 0.88$. The function is $Q = 30(0.88)^t$, and the graph is shown in Figure 1.9.

7. The graph shows a concave up function.

8. The graph shows a concave down function.

9. This graph is neither concave up or down.

10. The graph is concave up.

11. The function is increasing and concave up on the x-interval between D and E, and the x-interval between H and I. It is increasing and concave down on the x-interval between A and B, and the x-interval between E and F. It is decreasing and concave up on the x-interval between C and D, and the x-interval between G and H. Finally, it is decreasing and concave down on the x-interval between B and C, and the x-interval between F and G

Problems

12. (a) Let P represent the population of the world, and let t represent the number of years since 1999. Then we have $P = 6(1.013)^t$.

(b) According to this formula, the population of the world in the year 2020 (at $t = 21$) will be $P = 6(1.013)^{21} = 7.87$ billion people.

(c) The graph is shown in Figure 1.10. The population of the world has doubled when $P = 12$; we see on the graph that this occurs at approximately $t = 53.7$. Under these assumptions, the doubling time of the world's population is about 53.7 years.

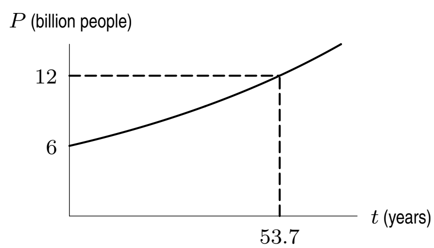


Figure 1.10

13. (a) We have

$$\text{Reduced size } = (0.80) \cdot \text{ Original size}$$

or

$$\text{Original size} = \frac{1}{(0.80)} \text{ Reduced size } = (1.25) \text{ Reduced size,}$$

so the copy must be enlarged by a factor of 1.25, which means it is enlarged to 125% of the reduced size.

(b) If a page is copied n times, then

$$\text{New size } = (0.80)^n \cdot \text{ Original.}$$

We want to solve for n so that

$$(0.80)^n = 0.15.$$

By trial and error, we find $(0.80)^8 = 0.168$ and $(0.80)^9 = 0.134$. So the page needs to be copied 9 times.

14. (a) Using $Q = Q_0(1 - r)^t$ for loss, we have

$$Q = 10{,}000(1 - 0.1)^{10} = 10{,}000(0.9)^{10} = 3486.78.$$

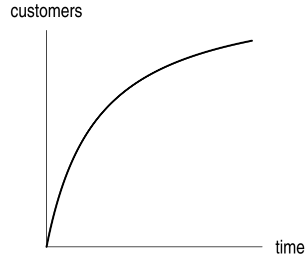
The investment was worth $\$3486.78$ after 10 years.

(b) Measuring time from the moment at which the stock begins to gain value and letting $Q_0 = 3486.78$, the value after t years is

$$Q = 3486.78(1 + 0.1)^t = 3486.78(1.1)^t.$$

We can estimate the value of t when $Q = 10{,}000$ by tracing along a graph of Q, giving $t \approx 11$. It will take about 11 years to get the investment back to $\$10{,}000$.

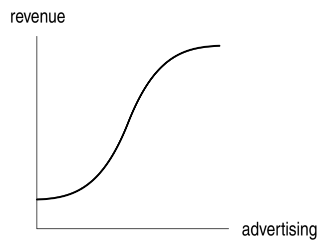
15. (a)

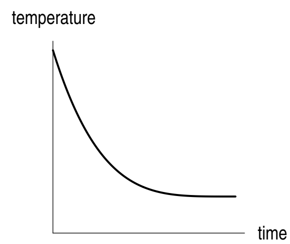


(b) "The rate at which new people try it" is the rate of change of the total number of people who have tried the product. Thus, the statement of the problem is telling you that the graph is concave down—the slope is positive but decreasing, as the graph shows.

16. (a) Advertising is generally cheaper in bulk; spending more money will give better and better marginal results initially, (Spending \$5,000 could give you a big newspaper ad reaching 200,000 people; spending \$100,000 could give you a series of TV spots reaching 50,000,000 people.) A graph is shown below, left.

(b) The temperature of a hot object decreases at a rate proportional to the difference between its temperature and the temperature of the air around it. Thus, the temperature of a very hot object decreases more quickly than a cooler object. The graph is decreasing and concave up. (We are assuming that the coffee is all at the same temperature.)





17. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(0, 3)$ and $(2, 12)$ are on the graph, so

$$3 = y_0 a^0 = y_0$$

and

$$12 = y_0 \cdot a^2 = 3 \cdot a^2, \quad \text{giving} \quad a = \pm 2.$$

Since $a > 0$, our equation is $y = 3(2^x)$.

18. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(-1, 8)$ and $(1, 2)$ are on the graph, so

$$8 = y_0 a^{-1} \quad \text{and} \quad 2 = y_0 a^1$$

Therefore $\dfrac{8}{2} = \dfrac{y_0 a^{-1}}{y_0 a} = \dfrac{1}{a^2}$, giving $a = \frac{1}{2}$, and so $2 = y_0 a^1 = y_0 \cdot \frac{1}{2}$, so $y_0 = 4$.
Hence $y = 4\left(\frac{1}{2}\right)^x = 4(2^{-x})$.

19. We look for an equation of the form $y = y_0 a^x$ since the graph looks exponential. The points $(1, 6)$ and $(2, 18)$ are on the graph, so

$$6 = y_0 a^1 \quad \text{and} \quad 18 = y_0 a^2$$

Therefore $a = \frac{y_0 a^2}{y_0 a} = \dfrac{18}{6} = 3$, and so $6 = y_0 a = y_0 \cdot 3$; thus, $y_0 = 2$. Hence $y = 2(3^x)$.

20. The difference, D, between the horizontal asymptote and the graph appears to decrease exponentially, so we look for an equation of the form

$$D = D_0 a^x$$

where $D_0 = 4 =$ difference when $x = 0$. Since $D = 4 - y$, we have

$$4 - y = 4a^x \quad \text{or} \quad y = 4 - 4a^x = 4(1 - a^x)$$

The point $(1, 2)$ is on the graph, so $2 = 4(1 - a^1)$, giving $a = \frac{1}{2}$.
Therefore $y = 4(1 - (\frac{1}{2})^x) = 4(1 - 2^{-x})$.

21. (a) Let $Q = Q_0 a^t$. Then $Q_0 a^5 = 75.94$ and $Q_0 a^7 = 170.86$. So

$$\frac{Q_0 a^7}{Q_0 a^5} = \frac{170.86}{75.94} = 2.25 = a^2.$$

So $a = 1.5$.

(b) Since $a = 1.5$, the growth rate is $r = 0.5 = 50\%$.

22. (a) Let $Q = Q_0 a^t$. Then $Q_0 a^{0.02} = 25.02$ and $Q_0 a^{0.05} = 25.06$. So

$$\frac{Q_0 a^{0.05}}{Q_0 a^{0.02}} = \frac{25.06}{25.02} = 1.001 = a^{0.03}.$$

So

$$a = (1.001)^{\frac{100}{3}} = 1.05.$$

(b) Since $a = 1.05$, the growth rate is $r = 0.05 = 5\%$.

23. Since $e^{0.25t} = \left(e^{0.25}\right)^t \approx (1.2840)^t$, we have $P = 15(1.2840)^t$. This is exponential growth since 0.25 is positive. We can also see that this is growth because $1.2840 > 1$.

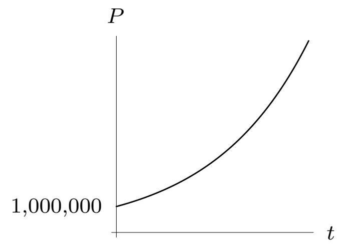
24. Since $e^{-0.5t} = (e^{-0.5})^t \approx (0.6065)^t$, we have $P = 2(0.6065)^t$. This is exponential decay since -0.5 is negative. We can also see that this is decay because $0.6065 < 1$.

25. $P = P_0(e^{0.2})^t = P_0(1.2214)^t$. Exponential growth because $0.2 > 0$ or $1.2214 > 1$.

26. $P = 7(e^{-\pi})^t = 7(0.0432)^t$. Exponential decay because $-\pi < 0$ or $0.0432 < 1$.

27. (a) We have $P_0 = 1$ million, and $k = 0.02$, so $P = (1{,}000{,}000)(e^{0.02t})$.

(b)



28. If the pressure at sea level is P_0, the pressure P at altitude h is given by

$$P = P_0\left(1 - \frac{0.4}{100}\right)^{\frac{h}{100}},$$

since we want the pressure to be multiplied by a factor of $(1 - \frac{0.4}{100}) = 0.996$ for each 100 feet we go up to make it decrease by 0.4% over that interval. At Mexico City $h = 7340$, so the pressure is

$$P = P_0(0.996)^{\frac{7340}{100}} \approx 0.745P_0.$$

So the pressure is reduced from P_0 to approximately $0.745P_0$, a decrease of 25.5%.

29. (a) We compound the daily inflation rate 30 times to get the desired monthly rate r:

$$\left(1 + \frac{r}{100}\right)^1 = \left(1 + \frac{1.3}{100}\right)^{30}.$$

Solving for r, we get $r = 47.3$, so the inflation rate for June was 47.3%.

(b) We compound the daily inflation rate 366 times to get a yearly rate R for 1988, a leap year:

$$\left(1 + \frac{R}{100}\right)^1 = \left(1 + \frac{1.3}{100}\right)^{366}.$$

Solving for R, we get $R = 111.994$, so the yearly rate was 11,199.4% during 1988. We could have obtained approximately the same result by compounding the monthly rate 12 times. Computing the annual rate from the monthly gives a lower result, because 12 months of 30 days each is only 360 days.

30. (a) The formula is $Q = Q_0\left(\frac{1}{2}\right)^{(t/1620)}$.

(b) The percentage left after 500 years is

$$\frac{Q_0(\frac{1}{2})^{(500/1620)}}{Q_0}.$$

The Q_0s cancel giving

$$\left(\frac{1}{2}\right)^{(500/1620)} \approx 0.807,$$

so 80.7% is left.

31. Let Q_0 be the initial quantity absorbed in 1960. Then the quantity, Q of strontium-90 left after t years is

$$Q = Q_0\left(\frac{1}{2}\right)^{(t/29)}.$$

Since $1990 - 1960 = 30$ years elapsed, the fraction of strontium-90 left in 1990 is

$$Q = \frac{Q_0\left(\frac{1}{2}\right)^{(30/29)}}{Q_0} = \left(\frac{1}{2}\right)^{(30/29)} \approx .488 = 48.8\%.$$

32. The doubling time t depends only on the growth rate; it is the solution to

$$2 = (1.02)^t,$$

since 1.02^t represents the factor by which the population has grown after time t. Trial and error shows that $(1.02)^{35} \approx 1.9999$ and $(1.02)^{36} \approx 2.0399$, so that the doubling time is about 35 years.

33. Because the population is growing exponentially, the time it takes to double is the same, regardless of the population levels we are considering. For example, the population is 20,000 at time 3.7, and 40,000 at time 6.0. This represents a doubling of the population in a span of $6.0 - 3.7 = 2.3$ years.

How long does it take the population to double a second time, from 40,000 to 80,000? Looking at the graph once again, we see that the population reaches 80,000 at time $t = 8.3$. This second doubling has taken $8.3 - 6.0 = 2.3$ years, the same amount of time as the first doubling.

Further comparison of any two populations on this graph that differ by a factor of two will show that the time that separates them is 2.3 years. Similarly, during any 2.3 year period, the population will double. Thus, the doubling time is 2.3 years.

Suppose $P = P_0 a^t$ doubles from time t to time $t + d$. We now have $P_0 a^{t+d} = 2P_0 a^t$, so $P_0 a^t a^d = 2P_0 a^t$. Thus, canceling P_0 and a^t, d must be the number such that $a^d = 2$, no matter what t is.

34. Direct calculation reveals that each 1000 foot increase in altitude results in a longer takeoff roll by a factor of about 1.096. Since the value of d when $h = 0$ (sea level) is $d = 670$, we are led to the formula

$$d = 670(1.096)^{h/1000},$$

where d is the takeoff roll, in feet, and h is the airport's elevation, in feet.

Alternatively, we can write

$$d = d_0 a^h,$$

where d_0 is the sea level value of d, $d_0 = 670$. In addition, when $h = 1000$, $d = 734$, so

$$734 = 670a^{1000}.$$

Solving for a gives

$$a = \left(\frac{734}{670}\right)^{1/1000} = 1.00009124,$$

so

$$d = 670(1.00009124)^h.$$

35. (a) This is the graph of a linear function, which increases at a constant rate, and thus corresponds to $k(t)$, which increases by 0.3 over each interval of 1.

(b) This graph is concave down, so it corresponds to a function whose increases are getting smaller, as is the case with $h(t)$, whose increases are 10, 9, 8, 7, and 6.

(c) This graph is concave up, so it corresponds to a function whose increases are getting bigger, as is the case with $g(t)$, whose increases are 1, 2, 3, 4, and 5.

36. (a) A linear function must change by exactly the same amount whenever x changes by some fixed quantity. While $h(x)$ decreases by 3 whenever x increases by 1, $f(x)$ and $g(x)$ fail this test, since both change by different amounts between $x = -2$ and $x = -1$ and between $x = -1$ and $x = 0$. So the only possible linear function is $h(x)$, so it will be given by a formula of the type: $h(x) = mx + b$. As noted, $m = -3$. Since the y-intercept of h is 31, the formula for $h(x)$ is $h(x) = 31 - 3x$.

(b) An exponential function must grow by exactly the same factor whenever x changes by some fixed quantity. Here, $g(x)$ increases by a factor of 1.5 whenever x increases by 1. Since the y-intercept of $g(x)$ is 36, $g(x)$ has the formula $g(x) = 36(1.5)^x$. The other two functions are not exponential; $h(x)$ is not because it is a linear function, and $f(x)$ is not because it both increases and decreases.

37. We see that $\frac{1.09}{1.06} \approx 1.03$, and therefore $h(s) = c(1.03)^s$; c must be 1. Similarly $\frac{2.42}{2.20} = 1.1$, and so $f(s) = a(1.1)^s$; $a = 2$. Lastly, $\frac{3.65}{3.47} \approx 1.05$, so $g(s) = b(1.05)^s$; $b \approx 3$.

38. Since f is linear, its slope is a constant

$$m = \frac{20 - 10}{2 - 0} = 5.$$

Thus f increases 5 units for unit increase in x, so

$$f(1) = 15, \quad f(3) = 25, \quad f(4) = 30.$$

Since g is exponential, its growth factor is constant. Writing $g(x) = ab^x$, we have $g(0) = a = 10$, so

$$g(x) = 10 \cdot b^x.$$

Since $g(2) = 10 \cdot b^2 = 20$, we have $b^2 = 2$ and since $b > 0$, we have

$$b = \sqrt{2}.$$

Thus g increases by a factor of $\sqrt{2}$ for unit increase in x, so

$$g(1) = 10\sqrt{2}, \quad g(3) = 10(\sqrt{2})^3 = 20\sqrt{2}, \quad g(4) = 10(\sqrt{2})^4 = 40.$$

Notice that the value of $g(x)$ doubles between $x = 0$ and $x = 2$ (from $g(0) = 10$ to $g(2) = 20$), so the doubling time of $g(x)$ is 2. Thus, $g(x)$ doubles again between $x = 2$ and $x = 4$, confirming that $g(4) = 40$.

39. (a) The slope is given by

$$m = \frac{P - P_1}{t - t_1} = \frac{100 - 50}{20 - 0} = \frac{50}{20} = 2.5.$$

We know $P = 50$ when $t = 0$, so

$$P = 2.5t + 50.$$

(b) Given $P = P_0 a^t$ and $P = 50$ when $t = 0$,

$$50 = P_0 a^0, \text{ so } P_0 = 50.$$

Then, using $P = 100$ when $t = 20$

$$100 = 50a^{20}$$
$$2 = a^{20}$$
$$a = 2^{1/20} = 1.035265.$$

And so we have

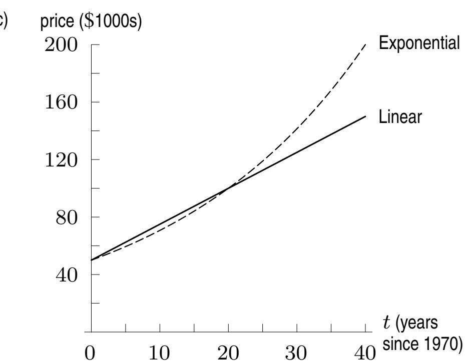
$$P = 50(1.035265)^t.$$

The completed table is found in Table 1.1.

Table 1.1 *The cost of a home*

t	(a) Linear Growth (price in $1000 units)	(b) Exponential Growth (price in $1000 units)
0	50	50
10	75	70.71
20	100	100
30	125	141.42
40	150	200

(c)



(d) Since economic growth (inflation, investments) are usually measured in percentage change per year, the exponential model is probably more realistic.

Solutions for Section 1.3

Exercises

1. (a) $f(g(1)) = f(1 + 1) = f(2) = 2^2 = 4$
(b) $g(f(1)) = g(1^2) = g(1) = 1 + 1 = 2$
(c) $f(g(x)) = f(x + 1) = (x + 1)^2$
(d) $g(f(x)) = g(x^2) = x^2 + 1$
(e) $f(t)g(t) = t^2(t + 1)$

2. (a) $f(g(1)) = f(1^2) = f(1) = \sqrt{1+4} = \sqrt{5}$
(b) $g(f(1)) = g(\sqrt{1+4}) = g(\sqrt{5}) = (\sqrt{5})^2 = 5$
(c) $f(g(x)) = f(x^2) = \sqrt{x^2+4}$
(d) $g(f(x)) = g(\sqrt{x+4}) = (\sqrt{x+4})^2 = x+4$
(e) $f(t)g(t) = (\sqrt{t+4})t^2 = t^2\sqrt{t+4}$

3. (a) $f(g(1)) = f(1^2) = f(1) = e^1 = e$
(b) $g(f(1)) = g(e^1) = g(e) = e^2$
(c) $f(g(x)) = f(x^2) = e^{x^2}$
(d) $g(f(x)) = g(e^x) = (e^x)^2 = e^{2x}$
(e) $f(t)g(t) = e^t t^2$

4. (a) $f(g(1)) = f(3 \cdot 1 + 4) = f(7) = \dfrac{1}{7}$
(b) $g(f(1)) = g(1/1) = g(1) = 7$
(c) $f(g(x)) = f(3x+4) = \dfrac{1}{3x+4}$
(d) $g(f(x)) = g\left(\dfrac{1}{x}\right) = 3\left(\dfrac{1}{x}\right) + 4 = \dfrac{3}{x} + 4$
(e) $f(t)g(t) = \dfrac{1}{t}(3t+4) = 3 + \dfrac{4}{t}$

5. This graph is the graph of $m(t)$ shifted upward by two units. See Figure 1.11.

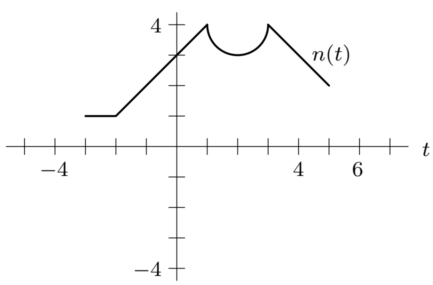


Figure 1.11

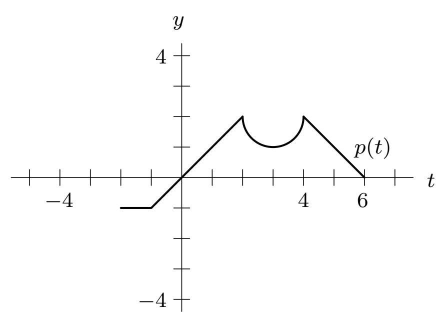


Figure 1.12

6. This graph is the graph of $m(t)$ shifted to the right by one unit. See Figure 1.12.

7. This graph is the graph of $m(t)$ shifted to the left by 1.5 units. See Figure 1.13.

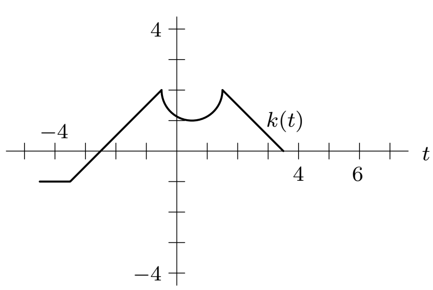


Figure 1.13

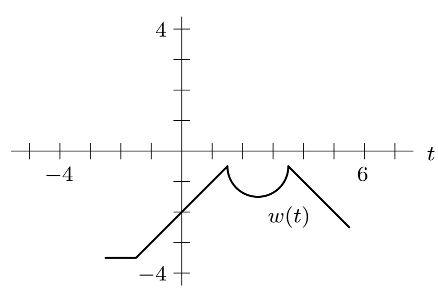


Figure 1.14

8. This graph is the graph of $m(t)$ shifted to the right by 0.5 units and downward by 2.5 units. See Figure 1.14.

9. For $f(-x)$, the graph is reflected in the y-axis. See Figure 1.15.

10. For $f(x) + 5$, the graph is shifted 5 upward. See Figure 1.16.

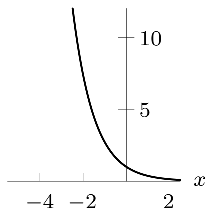


Figure 1.15

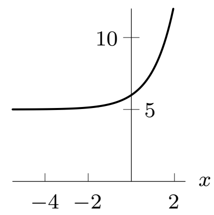


Figure 1.16

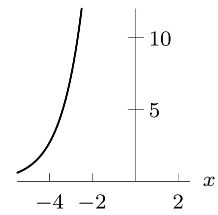


Figure 1.17

11. For $f(x+5)$, the graph is shifted 5 units to the left. See Figure 1.17.

12. For $5f(x)$, the values are 5 times as large. See Figure 1.18.

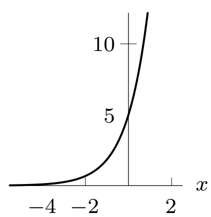


Figure 1.18

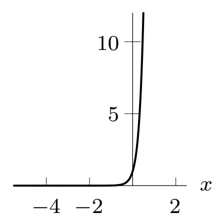


Figure 1.19

13. For $f(5x)$, the values increase 5 times as fast as we move right. See Figure 1.19.

14. (a) $g(2+h) = (2+h)^2 + 2(2+h) + 3 = 4 + 4h + h^2 + 4 + 2h + 3 = h^2 + 6h + 11.$
(b) $g(2) = 2^2 + 2(2) + 3 = 4 + 4 + 3 = 11$, which agrees with what we get by substituting $h = 0$ into (a).
(c) $g(2+h) - g(2) = (h^2 + 6h + 11) - (11) = h^2 + 6h.$

15. (a) $f(t+1) = (t+1)^2 + 1 = t^2 + 2t + 1 + 1 = t^2 + 2t + 2.$
(b) $f(t^2+1) = (t^2+1)^2 + 1 = t^4 + 2t^2 + 1 + 1 = t^4 + 2t^2 + 2.$
(c) $f(2) = 2^2 + 1 = 5.$
(d) $2f(t) = 2(t^2+1) = 2t^2 + 2.$
(e) $[f(t)]^2 + 1 = \left(t^2+1\right)^2 + 1 = t^4 + 2t^2 + 1 + 1 = t^4 + 2t^2 + 2.$

16. (a) $f(n) + g(n) = (3n^2 - 2) + (n+1) = 3n^2 + n - 1.$
(b) $f(n)g(n) = (3n^2 - 2)(n+1) = 3n^3 + 3n^2 - 2n - 2.$
(c) The domain of $f(n)/g(n)$ is defined everywhere where $g(n) \neq 0$, i.e. for all $n \neq -1$.
(d) $f(g(n)) = 3(n+1)^2 - 2 = 3n^2 + 6n + 1.$
(e) $g(f(n)) = (3n^2 - 2) + 1 = 3n^2 - 1.$

17. $m(z+1) - m(z) = (z+1)^2 - z^2 = 2z + 1.$

18. $m(z+h) - m(z) = (z+h)^2 - z^2 = 2zh + h^2.$

19. $m(z) - m(z-h) = z^2 - (z-h)^2 = 2zh - h^2.$

20. $m(z+h) - m(z-h) = (z+h)^2 - (z-h)^2 = z^2 + 2hz + h^2 - (z^2 - 2hz + h^2) = 4hz.$

21. (a) $f(25)$ is q corresponding to $p = 25$, or, in other words, the number of items sold when the price is 25.
(b) $f^{-1}(30)$ is p corresponding to $q = 30$, or the price at which 30 units will be sold.

22. (a) $f(10{,}000)$ represents the value of C corresponding to $A = 10{,}000$, or in other words the cost of building a 10,000 square-foot store.
(b) $f^{-1}(20{,}000)$ represents the value of A corresponding to $C = 20{,}000$, or the area in square feet of a store which would cost \$20,000 to build.

23. $f^{-1}(75)$ is the length of the column of mercury in the thermometer when the temperature is $75°\text{F}$.

24. (a) Since $m = f(A)$, we see that $f(100)$ represents the value of m when $A = 100$. Thus $f(100)$ is the minimum annual gross income needed (in thousands) to take out a 30-year mortgage loan of \$100,000 at an interest rate of 9%.
(b) Since $m = f(A)$, we have $A = f^{-1}(m)$. We see that $f^{-1}(75)$ represents the value of A when $m = 75$, or the size of a mortgage loan that could be obtained on an income of \$75,000.

25. The function is not invertible since there are many horizontal lines which hit the function twice.

26. The function is not invertible since there are horizontal lines which hit the function more than once.

27. (a) We find $f^{-1}(2)$ by finding the x value corresponding to $f(x) = 2$. Looking at the graph, we see that $f^{-1}(2) = -1$.
(b) We construct the graph of $f^{-1}(x)$ by reflecting the graph of $f(x)$ over the line $y = x$. The graphs of $f^{-1}(x)$ and $f(x)$ are shown together in Figure 1.20.

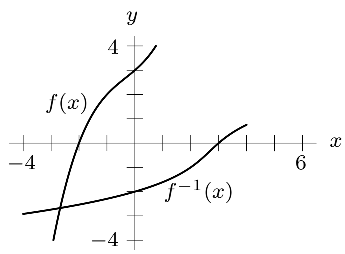


Figure 1.20

28. This looks like a shift of the graph $y = -x^2$. The graph is shifted to the left 1 unit and up 3 units, so a possible formula is $y = -(x+1)^2 + 3$.

29. This looks like a shift of the graph $y = x^3$. The graph is shifted to the right 2 units and down 1 unit, so a possible formula is $y = (x-2)^3 - 1$.

30.
$$f(-x) = (-x)^6 + (-x)^3 + 1 = x^6 - x^3 + 1.$$
Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, this function is neither even nor odd.

31.
$$f(-x) = (-x)^3 + (-x)^2 + (-x) = -x^3 + x^2 - x.$$
Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, this function is neither even nor odd.

32. Since
$$f(-x) = (-x)^4 - (-x)^2 + 3 = x^4 - x^2 + 3 = f(x),$$
we see f is even

33. Since
$$f(-x) = (-x)^3 + 1 = -x^3 + 1,$$
we see $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd

34. Since
$$f(-x) = 2(-x) = -2x = -f(x),$$
we see f is odd.

35. Since
$$f(-x) = e^{(-x)^2-1} = e^{x^2-1} = f(x),$$
we see f is even.

36. Since
$$f(-x) = (-x)((-x)^2 - 1) = -x(x^2 - 1) = -f(x),$$
we see f is odd

37. Since
$$f(-x) = e^{-x} + x,$$
we see $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd

Problems

38. (a) The equation is $y = 2x^2 + 1$. Note that its graph is narrower than the graph of $y = x^2$ which appears in gray. See Figure 1.21.

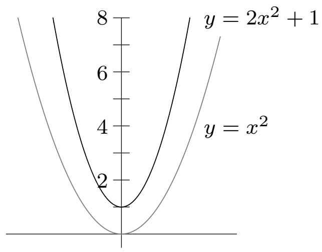


Figure 1.21

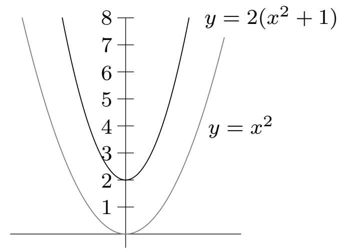


Figure 1.22

(b) $y = 2(x^2 + 1)$ moves the graph up one unit and *then* stretches it by a factor of two. See Figure 1.22.
(c) No, the graphs are not the same. Since $2(x^2 + 1) = (2x^2 + 1) + 1$, the second graph is always one unit higher than the first.

39. Not invertible. Given a certain number of customers, say $f(t) = 1500$, there could be many times, t, during the day at which that many people were in the store. So we don't know which time instant is the right one.

40. Probably not invertible. Since your calculus class probably has less than 363 students, there will be at least two days in the year, say a and b, with $f(a) = f(b) = 0$. Hence we don't know what to choose for $f^{-1}(0)$.

41. Invertible. Since at $4°$C, the mass of 1 liter of water is 1 kilogram, the mass of x liters is x kilograms. So $f(x) = x$ and therefore, $f^{-1}(x) = x$.

42. Not invertible, since it costs the same to mail a 50-gram letter as it does to mail a 51-gram letter.

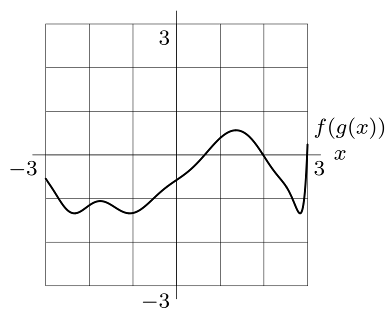
43. $f(g(1)) = f(2) \approx 0.4$.

44. $g(f(2)) \approx g(0.4) \approx 1.1$.

45. $f(f(1)) \approx f(-0.4) \approx -0.9$.

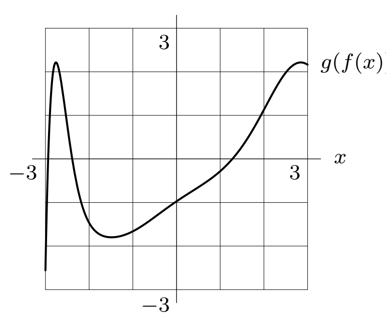
46. Computing $f(g(x))$ as in Problem 43, we get the following table. From it we graph $f(g(x))$.

x	$g(x)$	$f(g(x))$
-3	0.6	-0.5
-2.5	-1.1	-1.3
-2	-1.9	-1.2
-1.5	-1.9	-1.2
-1	-1.4	-1.3
-0.5	-0.5	-1
0	0.5	-0.6
0.5	1.4	-0.2
1	2	0.4
1.5	2.2	0.5
2	1.6	0
2.5	0.1	-0.7
3	-2.5	0.1



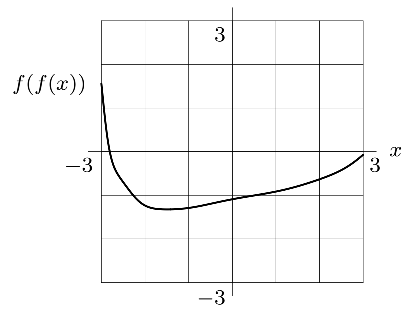
47. Using the same way to compute $g(f(x))$ as in Problem 44, we get the following table. Then we can plot the graph of $g(f(x))$.

x	$f(x)$	$g(f(x))$
-3	3	-2.6
-2.5	0.1	0.8
-2	-1	-1.4
-1.5	-1.3	-1.8
-1	-1.2	-1.7
-0.5	-1	-1.4
0	-0.8	-1
0.5	-0.6	-0.6
1	-0.4	-0.3
1.5	-0.1	0.3
2	0.3	1.1
2.5	0.9	2
3	1.6	2.2



48. Using the same way to compute $f(f(x))$ as in Problem 45, we get the following table. Then we can plot the graph of $f(f(x))$.

x	$f(x)$	$f(f(x))$
-3	3	1.6
-2.5	0.1	-0.7
-2	-1	-1.2
-1.5	-1.3	-1.3
-1	-1.2	-1.3
-0.5	-1	-1.2
0	-0.8	-1.1
0.5	-0.6	-1
1	-0.4	-0.9
1.5	-0.1	-0.8
2	0.3	-0.6
2.5	0.9	-0.4
3	1.6	0



49. $f(x) = x^3, \quad g(x) = x + 1.$

50. $f(x) = x + 1, \quad g(x) = x^3.$

51. Since $B = y - 1$ and $n = 2B^2 - B$, substitution gives

$$\text{Average number of leaves } = Bn = (y-1)(2(y-1)^2 - (y-1)) = 2(y-1)^3 - (y-1)^2.$$

52. (a) The function f tells us C in terms of q. To get its inverse, we want q in terms of C, which we find by solving for q:

$$C = 100 + 2q,$$
$$C - 100 = 2q,$$
$$q = \frac{C - 100}{2} = f^{-1}(C).$$

(b) The inverse function tells us the number of articles that can be produced for a given cost.

53. (a) For each 2.2 pounds of weight the object has, it has 1 kilogram of mass, so the conversion formula is

$$k = f(p) = \frac{1}{2.2}p.$$

(b) The inverse function is

$$p = 2.2k,$$

and it gives the weight of an object in pounds as a function of its mass in kilograms.

54.

x	$f(x)$	$g(x)$	$h(x)$
-3	0	0	0
-2	2	2	-2
-1	2	2	-2
0	0	0	0
1	2	-2	-2
2	2	-2	-2
3	0	0	0

Solutions for Section 1.4

Exercises

1. Using the identity $e^{\ln x} = x$, we have $e^{\ln(1/2)} = \frac{1}{2}$.

2. Using the identity $10^{\log x} = x$, we have

$$10^{\log(AB)} = AB$$

3. Using the identity $e^{\ln x} = x$, we have $5A^2$.

4. Using the identity $\ln(e^x) = x$, we have 2AB.

5. Using the rules for ln, we have

$$\begin{aligned}\ln\left(\frac{1}{e}\right) + \ln AB &= \ln 1 - \ln e + \ln A + \ln B \\ &= 0 - 1 + \ln A + \ln B \\ &= -1 + \ln A + \ln B.\end{aligned}$$

6. Using the rules for ln, we have $2A + 3e\ln B$.

7. Taking logs of both sides

$$\begin{aligned}\log 3^x &= x\log 3 = \log 11 \\ x &= \frac{\log 11}{\log 3} = 2.2.\end{aligned}$$

8. Taking logs of both sides

$$\begin{aligned}\log 17^x &= \log 2 \\ x\log 17 &= \log 2 \\ x &= \frac{\log 2}{\log 17} \approx 0.24.\end{aligned}$$

9. Isolating the exponential term

$$\begin{aligned}20 &= 50(1.04)^x \\ \frac{20}{50} &= (1.04)^x.\end{aligned}$$

Taking logs of both sides

$$\begin{aligned}\log\frac{2}{5} &= \log(1.04)^x \\ \log\frac{2}{5} &= x\log(1.04) \\ x &= \frac{\log(2/5)}{\log(1.04)} = -23.4.\end{aligned}$$

10.

$$\begin{aligned}\frac{4}{7} &= \frac{5^x}{3^x} \\ \frac{4}{7} &= \left(\frac{5}{3}\right)^x\end{aligned}$$

Taking logs of both sides

$$\begin{aligned}\log\left(\frac{4}{7}\right) &= x\log\left(\frac{5}{3}\right) \\ x &= \frac{\log(4/7)}{\log(5/3)} \approx -1.1.\end{aligned}$$

11.
$$\begin{aligned}\ln(2^x) &= \ln(e^{x+1}) \\ x\ln 2 &= (x+1)\ln e \\ x\ln 2 &= x+1 \\ 0.693x &= x+1 \\ x &= \frac{1}{0.693-1} \approx -3.26\end{aligned}$$

12. To solve for x, we first divide both sides by 600 and then take the natural logarithm of both sides.

$$\begin{aligned}\frac{50}{600} &= e^{-0.4x} \\ \ln(50/600) &= -0.4x \\ x &= \frac{\ln(50/600)}{-0.4} \approx 6.212.\end{aligned}$$

13.
$$\begin{aligned} \ln(2e^{3x}) &= \ln(4e^{5x}) \\ \ln 2 + \ln(e^{3x}) &= \ln 4 + \ln(e^{5x}) \\ 0.693 + 3x &= 1.386 + 5x \\ x &= -0.347 \end{aligned}$$

14. Using the rules for ln, we get

$$\begin{aligned} \ln 7^{x+2} &= \ln e^{17x} \\ (x+2)\ln 7 &= 17x \\ x(\ln 7 - 17) &= -2\ln 7 \\ x &= \frac{-2\ln 7}{\ln 7 - 17} \approx 0.26. \end{aligned}$$

15.
$$\begin{aligned} \ln(10^{x+3}) &= \ln(5e^{7-x}) \\ (x+3)\ln 10 &= \ln 5 + (7-x)\ln e \\ 2.303(x+3) &= 1.609 + (7-x) \\ 3.303x &= 1.609 + 7 - 2.303(3) \\ x &= 0.515 \end{aligned}$$

16. Using the rules for ln, we have

$$\begin{aligned} 2x - 1 &= x^2 \\ x^2 - 2x + 1 &= 0 \\ (x-1)^2 &= 0 \\ x &= 1. \end{aligned}$$

17. Using the rules for ln, we get

$$\begin{aligned} \ln 9^x &= \ln 2e^{x^2} \\ x\ln 9 &= \ln 2 + x^2 \\ x^2 - x\ln 9 + \ln 2 &= 0. \end{aligned}$$

We can use the quadratic formula to get $x = \dfrac{\ln 9 \pm \sqrt{(\ln 9)^2 - 4\ln 2}}{2}$, so $x \approx 0.382$, $x \approx 1.815$.

18. $t = \dfrac{\log a}{\log b}$.

19. $t = \dfrac{\log\left(\frac{P}{P_0}\right)}{\log a} = \dfrac{\log P - \log P_0}{\log a}$.

20. Taking logs of both sides yields

$$nt = \frac{\log\left(\frac{Q}{Q_0}\right)}{\log a}.$$

Hence

$$t = \frac{\log\left(\frac{Q}{Q_0}\right)}{n\log a} = \frac{\log Q - \log Q_0}{n\log a}.$$

21. Collecting similar terms yields

$$\left(\frac{a}{b}\right)^t = \frac{Q_0}{P_0}.$$

Hence

$$t = \frac{\log\left(\frac{Q_0}{P_0}\right)}{\log\left(\frac{a}{b}\right)}.$$

22. $t = \ln \dfrac{a}{b}$.

23. $\ln \dfrac{P}{P_0} = kt$, so $t = \dfrac{\ln \frac{P}{P_0}}{k}$.

24. Since we want $(1.5)^t = e^{kt} = (e^k)^t$, so $1.5 = e^k$, and $k = \ln 1.5 = 0.4055$. Thus, $P = 15e^{0.4055t}$. Since 0.4055 is positive, this is exponential growth.

25. We want $1.7^t = e^{kt}$ so $1.7 = e^k$ and $k = \ln 1.7 = 0.5306$. Thus $P = 10e^{0.5306t}$.

26. We want $0.9^t = e^{kt}$ so $0.9 = e^k$ and $k = \ln 0.9 = -0.1054$. Thus $P = 174e^{-0.1054t}$.

27. Since we want $(0.55)^t = e^{kt} = (e^k)^t$, so $0.55 = e^k$, and $k = \ln 0.55 = -0.5978$. Thus $P = 4e^{-0.5978t}$. Since -0.5978 is negative, this represents exponential decay.

28. If $p(t) = (1.04)^t$, then, for p^{-1} the inverse of p, we should have

$$\begin{aligned} (1.04)^{p^{-1}(t)} &= t, \\ p^{-1}(t)\log(1.04) &= \log t, \\ p^{-1}(t) &= \frac{\log t}{\log(1.04)} \approx 58.708 \log t. \end{aligned}$$

29. Since f is increasing, f has an inverse. To find the inverse of $f(t) = 50e^{0.1t}$, we replace t with $f^{-1}(t)$, and, since $f(f^{-1}(t)) = t$, we have

$$t = 50e^{0.1f^{-1}(t)}.$$

We then solve for $f^{-1}(t)$:

$$\begin{aligned} t &= 50e^{0.1f^{-1}(t)} \\ \frac{t}{50} &= e^{0.1f^{-1}(t)} \\ \ln\left(\frac{t}{50}\right) &= 0.1f^{-1}(t) \\ f^{-1}(t) &= \frac{1}{0.1}\ln\left(\frac{t}{50}\right) = 10\ln\left(\frac{t}{50}\right). \end{aligned}$$

30. Using $f(f^{-1}(t)) = t$, we see

$$f(f^{-1}(t)) = 1 + \ln f^{-1}(t) = t.$$

So

$$\begin{aligned} \ln f^{-1}(t) &= t - 1 \\ f^{-1}(t) &= e^{t-1}. \end{aligned}$$

Problems

31. The function e^x has a vertical intercept of 1, so must be A. The function $\ln x$ has an x-intercept of 1, so must be D. The graphs of x^2 and $x^{1/2}$ go through the origin. The graph of $x^{1/2}$ is concave down so it corresponds to graph C and the graph of x^2 is concave up so it corresponds to graph B.

32. The population has increased by a factor of $56{,}000{,}000/40{,}000{,}000 = 1.4$ in 10 years. Thus we have the formula

$$P = 40{,}000{,}000(1.4)^{t/10},$$

and $t/10$ gives the number of 10-year periods that have passed since 1980.

In 1980, $t/10 = 0$, so we have $P = 40{,}000{,}000$.

In 1990, $t/10 = 1$, so $P = 40{,}000{,}000(1.4) = 56{,}000{,}000$.

In 2000, $t/10 = 2$, so $P = 40{,}000{,}000(1.4)^2 = 78{,}400{,}000$.

To find the doubling time, solve $80{,}000{,}000 = 40{,}000{,}000(1.4)^{t/10}$, to get $t \approx 20.6$ years.

33. Since the factor by which the prices have increased after time t is given by $(1.05)^t$, the time after which the prices have doubled solves

$$\begin{aligned} 2 &= (1.05)^t \\ \log 2 &= \log(1.05^t) = t\log(1.05) \\ t &= \frac{\log 2}{\log 1.05} \approx 14.21 \text{ years}. \end{aligned}$$

34. Given the doubling time of 5 hours, we can solve for the bacteria's growth rate;

$$\begin{aligned} 2P_0 &= P_0e^{k5} \\ k &= \frac{\ln 2}{5}. \end{aligned}$$

So the growth of the bacteria population is given by:

$$P = P_0e^{\ln(2)t/5}.$$

We want to find t such that

$$3P_0 = P_0e^{\ln(2)t/5}.$$

Therefore we cancel P_0 and apply $\ln$. We get

$$t = \frac{5\ln(3)}{\ln(2)} = 7.925 \text{ hours.}$$

35. In ten years, the substance has decayed to 40% of its original mass. In another ten years, it will decay by an additional factor of 40%, so the amount remaining after 20 years will be $100 \cdot 40\% \cdot 40\% = 16$ kg.

36. Using the exponential decay equation $P = P_0e^{-kt}$, we can solve for the substance's decay constant k:

$$\begin{aligned} (P_0 - 0.3P_0) &= P_0e^{-20k} \\ k &= \frac{\ln(0.7)}{-20}. \end{aligned}$$

Knowing this decay constant, we can solve for the half-life t using the formula

$$\begin{aligned} 0.5P_0 &= P_0e^{\ln(0.7)t/20} \\ t &= \frac{20\ln(0.5)}{\ln(0.7)} \approx 38.87 \text{ hours.} \end{aligned}$$

37. Let B represent the sales (in millions of dollars) at Borders bookstores t years since 1991. Since $B = 78$ when $t = 0$ and we want the continuous growth rate, we write $B = 78e^{kt}$. We use the information from 1994, that $B = 412$ when $t = 3$, to find k:

$$\begin{aligned} 412 &= 78e^{k\cdot 3} \\ 5.282 &= e^{3k} \\ \ln(5.282) &= 3k \\ k &= 0.555. \end{aligned}$$

We have $B = 78e^{0.555t}$, which represents a continuous growth rate of 55.5% per year.

38. Let n be the infant mortality of Senegal. As a function of time t, n is given by

$$n = n_0(0.90)^t.$$

To find when $n = 0.50n_0$ (so the number of cases has been reduced by 50%), we solve

$$\begin{aligned} 0.50 &= (0.90)^t, \\ \log(0.50) &= t\log(0.90), \\ t &= \frac{\log(0.50)}{\log(0.90)} \approx 6.58 \text{ years.} \end{aligned}$$

39. We know that the y-intercept of the line is at (0,1), so we need one other point to determine the equation of the line. We observe that it intersects the graph of $f(x) = 10^x$ at the point $x = \log 2$. The y-coordinate of this point is then

$$y = 10^x = 10^{\log 2} = 2,$$

so $(\log 2, 2)$ is the point of intersection. We can now find the slope of the line:

$$m = \frac{2-1}{\log 2 - 0} = \frac{1}{\log 2}.$$

Plugging this into the point-slope formula for a line, we have

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{1}{\log 2}(x - 0) \\ y &= \frac{1}{\log 2}x + 1 \approx 3.3219x + 1. \end{aligned}$$

40. (a) The initial dose is 10 mg.
(b) Since $0.82 = 1 - 0.18$, the decay rate is 0.18, so 18% leaves the body each hour.
(c) When $t = 6$, we have $A = 10(0.82)^6 = 3.04$. The amount in the body after 6 hours is 3.04 mg.
(d) We want to find the value of t when $A = 1$. Using logarithms:

$$\begin{aligned} 1 &= 10(0.82)^t \\ 0.1 &= (0.82)^t \\ \ln(0.1) &= t \ln(0.82) \\ t &= 11.60 \text{ hours}. \end{aligned}$$

After 11.60 hours, the amount is 1 mg.

41. (a) Since the initial amount of caffeine is 100 mg and the exponential decay rate is -0.17, we have $A = 100e^{-0.17t}$.
(b) See Figure 1.23. We estimate the half-life by estimating t when the caffeine is reduced by half (so $A = 50$); this occurs at approximately $t = 4$ hours.

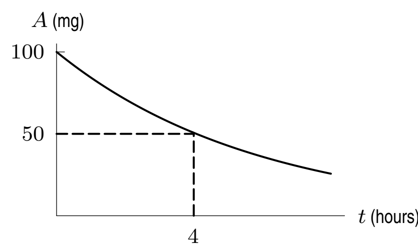


Figure 1.23

(c) We want to find the value of t when $A = 50$:

$$\begin{aligned} 50 &= 100e^{-0.17t} \\ 0.5 &= e^{-0.17t} \\ \ln 0.5 &= -0.17t \\ t &= 4.077. \end{aligned}$$

The half-life of caffeine is about 4.077 hours. This agrees with what we saw in Figure 1.23.

42. Let t = number of years since 1980. Then the number of vehicles, V, in millions, at time t is given by

$$V = 170(1.04)^t$$

and the number of people, P, in millions, at time t is given by

$$P = 227(1.01)^t.$$

There is an average of one vehicle per person when $\frac{V}{P} = 1$, or $V = P$. Thus, we must solve for t the equation:

$$170(1.04)^t = 227(1.01)^t,$$

which implies

$$\left(\frac{1.04}{1.01}\right)^t = \frac{(1.04)^t}{(1.01)^t} = \frac{227}{170}$$

Taking logs on both sides,

$$t \log \frac{1.04}{1.01} = \log \frac{227}{170}.$$

Therefore,

$$t = \frac{\log\left(\frac{227}{170}\right)}{\log\left(\frac{1.04}{1.01}\right)} \approx 9.9 \text{ years.}$$

So there was, according to this model, about one vehicle per person in 1990.

43. (a) We know the decay follows the equation

$$P = P_0 e^{-kt},$$

and that 10% of the pollution is removed after 5 hours (meaning that 90% is left). Therefore,

$$\begin{aligned} 0.90P_0 &= P_0 e^{-5k} \\ k &= -\frac{1}{5}\ln(0.90). \end{aligned}$$

Thus, after 10 hours:

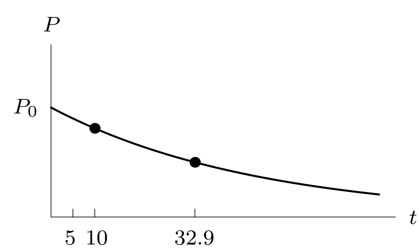
$$\begin{aligned} P &= P_0 e^{-10((-0.2)\ln 0.90)} \\ P &= P_0(0.9)^2 = 0.81P_0 \end{aligned}$$

so 81% of the original amount is left.

(b) We want to solve for the time when $P = 0.50P_0$:

$$\begin{aligned} 0.50P_0 &= P_0 e^{t((0.2)\ln 0.90)} \\ 0.50 &= e^{\ln(0.90^{0.2t})} \\ 0.50 &= 0.90^{0.2t} \\ t &= \frac{5\ln(0.50)}{\ln(0.90)} \approx 32.9 \text{ hours.} \end{aligned}$$

(c)



(d) When highly polluted air is filtered, there is more pollutant per liter of air to remove. If a fixed amount of air is cleaned every day, there is a higher amount of pollutant removed earlier in the process.

44. (a) The pressure P at 6198 meters is given in terms of the pressure P_0 at sea level to be

$$\begin{aligned} P &= P_0 e^{-0.00012h} \\ &= P_0 e^{(-0.00012)6198} \\ &= P_0 e^{-0.74376} \\ &\approx 0.4753P_0 \quad \text{or about 47.5\% of sea level pressure.} \end{aligned}$$

(b) At $h = 12{,}000$ meters, we have

$$\begin{aligned} P &= P_0 e^{-0.00012h} \\ &= P_0 e^{(-0.00012)12{,}000} \\ &= P_0 e^{-1.44} \\ &\approx 0.2369P_0 \quad \text{or about 23.7\% of sea level pressure.} \end{aligned}$$

45. Since the amount of strontium-90 remaining halves every 29 years, we can solve for the decay constant;

$$0.5P_0 = P_0 e^{-29k}$$
$$k = \frac{\ln(1/2)}{-29}.$$

Knowing this, we can look for the time t in which $P = 0.10P_0$, or

$$0.10P_0 = P_0 e^{\ln(0.5)t/29}$$
$$t = \frac{29\ln(0.10)}{\ln(0.5)} = 96.34 \text{ years.}$$

46. We assume exponential decay and solve for k using the half-life:

$$e^{-k(5730)} = 0.5 \quad \text{so} \quad k = 1.21 \cdot 10^{-4}.$$

Now find t, the age of the painting:

$$e^{-1.21\cdot 10^{-4}t} = 0.995, \quad \text{so} \quad t = \frac{\ln 0.995}{-1.21\cdot 10^{-4}} = 41.43 \text{ years.}$$

Since Vermeer died in 1675, the painting is a fake.

Solutions for Section 1.5

Exercises

1. See Figure 1.24.

$$\sin\left(\frac{3\pi}{2}\right) = -1 \quad \text{is negative.}$$
$$\cos\left(\frac{3\pi}{2}\right) = 0$$
$$\tan\left(\frac{3\pi}{2}\right) \quad \text{is undefined.}$$

2. See Figure 1.25.

$$\sin(2\pi) = 0$$
$$\cos(2\pi) = 1 \text{ is positive.}$$
$$\tan(2\pi) = 0$$

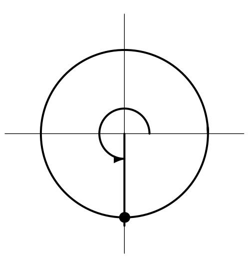


Figure 1.24

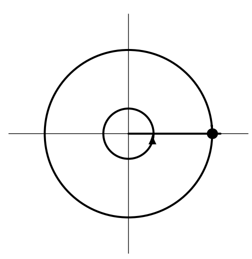


Figure 1.25

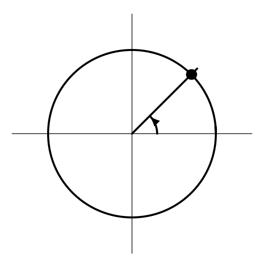


Figure 1.26

3. See Figure 1.26.

$$\sin\frac{\pi}{4} \quad \text{is positive}$$
$$\cos\frac{\pi}{4} \quad \text{is positive}$$
$$\tan\frac{\pi}{4} \quad \text{is positive}$$

4. See Figure 1.27.

$$\begin{aligned}\sin 3\pi &= 0\\ \cos 3\pi &= -1 \text{ is negative}\\ \tan 3\pi &= 0\end{aligned}$$

5. See Figure 1.28.

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &\text{ is positive.}\\ \cos\left(\frac{\pi}{6}\right) &\text{ is positive.}\\ \tan\left(\frac{\pi}{6}\right) &\text{ is positive.}\end{aligned}$$

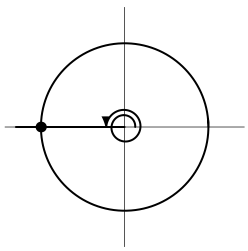


Figure 1.27

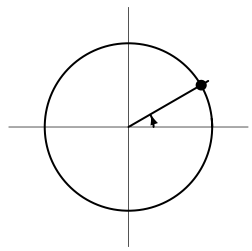


Figure 1.28

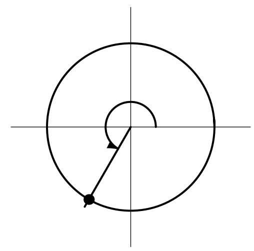


Figure 1.29

6. See Figure 1.29.

$$\begin{aligned}\sin\frac{4\pi}{3} &\quad \text{is negative}\\ \cos\frac{4\pi}{3} &\quad \text{is negative}\\ \tan\frac{4\pi}{3} &\quad \text{is positive}\end{aligned}$$

7. See Figure 1.30.

$$\begin{aligned}\sin\left(\frac{-4\pi}{3}\right) &\text{ is positive.}\\ \cos\left(\frac{-4\pi}{3}\right) &\text{ is negative.}\\ \tan\left(\frac{-4\pi}{3}\right) &\text{ is negative.}\end{aligned}$$

8. $4 \text{ radians} \cdot \dfrac{180^\circ}{\pi \text{ radians}} = \left(\dfrac{720}{\pi}\right)^\circ \approx 240^\circ$. See Figure 1.31.

$$\begin{aligned}\sin 4 &\quad \text{is negative}\\ \cos 4 &\quad \text{is negative}\\ \tan 4 &\quad \text{is positive.}\end{aligned}$$

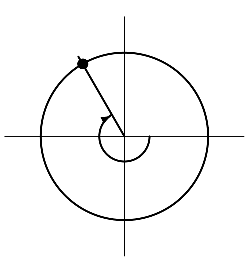


Figure 1.30

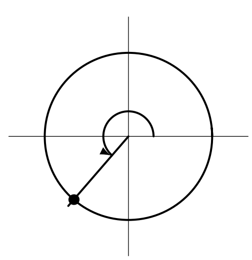


Figure 1.31

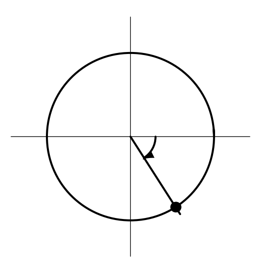
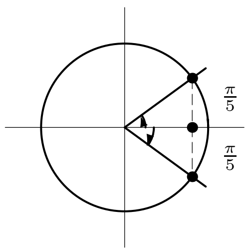


Figure 1.32

9. $-1 \text{ radian} \cdot \frac{180^\circ}{\pi \text{ radians}} = -\left(\frac{180^\circ}{\pi}\right) \approx -60^\circ$. See Figure 1.32.

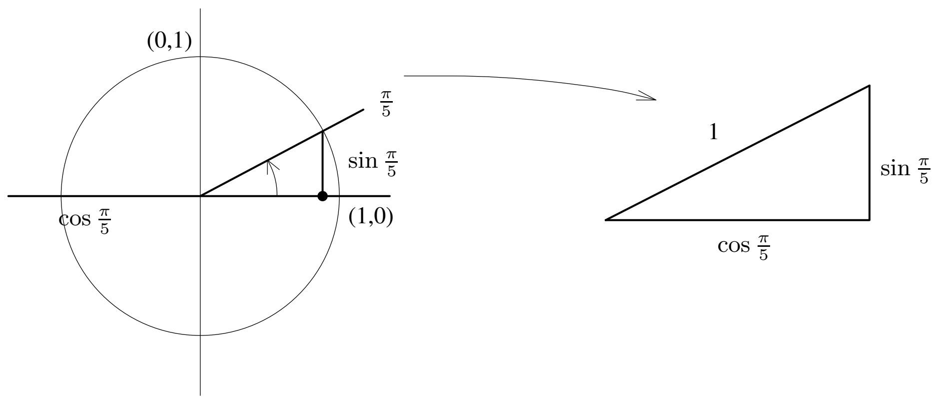
$$\begin{aligned} \sin(-1) &\quad \text{is negative} \\ \cos(-1) &\quad \text{is positive} \\ \tan(-1) &\quad \text{is negative.} \end{aligned}$$

10.



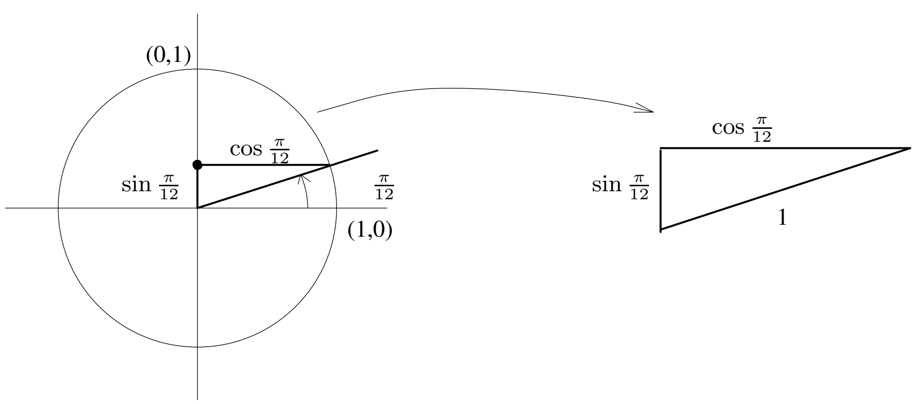
$$\cos\left(-\frac{\pi}{5}\right) = \cos\frac{\pi}{5} \quad \text{(by picture)}$$
$$= 0.809.$$

11.



By the Pythagorean Theorem, $(\cos\frac{\pi}{5})^2 + (\sin\frac{\pi}{5})^2 = 1^2$;
so $(\sin\frac{\pi}{5})^2 = 1 - (\cos\frac{\pi}{5})^2$, and $\sin\frac{\pi}{5} = \sqrt{1-(\cos\frac{\pi}{5})^2} = \sqrt{1-(0.809)^2} \approx 0.588$.
We take the positive square root since by the picture we know that $\sin\frac{\pi}{5}$ is positive.

12.

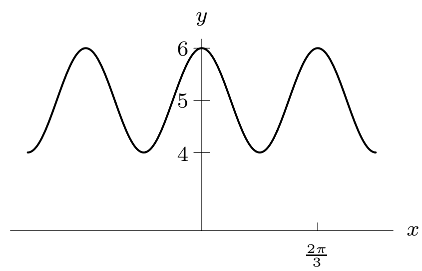


By the Pythagorean Theorem, $(\cos\frac{\pi}{12})^2+(\sin\frac{\pi}{12})^2 = 1^2$; so $(\cos\frac{\pi}{12})^2 = 1-(\sin\frac{\pi}{12})^2$ and $\cos\frac{\pi}{12} = \sqrt{1-(\sin\frac{\pi}{12})^2} = \sqrt{1-(0.259)^2} \approx 0.966$. We take the positive square root since by the picture we know that $\cos\frac{\pi}{12}$ is positive.

13. (a) We determine the amplitude of y by looking at the coefficient of the cosine term. Here, the coefficient is 1, so the amplitude of y is 1. Note that the constant term does not affect the amplitude.

(b) We know that the cosine function $\cos x$ repeats itself at $x = 2\pi$, so the function $\cos(3x)$ must repeat itself when $3x = 2\pi$, or at $x = 2\pi/3$. So the period of y is $2\pi/3$. Here as well the constant term has no effect.

(c) The graph of y is shown in the figure below.



14. The period is $2\pi/3$, because when t varies from 0 to $2\pi/3$, the quantity $3t$ varies from 0 to 2π. The amplitude is 7, since the value of the function oscillates between -7 and 7.

15. The period is $2\pi/(1/4) = 8\pi$, because when u varies from 0 to 8π, the quantity $u/4$ varies from 0 to 2π. The amplitude is 3, since the function oscillates between 2 and 8.

16. The period is $2\pi/2 = \pi$, because as x varies from $-\pi/2$ to $\pi/2$, the quantity $2x+\pi$ varies from 0 to 2π. The amplitude is 4, since the function oscillates between 4 and 12.

17. The period is $2\pi/\pi = 2$, since when t increases from 0 to 2, the value of πt increases from 0 to 2π. The amplitude is 0.1, since the function oscillates between 1.9 and 2.1.

18. (a) $h(t) = 2\cos(t - \pi/2)$
(b) $f(t) = 2\cos t$
(c) $g(t) = 2\cos(t + \pi/2)$

19. This graph is a sine curve with period 8π and amplitude 2, so it is given by $f(x) = 2\sin\left(\frac{x}{4}\right)$.

20. This graph is a cosine curve with period 6π and amplitude 5, so it is given by $f(x) = 5\cos\left(\frac{x}{3}\right)$.

21. This graph is an inverted sine curve with amplitude 4 and period π, so it is given by $f(x) = -4\sin(2x)$.

22. This graph is an inverted cosine curve with amplitude 8 and period 20π, so it is given by $f(x) = -8\cos\left(\frac{x}{10}\right)$.

23. This graph has period 6, amplitude 5 and no vertical or horizontal shift, so it is given by

$$f(x) = 5\sin\left(\frac{2\pi}{6}x\right) = 5\sin\left(\frac{\pi}{3}x\right).$$

24. The graph is a cosine curve with period $2\pi/5$ and amplitude 2, so it is given by $f(x) = 2\cos(5x)$.

25. The graph is an inverted sine curve with amplitude 1 and period 2π, shifted up by 2, so it is given by $f(x) = 2 - \sin x$.

26. The graph is a sine curve which has been shifted up by 2, so $f(x) = (\sin x) + 2$.

27. This graph is the same as in Problem 19 but shifted up by 2, so it is given by $f(x) = 2\sin\left(\frac{x}{4}\right) + 2$.

28. This graph has period 8, amplitude 3, and a vertical shift of 3 with no horizontal shift. It is given by

$$f(x) = 3 + 3\sin\left(\frac{2\pi}{8}x\right) = 3 + 3\sin\left(\frac{\pi}{4}x\right).$$

29. We first divide by 5 and then use inverse sine:

$$\begin{aligned}
\frac{2}{5} &= \sin(3x) \\
\sin^{-1}(2/5) &= 3x \\
x &= \frac{\sin^{-1}(2/5)}{3} \approx 0.1372.
\end{aligned}$$

There are infinitely many other possible solutions since the sine is periodic.

30. We first isolate $\cos(2x+1)$ and then use inverse cosine:

$$\begin{aligned} 1 &= 8\cos(2x+1) - 3 \\ 4 &= 8\cos(2x+1) \\ 0.5 &= \cos(2x+1) \\ \cos^{-1}(0.5) &= 2x+1 \\ x &= \frac{\cos^{-1}(0.5) - 1}{2} \approx 0.0236. \end{aligned}$$

There are infinitely many other possible solutions since the cosine is periodic.

31. We first isolate $\tan(5x)$ and then use inverse tangent:

$$\begin{aligned} 8 &= 4\tan(5x) \\ 2 &= \tan(5x) \\ \tan^{-1} 2 &= 5x \\ x &= \frac{\tan^{-1} 2}{5} = 0.221. \end{aligned}$$

There are infinitely many other possible solutions since the tangent is periodic.

32. We first isolate $(2x+1)$ and then use inverse tangent:

$$\begin{aligned} 1 &= 8\tan(2x+1) - 3 \\ 4 &= 8\tan(2x+1) \\ 0.5 &= \tan(2x+1) \\ \arctan(0.5) &= 2x+1 \\ x &= \frac{\arctan(0.5) - 1}{2} = -0.268. \end{aligned}$$

There are infinitely many other possible solutions since the tangent is periodic.

33. We first isolate $\sin(5x)$ and then use inverse sine:

$$\begin{aligned} 8 &= 4\sin(5x) \\ 2 &= \sin(5x). \end{aligned}$$

But this equation has no solution since $-1 \le \sin(5x) \le 1$.

Problems

34. Using the fact that 1 revolution $= 2\pi$ radians and 1 minute = 60 seconds, we have

$$\begin{aligned} 200\frac{\text{rev}}{\text{min}} &= (200)\cdot 2\pi \frac{\text{rad}}{\text{min}} = 200 \cdot 2\pi \frac{1}{60}\frac{\text{rad}}{\text{sec}} \\ &\approx \frac{(200)(6.283)}{60} \\ &\approx 20.94 \text{ radians per second.} \end{aligned}$$

Similarly, 500 rpm is equivalent to 52.36 radians per second.

35. 200 revolutions per minute is $\frac{1}{200}$ minutes per revolution, so the period is $\frac{1}{200}$ minutes, or 0.3 seconds.

36. $\sin x^2$ is by convention $\sin(x^2)$, which means you square the x first and then take the sine.
$\sin^2 x = (\sin x)^2$ means find $\sin x$ and then square it.
$\sin(\sin x)$ means find $\sin x$ and then take the sine of that.
Expressing each as a composition: If $f(x) = \sin x$ and $g(x) = x^2$, then
$\sin x^2 = f(g(x))$
$\sin^2 x = g(f(x))$
$\sin(\sin x) = f(f(x))$.

37. Suppose P is at the point $(3\pi/2, -1)$ and Q is at the point $(5\pi/2, 1)$. Then

$$\text{Slope} = \frac{1-(-1)}{5\pi/2 - 3\pi/2} = \frac{2}{\pi}.$$

If P had been picked to the right of Q, the slope would have been $-2/\pi$.

38. (a) $D =$ the average depth of the water.
(b) $A =$ the amplitude $= 15/2 = 7.5$.
(c) Period $= 12.4$ hours. Thus $(B)(12.4) = 2\pi$ so $B = 2\pi/12.4 \approx 0.507$.
(d) C is the time of a high tide.

39. (a) The amplitude, y_0, is the maximum value of y; that is, the maximum displacement from the equilibrium position.
(b) One complete cycle is executed when

$$2\pi\omega t = 2\pi, \quad \text{so} \quad t = \frac{1}{\omega}.$$

Therefore, the period is $1/\omega$ seconds, and the number of complete oscillations that take place in 1 second is ω.

40. (a) V_0 represents the maximum voltage.
(b) The period is $2\pi/(120\pi) = 1/60$ second.
(c) Since each oscillation takes $1/60$ second, in 1 second there are 60 complete oscillations.

41. The US voltage has a maximum value of 156 volts and has a period of $1/60$ of a second, so it executes 60 cycles a second.

The European voltage has a higher maximum of 339 volts, and a slightly longer period of $1/50$ seconds, so it oscillates at 50 cycles per second.

42. The function R has period of π, so its graph is as shown in Figure 1.33. The maximum value of the range is v_0^2/g and occurs when $\theta = \pi/4$.

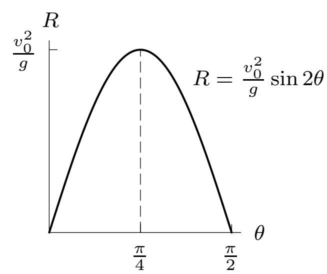


Figure 1.33

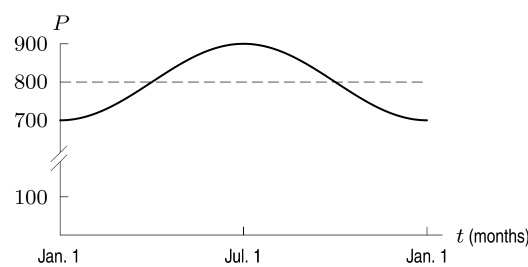


Figure 1.34

43. (a) See Figure 1.34.
(b) Average value of population $= \frac{700+900}{2} = 800$, amplitude $= \frac{900-700}{2} = 100$, and period $= 12$ months, so $B = 2\pi/12 = \pi/6$. Since the population is at its minimum when $t = 0$, we use a negative cosine:

$$P = 800 - 100\cos\left(\frac{\pi t}{6}\right).$$

44. We use a cosine of the form

$$H = A\cos(Bt) + C$$

and choose B so that the period is 24 hours, so $2\pi/B = 24$ giving $B = \pi/12$.

The temperature oscillates around an average value of 60° F, so $C = 60$. The amplitude of the oscillation is 20° F. To arrange that the temperature be at its lowest when $t = 0$, we take A negative, so $A = -20$. Thus

$$A = 60 - 20\cos\left(\frac{\pi}{12}t\right).$$

45. (a) Reading the graph of θ against t shows that $\theta \approx 5.2$ when $t = 1.5$. Since the coordinates of P are $x = 5\cos\theta$, $y = 5\sin\theta$, when $t = 1.5$ the coordinates are

$$(x, y) \approx (5\cos 5.2, 5\sin 5.2) = (2.3, -4.4).$$

(b) As t increases from 0 to 5, the angle θ increases from 0 to about 6.3 and then decreases to 0 again. Since $6.3 \approx 2\pi$, this means that P starts on the x-axis at the point $(5, 0)$, moves counterclockwise the whole way around the circle (at which time $\theta \approx 2\pi$), and then moves back clockwise to its starting point.

46.

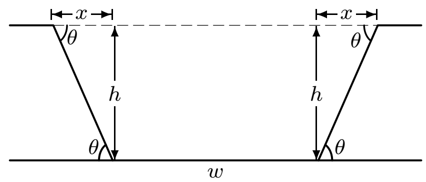


Figure 1.35

Figure 1.35 shows that the cross-sectional area is one rectangle of area hw and two triangles. Each triangle has height h and base x, where

$$\frac{h}{x} = \tan\theta \quad \text{so} \quad x = \frac{h}{\tan\theta}.$$

$$\text{Area of triangle} = \frac{1}{2}xh = \frac{h^2}{2\tan\theta}$$

$$\begin{aligned}\text{Total area} &= \text{Area of rectangle} + 2(\text{Area of triangle})\\ &= hw + 2\cdot\frac{h^2}{2\tan\theta} = hw + \frac{h^2}{\tan\theta}.\end{aligned}$$

47. (a) A table of values for $g(x)$ is given below.

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
$\arccos x$	3.14	2.50	2.21	1.98	1.77	1.57	1.37	1.16	0.93	0.64	0

(b)

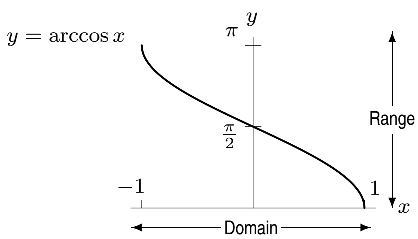


Figure 1.36

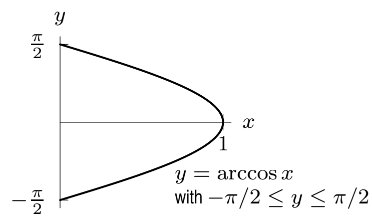


Figure 1.37

(c) The domain of arccos and arcsin are the same, $-1 \le x \le 1$, since their inverses (sine and cosine) only take on values in this range.

(d) The domain of the original sine function was restricted to the the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to construct the arcsine function. Hence, the range of arcsine is also $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Now, if we restrict the domain of cosine in the same way, we obtain an arccosine curve which is not a function. (See Figure 1.37.) For example, for $x = 0$, $y = \arccos x$ will have two values, $-\frac{\pi}{2}$, and $\frac{\pi}{2}$. Also, it gives no values for $x < 0$, so it is not very useful. The domain of cosine should instead be restricted to $[0, \pi]$, so that $y = \arccos x$ gives a unique y for each value of x.

Solutions for Section 1.6

Exercises

1. Exponential growth dominates power growth as $x \to \infty$, so $10 \cdot 2^x$ is larger.

2. As $x \to \infty$, $0.25x^{1/2}$ is larger than $25{,}000x^{-3}$.

3. As $x \to \infty$, $y \to \infty$.
As $x \to -\infty$, $y \to -\infty$.

4. As $x \to \infty$, $y \to \infty$.
As $x \to -\infty$, $y \to 0$.

5. (I) **(a)** Minimum degree is 3 because graph turns around twice.
(b) Leading coefficient is negative because $y \to -\infty$ as $x \to \infty$.
(II) **(a)** Minimum degree is 4 because graph turns around three times.
(b) Leading coefficient is positive because $y \to \infty$ as $x \to \infty$.

(III) **(a)** Minimum degree is 4 because graph turns around three times.
(b) Leading coefficient is negative because $y \to -\infty$ as $x \to \infty$.

(IV) **(a)** Minimum degree is 5 because graph turns around four times.
(b) Leading coefficient is negative because $y \to -\infty$ as $x \to \infty$.

(V) **(a)** Minimum degree is 5 because graph turns around four times.
(b) Leading coefficient is positive because $y \to \infty$ as $x \to \infty$.

6. (a) A polynomial has the same end behavior as its leading term, so this polynomial behaves as $-5x^4$ globally. Thus we have:
$f(x) \to -\infty$ as $x \to -\infty$, and $f(x) \to -\infty$ as $x \to +\infty$.
(b) Polynomials behave globally as their leading term, so this rational function behaves globally as $(3x^2)/(2x^2)$, or $3/2$. Thus we have:
$f(x) \to 3/2$ as $x \to -\infty$, and $f(x) \to 3/2$ as $x \to +\infty$.
(c) We see from a graph of $y = e^x$ that
$f(x) \to 0$ as $x \to -\infty$, and $f(x) \to +\infty$ as $x \to +\infty$.

7. $f(x) = k(x+3)(x-1)(x-4) = k(x^3 - 2x^2 - 11x + 12)$, where $k < 0$. ($k \approx -\frac{1}{6}$ if the horizontal and vertical scales are equal; otherwise one can't tell how large k is.)

8. $f(x) = kx(x+3)(x-4) = k(x^3 - x^2 - 12x)$, where $k < 0$. ($k \approx -\frac{2}{9}$ if the horizontal and vertical scales are equal; otherwise one can't tell how large k is.)

9. $f(x) = k(x+2)(x-1)(x-3)(x-5) = k(x^4 - 7x^3 + 5x^2 + 31x - 30)$, where $k > 0$. ($k \approx \frac{1}{15}$ if the horizontal and vertical scales are equal; otherwise one can't tell how large k is.)

10. $f(x) = k(x+2)(x-2)^2(x-5) = k(x^4 - 7x^3 + 6x^2 + 28x - 40)$, where $k < 0$. ($k \approx -\frac{1}{15}$ if the scales are equal; otherwise one can't tell how large k is.)

11. Figure 1.38 shows the appropriate graphs. Note that asymptotes are shown as dashed lines and x- or y-intercepts are shown as filled circles.

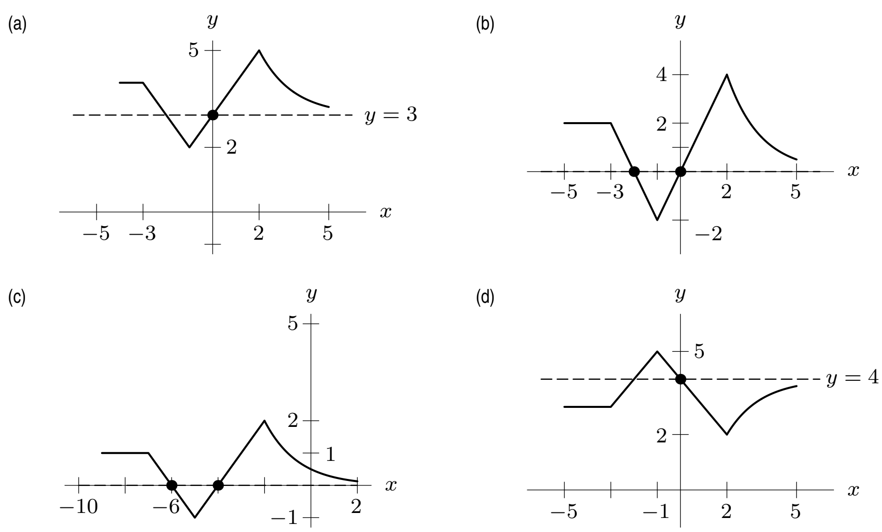


Figure 1.38

Problems

12. (a) From the x-intercepts, we know the equation has the form

$$y = k(x+2)(x-1)(x-5).$$

Since $y = 2$ when $x = 0$,

$$2 = k(2)(-1)(-5) = k \cdot 10$$
$$k = \frac{1}{5}.$$

Thus we have

$$y = \frac{1}{5}(x+2)(x-1)(x-5).$$

13. (a) Because our cubic has a root at 2 and a double root at -2, it has the form

$$y = k(x+2)(x+2)(x-2).$$

Since $y = 4$ when $x = 0$,

$$4 = k(2)(2)(-2) = -8k,$$
$$k = -\frac{1}{2}.$$

Thus our equation is

$$y = -\frac{1}{2}(x+2)^2(x-2).$$

14. (a) II and III because in both cases, the numerator and denominator each have x^2 as the highest power, with coefficient $= 1$. Therefore,

$$y \to \frac{x^2}{x^2} = 1 \quad \text{as } x \to \pm\infty.$$

(b) I, since

$$y \to \frac{x}{x^2} = 0 \quad \text{as } x \to \pm\infty.$$

(c) II and III, since replacing x by $-x$ leaves the graph of the function unchanged.
(d) None
(e) III, since the denominator is zero and $f(x)$ tends to $\pm\infty$ when $x = \pm 1$.

15. Substituting $w = 65$ and $h = 160$, we have

(a)

$$s = 0.01(65^{0.25})(160^{0.75}) = 1.3 \text{ m}^2.$$

(b) We substitute $s = 1.5$ and $h = 180$ and solve for w:

$$1.5 = 0.01w^{0.25}(180^{0.75}).$$

We have

$$w^{0.25} = \frac{1.5}{0.01(180^{0.75})} = 3.05.$$

Since $w^{0.25} = w^{1/4}$, we take the fourth power of both sides, giving

$$w = 86.8 \text{ kg}.$$

(c) We substitute $w = 70$ and solve for h in terms of s:

$$s = 0.01(70^{0.25})h^{0.75},$$

so

$$h^{0.75} = \frac{s}{0.01(70^{0.25})}.$$

Since $h^{0.75} = h^{3/4}$, we take the $4/3$ power of each side, giving

$$h = \left(\frac{s}{0.01(70^{0.25})}\right)^{4/3} = \frac{s^{4/3}}{(0.01^{4/3})(70^{1/3})}$$

so

$$h = 112.6s^{4/3}.$$

16. Let us represent the height by h. Since the volume is V, we have

$$x^2h = V.$$

Solving for h gives

$$h = \frac{V}{x^2}.$$

The graph is in Figure 1.39. We are assuming V is a positive constant.

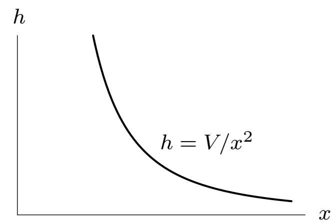


Figure 1.39

17. Let $D(v)$ be the stopping distance required by an Alpha Romeo as a function of its velocity. The assumption that stopping distance is proportional to the square of velocity is equivalent to the equation

$$D(v) = kv^2$$

where k is a constant of proportionality. To determine the value of k, we use the fact that $D(70) = 177$.

$$D(70) = k(70)^2 = 177.$$

Thus,

$$k = \frac{177}{70^2} \approx 0.0361.$$

It follows that

$$D(35) = \left(\frac{177}{70^2}\right)(35)^2 = \frac{177}{4} = 44.25 \text{ ft}$$

and

$$D(140) = \left(\frac{177}{70^2}\right)(140)^2 = 708 \text{ ft}.$$

Thus, at half the speed it requires one fourth the distance, whereas at twice the speed it requires four times the distance, as we would expect from the equation. (We could in fact have figured it out that way, without solving for k explicitly.)

18. (a) (i) The water that has flowed out of the pipe in 1 second is a cylinder of radius r and length 3 cm. Its volume is

$$V = \pi r^2(3) = 3\pi r^2.$$

(ii) If the rate of flow is k cm/sec instead of 3 cm/sec, the volume is given by

$$V = \pi r^2(k) = \pi r^2 k.$$

(b) (i) The graph of V as a function of r is a quadratic. See Figure 1.40.

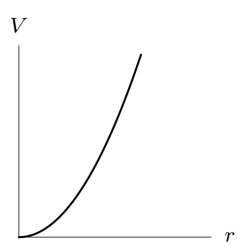


Figure 1.40

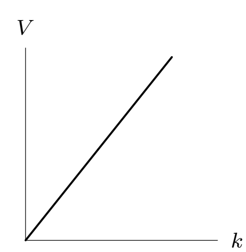


Figure 1.41

(ii) The graph of V as a function of k is a line. See Figure 1.41.

19. (a) Since the rate R varies directly with the fourth power of the radius r, we have the formula

$$R = kr^4$$

where k is a constant.

(b) Given $R = 400$ for $r = 3$, we can determine the constant k.

$$\begin{aligned} 400 &= k(3)^4 \\ 400 &= k(81) \\ k &= \frac{400}{81} \approx 4.938. \end{aligned}$$

So the formula is

$$R = 4.938r^4$$

(c) Evaluating the formula above at $r = 5$ yields

$$R = 4.928(5)^4 = 3086.42\frac{\text{cm}^3}{\text{sec}}.$$

20. (a) The object starts at $t = 0$, when $s = v_0(0) - g(0)^2/2 = 0$. Thus it starts on the ground, with zero height.

(b) The object hits the ground when $s = 0$. This is satisfied at $t = 0$, before it has left the ground, and at some later time t that we must solve for.

$$0 = v_0 t - gt^2/2 = t\,(v_0 - gt/2)$$

Thus $s = 0$ when $t = 0$ and when $v_0 - gt/2 = 0$, i.e., when $t = 2v_0/g$. The starting time is $t = 0$, so it must hit the ground at time $t = 2v_0/g$.

(c) The object reaches its maximum height halfway between when it is released and when it hits the ground, or at

$$t = (2v_0/g)/2 = v_0/g.$$

(d) Since we know the time at which the object reaches its maximum height, to find the height it actually reaches we just use the given formula, which tells us s at any given t. Substituting $t = v_0/g$,

$$\begin{aligned} s &= v_0\left(\frac{v_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0^2}{g^2}\right) = \frac{v_0^2}{g} - \frac{v_0^2}{2g} \\ &= \frac{2v_0^2 - v_0^2}{2g} = \frac{v_0^2}{2g}. \end{aligned}$$

21. The pomegranate is at ground level when $f(t) = -16t^2 + 64t = -16t(t-4) = 0$, so when $t = 0$ or $t = 4$. At time $t = 0$ it is thrown, so it must hit the ground at $t = 4$ seconds. The symmetry of its path with respect to time may convince you that it reaches its maximum height after 2 seconds. Alternatively, we can think of the graph of $f(t) = -16t^2 + 64t = -16(t-2)^2 + 64$, which is a downward parabola with vertex (i.e., highest point) at $(2, 64)$. The maximum height is $f(2) = 64$ feet.

22. (a) (i) If $(1, 1)$ is on the graph, we know that

$$1 = a(1)^2 + b(1) + c = a + b + c.$$

(ii) If $(1, 1)$ is the vertex, then the axis of symmetry is $x = 1$, so

$$-\frac{b}{2a} = 1,$$

and thus

$$a = -\frac{b}{2}, \text{ so } b = -2a.$$

But to be the vertex, $(1, 1)$ must also be on the graph, so we know that $a + b + c = 1$. Substituting $b = -2a$, we get $-a + c = 1$, which we can rewrite as $a = c - 1$, or $c = 1 + a$.

(iii) For $(0, 6)$ to be on the graph, we must have $f(0) = 6$. But $f(0) = a(0^2) + b(0) + c = c$, so $c = 6$.

(b) To satisfy all the conditions, we must first, from (a)(iii), have $c = 6$. From (a)(ii), $a = c - 1$ so $a = 5$. Also from (a)(ii), $b = -2a$, so $b = -10$. Thus the completed equation is

$$y = f(x) = 5x^2 - 10x + 6,$$

which satisfies all the given conditions.

23. $h(t)$ cannot be of the form ct^2 or kt^3 since $h(0.0) = 2.04$. Therefore $h(t)$ must be the exponential, and we see that the ratio of successive values of h is approximately 1.5. Therefore $h(t) = 2.04(1.5)^t$. If $g(t) = ct^2$, then $c = 3$ since $g(1.0) = 3.00$. However, $g(2.0) = 24.00 \neq 3 \cdot 2^2$. Therefore $g(t) = kt^3$, and using $g(1.0) = 3.00$, we obtain $g(t) = 3t^3$. Thus $f(t) = ct^2$, and since $f(2.0) = 4.40$, we have $f(t) = 1.1t^2$.

24. Looking at g, we see that the ratio of the values is:

$$\frac{3.12}{3.74} \approx \frac{3.74}{4.49} \approx \frac{4.49}{5.39} \approx \frac{5.39}{6.47} \approx \frac{6.47}{7.76} \approx 0.83.$$

Thus g is an exponential function, and so f and k are the power functions. Each is of the form ax^2 or ax^3, and since $k(1.0) = 9.01$ we see that for k, the constant coefficient is 9.01. Trial and error gives

$$k(x) = 9.01x^2,$$

since $k(2.2) = 43.61 \approx 9.01(4.84) = 9.01(2.2)^2$. Thus $f(x) = ax^3$ and we find a by noting that $f(9) = 7.29 = a(9^3)$ so

$$a = \frac{7.29}{9^3} = 0.01$$

and $f(x) = 0.01x^3$.

25. The function is a cubic polynomial with positive leading coefficient. Since the figure given in the text shows that the function turns around once, we know that the function has the shape shown in Figure 1.42. The function is below the x-axis for $x = 5$ in the given graph, and we know that it goes to $+\infty$ as $x \to +\infty$ because the leading coefficient is positive. Therefore, there are exactly three zeros. Two zeros are shown, and occur at approximately $x = -1$ and $x = 3$. The third zero must be to the right of $x = 10$ and so occurs for some $x > 10$.

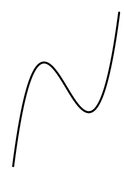


Figure 1.42

26. Consider the end behavior of the graph; that is, as $x \to +\infty$ and $x \to -\infty$. The ends of a degree 5 polynomial are in Quadrants I and III if the leading coefficient is positive or in Quadrants II and IV if the leading coefficient is negative. Thus, there must be at least one root. Since the degree is 5, there can be no more than 5 roots. Thus, there may be 1, 2, 3, 4, or 5 roots. Graphs showing these five possibilities are shown in Figure 1.43.

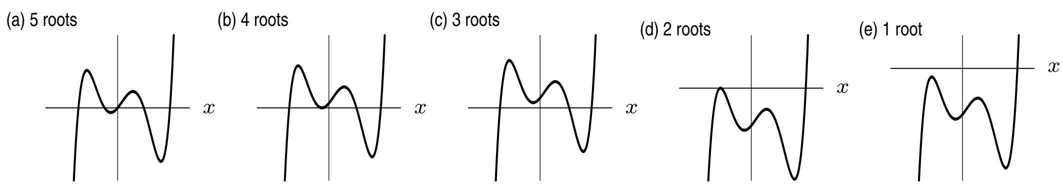


Figure 1.43

27. (a) Since $a_1 = -2a_2 = 2c$, we have $a_1 = 2c$ and $a_2 = -c$. So, $f(x) = (x+2c)^2$ and $g(x) = (x-c)^2$. Then we have

$$\begin{aligned} f(x) &= g(x) \\ (x+2c)^2 &= (x-c)^2 \\ x^2 + 4cx + 4c^2 &= x^2 - 2cx + c^2 \\ 4cx + 4c^2 &= -2cx + c^2 \\ 6cx &= -3c^2 \\ x &= \frac{-3c^2}{6c} \\ x &= \frac{-c}{2}. \end{aligned}$$

(b) Since $c > 0$, the graph of $f(x)$ is the parabola $y = x^2$ shifted left $2c$; the graph of $g(x)$ is the parabola $y = x^2$ shifted right c. We have

$$y\text{-intercept of } g(x) = g(0) = c^2,$$
$$y\text{-intercept of } f(x) = f(0) = 4c^2.$$

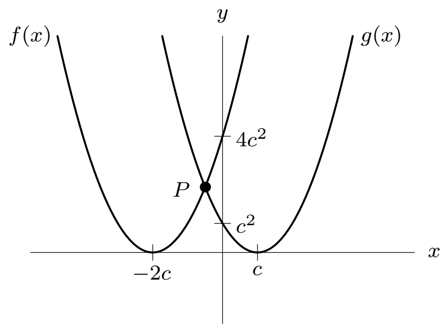


Figure 1.44

(c) The coordinates of the point of intersection P are given by $x = -c/2$ and $y = f(-c/2) = 9c^2/4$. As c increases, the x-coordinate of P, $x = -c/2$, moves left and the y-coordinate, $y = 9c^2/4$, moves up.

28. $g(x) = 2x^2$, $h(x) = x^2 + k$ for any $k > 0$. Notice that the graph is symmetric about the y-axis and $\lim_{x\to\infty} f(x) = 2$.

29. (a) III
(b) IV
(c) I
(d) II

30. The graphs of both these functions will resemble that of x^3 on a large enough window. One way to tackle the problem is to graph them both (along with x^3 if you like) in successively larger windows until the graphs come together. In Figure 1.45, f, g and x^3 are graphed in four windows. In the largest of the four windows the graphs are indistinguishable, as required. Answers may vary.

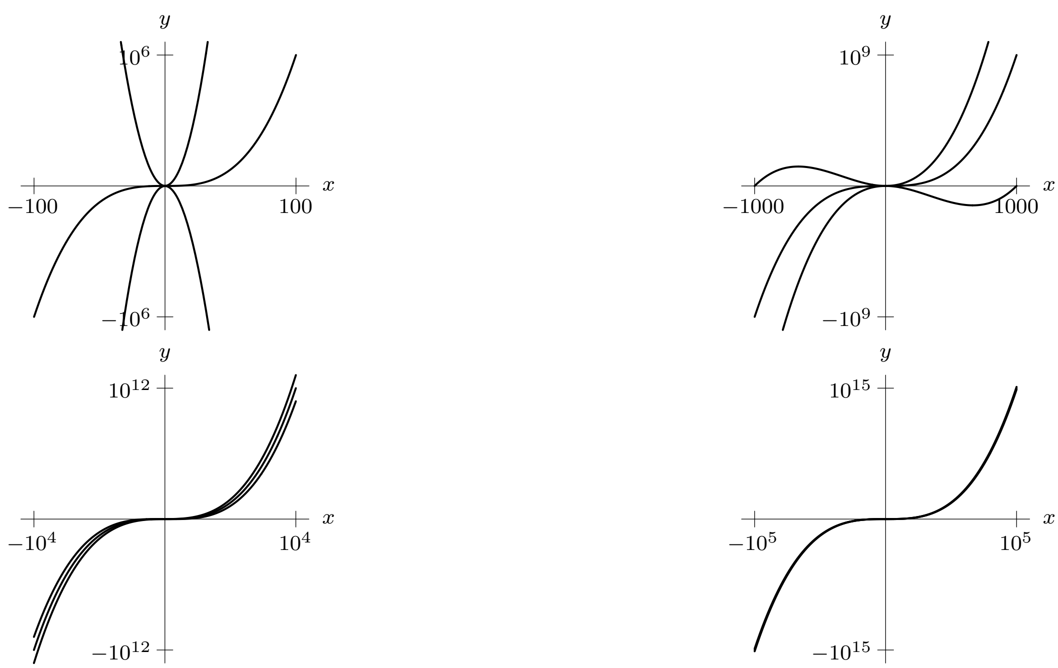


Figure 1.45

31. The graphs are shown in Figure 1.46.

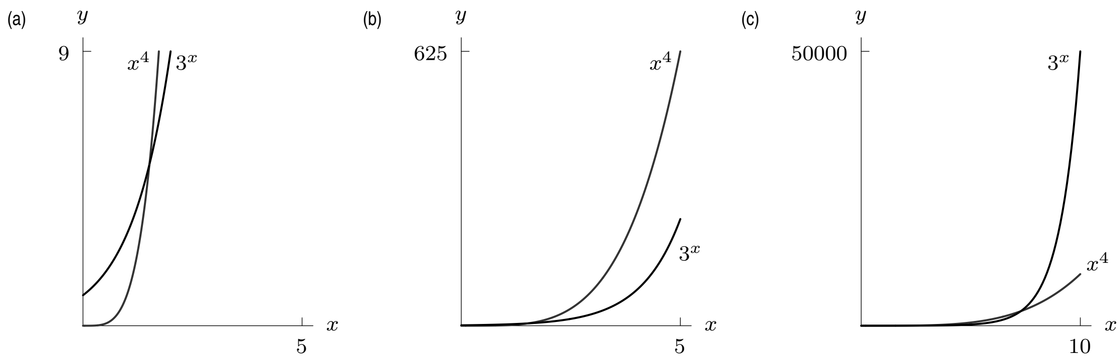


Figure 1.46

32. (a) $a(v) = \frac{1}{m}(\text{ENGINE} - \text{WIND}) = \frac{1}{m}(F_E - kv^2)$, where k is a positive constant.
(b) A possible graph is in Figure 1.47.

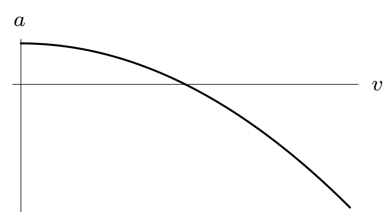


Figure 1.47

Solutions for Section 1.7

Exercises

1. Yes, because $2x + x^{2/3}$ is defined for all x.

2. No, because $2x + x^{-1}$ is undefined at $x = 0$.

3. Yes, because $x - 2$ is not zero on this interval.

4. No, because $x - 2 = 0$ at $x = 2$.

5. Yes, because $2x - 5$ is positive for $3 \leq x \leq 4$.

6. Yes, because the denominator is never zero.

7. No, because $\cos(\pi/2) = 0$.

8. No, because $\sin 0 = 0$.

9. No, because $e^x - 1 = 0$ at $x = 0$.

10. Yes, because $\cos\theta$ is not zero on this interval.

11. We have that $f(0) = -1 < 0$ and $f(1) = 1 > 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0, 1]$ such that $f(c) = k = 0$.

12. We have that $f(0) = 1 > 0$ and $f(1) = e - 3 < 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0, 1]$ such that $f(c) = k = 0$.

13. We have that $f(0) = -1 < 0$ and $f(1) = 1 - \cos 1 > 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0, 1]$ such that $f(c) = k = 0$.

14. Since f is not continuous at $x = 0$, we consider instead the smaller interval $[0.01, 1]$. We have that $f(0.01) = 2^{0.01} - 100 < 0$ and $f(1) = 2 - 1/1 = 1 > 0$ and that f is continuous. Thus, by the Intermediate Value Theorem applied to $k = 0$, there is a number c in $[0.01, 1]$, and hence in $[0, 1]$, such that $f(c) = k = 0$.

Problems

15. The voltage $f(t)$ is graphed in Figure 1.48.

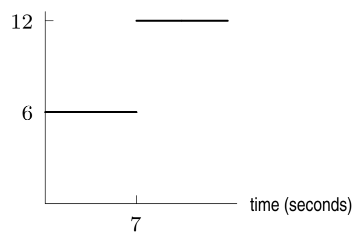


Figure 1.48: Voltage change from 6V to 12V

Using formulas, the voltage, $f(t)$, is represented by

$$f(t) = \begin{cases} 6, & 0 < t \le 7 \\ 12, & 7 < t \end{cases}$$

Although a real physical voltage is continuous, the voltage in this circuit is well-approximated by the function $f(t)$, which is not continuous on any interval around 7 seconds.

16. Two possible graphs are shown in Figures 1.49 and 1.50.

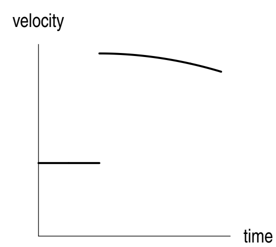


Figure 1.49: Velocity of the car

Figure 1.50: Distance

The distance moved by the car is continuous. (Figure 1.50 has no breaks in it.) In actual fact, the velocity of the car is also continuous; however, in this case, it is well-approximated by the function in Figure 1.49, which is not continuous on any interval containing the moment of impact.

17. For any value of k, the function is continuous at every point except $x = 2$. We choose k to make the function continuous at $x = 2$.

Since $3x^2$ takes the value $3(2^2) = 12$ at $x = 2$, we choose k so that kx goes through the point $(2, 12)$. Thus $k = 6$.

18. For $x > 0$, we have $|x| = x$, so $f(x) = 1$. For $x < 0$, we have $|x| = -x$, so $f(x) = -1$. Thus, the function is given by

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 1 \end{cases},$$

so f is not continuous on any interval containing $x = 0$.

19. For any values of k, the function is continuous on any interval that does not contain $x = 2$.

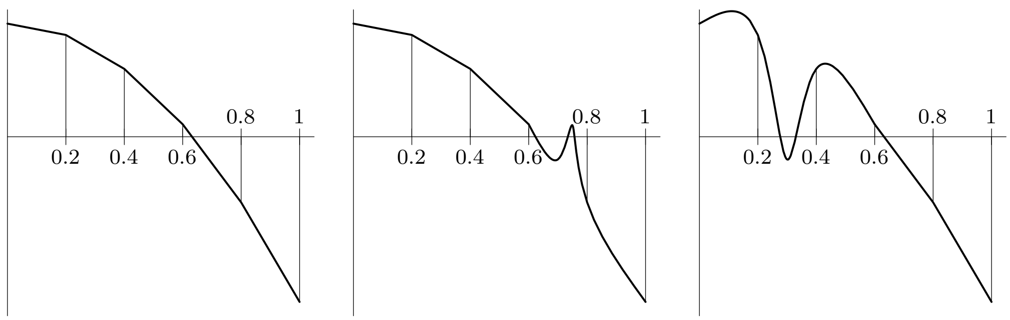
Since $5x^3 - 10x^2 = 5x^2(x - 2)$, we can cancel $(x - 2)$ provided $x \neq 2$, giving

$$f(x) = \frac{5x^3 - 10x^2}{x - 2} = 5x^2 \qquad x \neq 2.$$

Thus, if we pick $k = 5(2)^2 = 20$, the function is continuous.

20. The graph of g suggests that g is not continuous on any interval containing $\theta = 0$, since $g(0) = 1/2$.

21.



22. (a) Figure 1.51 shows a possible graph of $f(x)$, yours may be different.

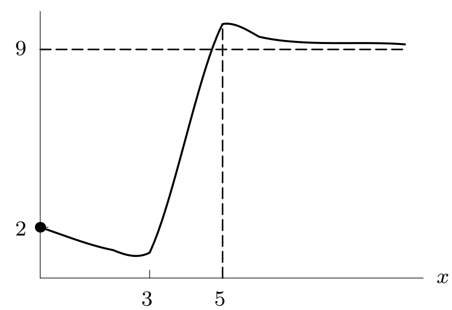


Figure 1.51

(b) In order for f to approach the horizontal asymptote at 9 from above it is necessary that f eventually become concave up. It is therefore not possible for f to be concave down for all $x > 6$.

23. The drug first increases linearly for half a second, at the end of which time there is 0.6 ml in the body. Thus, for $0 \leq t \leq 0.5$, the function is linear with slope $0.6/0.5 = 1.2$:

$$Q = 1.2t \quad \text{for} \quad 0 \leq t \leq 0.5.$$

At $t = 0.5$, we have $Q = 0.6$. For $t > 0.5$, the quantity decays exponentially at a continuous rate of 0.002, so Q has the form

$$Q = Ae^{-0.002t} \qquad 0.5 < t.$$

We choose A so that $Q = 0.6$ when $t = 0.5$:

$$\begin{aligned} 0.6 &= Ae^{-0.002(0.5)} = Ae^{-0.001} \\ A &= 0.6e^{0.001}. \end{aligned}$$

Thus

$$Q = \begin{cases} 1.2t & 0 \leq t \leq 0.5 \\ 0.6e^{0.001}e^{-.002t} & 0.5 < t. \end{cases}$$

24. The functions $y(x) = \sin x$ and $z_k(x) = ke^{-x}$ for $k = 1, 2, 4, 6, 8, 10$ are shown in Figure 1.52. The values of $f(k)$ for $k = 1, 2, 4, 6, 8, 10$ are given in Table 1.2. These values can be obtained using either tracing or a numerical root finder on a calculator or computer.

From Figure 1.52 it is clear that the smallest solution of $\sin x = ke^{-x}$ for $k = 1, 2, 4, 6$ occurs on the first period of the sine curve. For small changes in k, there are correspondingly small changes in the intersection point. For $k = 8$ and $k = 10$, the solution jumps to the second period because $\sin x < 0$ between π and 2π, but ke^{-x} is uniformly positive. Somewhere in the interval $6 \leq k \leq 8$, $f(k)$ has a discontinuity.

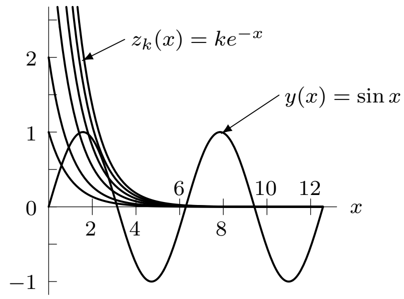


Figure 1.52

Table 1.2

k	$f(k)$
1	0.588
2	0.921
4	1.401
6	1.824
8	6.298
10	6.302

Solutions for Section 1.8

Exercises

1. (a) As x approaches -2 from either side, the values of $f(x)$ get closer and closer to 3, so the limit appears to be about 3.
(b) As x approaches 0 from either side, the values of $f(x)$ get closer and closer to 7. (Recall that to find a limit, we are interested in what happens to the function near x but not at x.) The limit appears to be about 7.
(c) As x approaches 2 from either side, the values of $f(x)$ get closer and closer to 3 on one side of $x = 2$ and get closer and closer to 2 on the other side of $x = 2$. Thus the limit does not exist.
(d) As x approaches 4 from either side, the values of $f(x)$ get closer and closer to 8. (Again, recall that we don't care what happens right at $x = 4$.) The limit appears to be about 8.

2. From the graphs of f and g, we estimate $\lim_{x\to 1^-} f(x) = 3$, $\lim_{x\to 1^-} g(x) = 5$, $\lim_{x\to 1^+} f(x) = 4$, $\lim_{x\to 1^+} g(x) = 1$.
(a) $\lim_{x\to 1^-} (f(x) + g(x)) = 3 + 5 = 8$
(b) $\lim_{x\to 1^+} (f(x) + 2g(x)) = \lim_{x\to 1^+} f(x) + 2 \lim_{x\to 1^+} g(x) = 4 + 2(1) = 6$
(c) $\lim_{x\to 1^-} (f(x)g(x)) = (\lim_{x\to 1^-} f(x))(\lim_{x\to 1^-} g(x)) = (3)(5) = 15$
(d) $\lim_{x\to 1^+} (f(x)/g(x)) = \left(\lim_{x\to 1^+} f(x)\right) / \left(\lim_{x\to 1^+} g(x)\right) = 4/1 = 4$

3. For $-0.5 \le \theta \le 0.5$, $0 \le y \le 3$, the graph of $y = \frac{\sin(2\,\theta)}{\theta}$ is shown in Figure 1.53. Therefore, $\lim_{\theta\to 0} \frac{\sin(2\,\theta)}{\theta} = 2$.

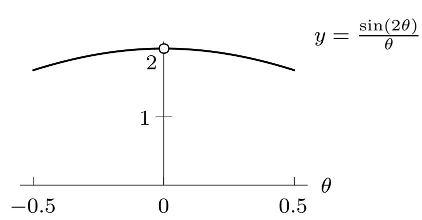


Figure 1.53

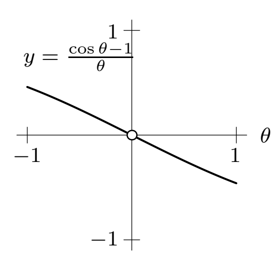


Figure 1.54

4. For $-1 \le \theta \le 1$, $-1 \le y \le 1$, the graph of $y = \frac{\cos\theta - 1}{\theta}$ is shown in Figure 1.54. Therefore, $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$.

5. For $-90^\circ \le \theta \le 90^\circ$, $0 \le y \le 0.02$, the graph of $y = \frac{\sin\theta}{\theta}$ is shown in Figure 1.55. Therefore, by tracing along the curve, we see that in degrees, $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 0.01745\ldots$.

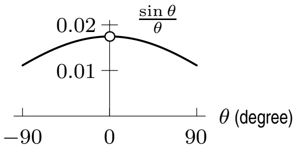


Figure 1.55

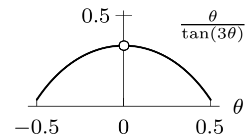


Figure 1.56

6. For $-0.5 \leq \theta \leq 0.5$, $0 \leq y \leq 0.5$, the graph of $y = \dfrac{\theta}{\tan(3\theta)}$ is shown in Figure 1.56. Therefore, by tracing along the curve, we see that $\lim_{\theta\to 0} \dfrac{\theta}{\tan(3\theta)} = 0.3333\ldots$.

7. From Table 1.3, it appears the limit is 1. This is confirmed by Figure 1.57. An appropriate window is $-0.0033 < x < 0.0033$, $0.99 < y < 1.01$.

Table 1.3

x	$f(x)$
0.1	1.3
0.01	1.03
0.001	1.003
0.0001	1.0003

x	$f(x)$
−0.0001	0.9997
−0.001	0.997
−0.01	0.97
−0.1	0.7

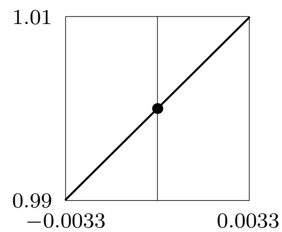


Figure 1.57

8. From Table 1.4, it appears the limit is −1. This is confirmed by Figure 1.58. An appropriate window is $-0.099 < x < 0.099$, $-1.01 < y < -0.99$.

Table 1.4

x	$f(x)$
0.1	−0.99
0.01	−0.9999
0.001	−0.999999
0.0001	−0.99999999

x	$f(x)$
−0.0001	−0.99999999
−0.001	−0.999999
−0.01	−0.9999
−0.1	−0.99

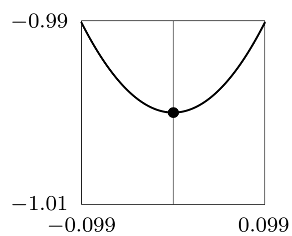


Figure 1.58

9. From Table 1.5, it appears the limit is 0. This is confirmed by Figure 1.59. An appropriate window is $-0.005 < x < 0.005$, $-0.01 < y < 0.01$.

Table 1.5

x	$f(x)$
0.1	0.1987
0.01	0.0200
0.001	0.0020
0.0001	0.0002

x	$f(x)$
−0.0001	−0.0002
−0.001	−0.0020
−0.01	−0.0200
−0.1	−0.1987

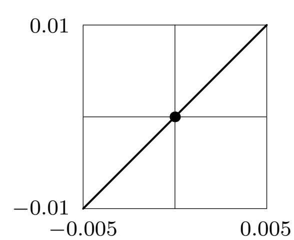


Figure 1.59

10. From Table 1.6, it appears the limit is 0. This is confirmed by Figure 1.60. An appropriate window is $-0.0033 < x < 0.0033$, $-0.01 < y < 0.01$.

Table 1.6

x	$f(x)$
0.1	0.2955
0.01	0.0300
0.001	0.0030
0.0001	0.0003

x	$f(x)$
−0.0001	−0.0003
−0.001	−0.0030
−0.01	−0.0300
−0.1	−0.2955

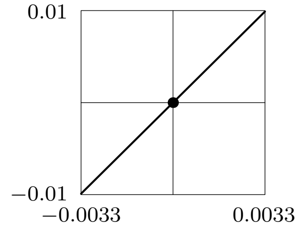


Figure 1.60

11. From Table 1.7, it appears the limit is 2. This is confirmed by Figure 1.61. An appropriate window is $-0.0865 < x < 0.0865$, $1.99 < y < 2.01$.

Table 1.7

x	$f(x)$
0.1	1.9867
0.01	1.9999
0.001	2.0000
0.0001	2.0000

x	$f(x)$
−0.0001	2.0000
−0.001	2.0000
−0.01	1.9999
−0.1	1.9867

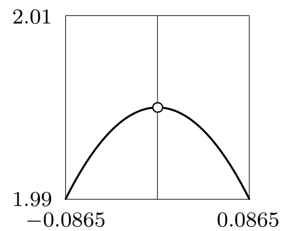


Figure 1.61

12. From Table 1.8, it appears the limit is 3. This is confirmed by Figure 1.62. An appropriate window is $-0.047 < x < 0.047$, $2.99 < y < 3.01$.

Table 1.8

x	$f(x)$
0.1	2.9552
0.01	2.9996
0.001	3.0000
0.0001	3.0000

x	$f(x)$
−0.0001	3.0000
−0.001	3.0000
−0.01	2.9996
−0.1	2.9552

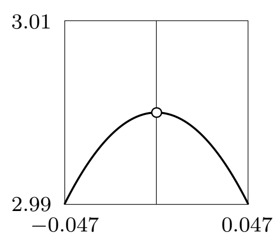


Figure 1.62

13. From Table 1.9, it appears the limit is 1. This is confirmed by Figure 1.63. An appropriate window is $-0.0198 < x < 0.0198$, $0.99 < y < 1.01$.

Table 1.9

x	$f(x)$
0.1	1.0517
0.01	1.0050
0.001	1.0005
0.0001	1.0001

x	$f(x)$
−0.0001	1.0000
−0.001	0.9995
−0.01	0.9950
−0.1	0.9516

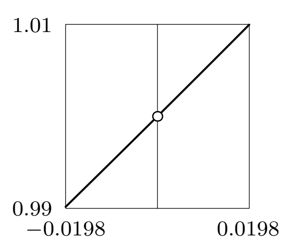


Figure 1.63

14. From Table 1.10, it appears the limit is 2. This is confirmed by Figure 1.64. An appropriate window is $-0.0049 < x < 0.0049$, $1.99 < y < 2.01$.

Table 1.10

x	$f(x)$
0.1	2.2140
0.01	2.0201
0.001	2.0020
0.0001	2.0002

x	$f(x)$
-0.0001	1.9998
-0.001	1.9980
-0.01	1.9801
-0.1	1.8127

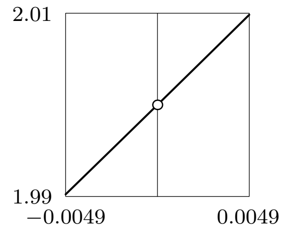


Figure 1.64

15. From Table 1.11, it appears the limit is 4. Figure 1.65 confirms this. An appropriate window is $1.99 < x < 2.01$, $3.99 < y < 4.01$.

Table 1.11

x	$f(x)$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001
1.9999	3.9999
1.999	3.999
1.99	3.99
1.9	3.9

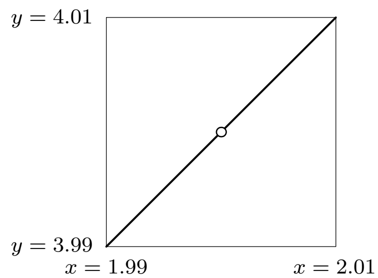


Figure 1.65

16. From Table 1.12, it appears the limit is 6. Figure 1.66 confirms this. An appropriate window is $2.99 < x < 3.01$, $5.99 < y < 6.01$.

Table 1.12

x	$f(x)$
3.1	6.1
3.01	6.01
3.001	6.001
3.0001	6.0001
2.9999	5.9999
2.999	5.999
2.99	5.99
2.9	5.9

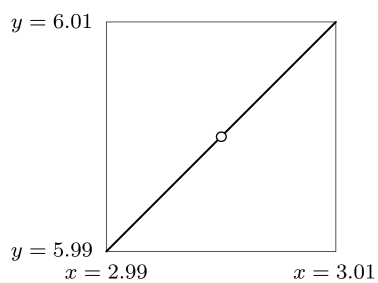


Figure 1.66

17. From Table 1.13, it appears the limit is 0. Figure 1.67 confirms this. An appropriate window is $1.55 < x < 1.59$, $-0.01 < y < 0.01$.

Table 1.13

x	$f(x)$
1.6708	-0.0500
1.5808	-0.0050
1.5718	-0.0005
1.5709	-0.0001
1.5707	0.0001
1.5698	0.0005
1.5608	0.0050
1.4708	0.0500

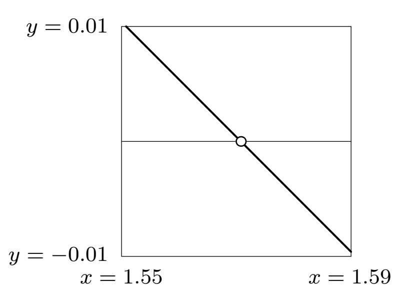


Figure 1.67

18. From Table 1.14, it appears the limit is 2. Figure 1.68 confirms this. An appropriate window is $0.995 < x < 1.004$, $1.99 < y < 2.01$.

Table 1.14

x	$f(x)$
1.1	2.2140
1.01	2.0201
1.001	2.0020
1.0001	2.0002
0.9999	1.9998
0.999	1.9980
0.99	1.9801
0.9	1.8127

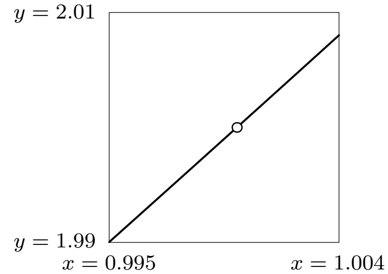


Figure 1.68

19. $f(x) = \frac{|x-4|}{x-4} = \begin{cases} \frac{x-4}{x-4} & x > 4 \\ -\frac{x-4}{x-4} & x < 4 \end{cases} = \begin{cases} 1 & x > 4 \\ -1 & x < 4 \end{cases}$

Figure 1.69 confirms that $\lim_{x \to 4^+} f(x) = 1$, $\lim_{x \to 4^-} f(x) = -1$ so $\lim_{x \to 4} f(x)$ does not exist.

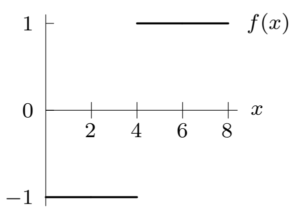


Figure 1.69

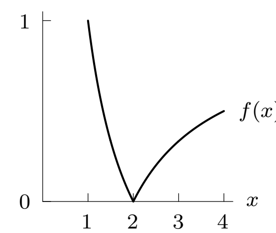


Figure 1.70

20. $f(x) = \frac{|x-2|}{x} = \begin{cases} \frac{x-2}{x}, & x > 2 \\ -\frac{x-2}{x}, & x < 2 \end{cases}$

Figure 1.70 confirms that $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2} f(x) = 0$.

21. $f(x) = \begin{cases} x^2 - 2 & 0 < x < 3 \\ 2 & x = 3 \\ 2x + 1 & 3 < x \end{cases}$

Figure 1.71 confirms that $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 2) = 7$ and that $\lim_{x \to 3+} f(x) = \lim_{x \to 3+} (2x + 1) = 7$, so $\lim_{x \to 3} f(x) = 7$. Note, however, that $f(x)$ is not continuous at $x = 3$ since $f(3) = 2$.

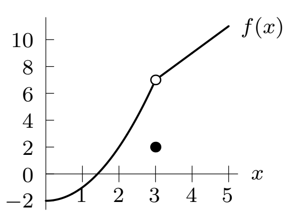


Figure 1.71

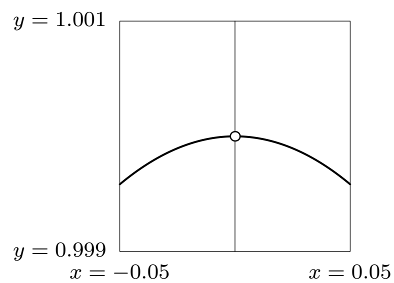


Figure 1.72: Graph of $(\sin\theta)/\theta$ with $-0.05 < \theta < 0.05$

22. The graph in Figure 1.72 suggests that

$$\text{if } -0.05 < \theta < 0.05, \quad \text{then} \quad 0.999 < (\sin\theta)/\theta < 1.001.$$

Thus, if θ is within 0.05 of 0, we see that $(\sin\theta)/\theta$ is within 0.001 of 1.

23. The statement

$$\lim_{h\to a} g(h) = K$$

means that we can make the value of $g(h)$ as close to K as we want by choosing h sufficiently close to, but not equal to, a.

In symbols, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|g(h) - K| < \epsilon \quad \text{for all } 0 < |h-a| < \delta.$$

Problems

24. The only change is that, instead of considering all x near c, we only consider x near to and greater than c. Thus the phrase "$|x-c| < \delta$" must be replaced by "$c < x < c + \delta$." Thus, we define

$$\lim_{x\to c^+} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $c < x < c+\delta$, then $|f(x) - L| < \epsilon$.

25. The only change is that, instead of considering all x near c, we only consider x near to and less than c. Thus the phrase "$|x-c| < \delta$" must be replaced by "$c - \delta < x < c$." Thus, we define

$$\lim_{x\to c^-} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $c - \delta < x < c$, then $|f(x) - L| < \epsilon$.

26. Instead of being "sufficiently close to c," we want x to be "sufficiently large." Using N to measure how large x must be, we define

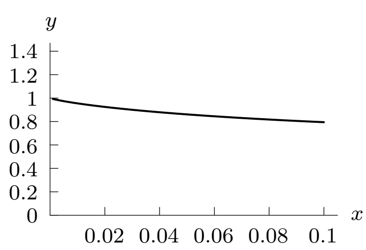
$$\lim_{x\to\infty} f(x) = L$$

to mean that for any $\epsilon > 0$ (as small as we want), there is a $N > 0$ (sufficiently large) such that if $x > N$, then $|f(x) - L| < \epsilon$.

27. If $x > 1$ and x approaches 1, then $p(x) = 55$. If $x < 1$ and x approaches 1, then $p(x) = 34$. There is not a single number that $p(x)$ approaches as x approaches 1, so we say that $\lim\limits_{x\to 1} p(x)$ does not exist.

28. We use values of h approaching, but not equal to, zero. If we let $h = 0.01, 0.001, 0.0001, 0.00001$, we calculate the values 2.7048, 2.7169, 2.7181, and 2.7183. If we let $h = -0.01\ -0.001, -0.0001, -0.00001$, we get values 2.7320, 2.7196, 2.7184, and 2.7183. These numbers suggest that the limit is the number $e = 2.71828\ldots$. However, these calculations cannot tell us that the limit is exactly e; for that a proof is needed.

29. The limit appears to be 1; a graph and table of values is shown below.



x	x^x
0.1	0.7943
0.01	0.9550
0.001	0.9931
0.0001	0.9990
0.00001	0.9999

30. Divide numerator and denominator by x:

$$f(x) = \frac{x+3}{2-x} = \frac{1+3/x}{2/x - 1},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+3/x}{2/x-1} = \frac{\lim_{x\to\infty}(1+3/x)}{\lim_{x\to\infty}(2/x-1)} = \frac{1}{-1} = -1.$$

31. Divide numerator and denominator by x^2, giving

$$f(x) = \frac{x^2 + 2x - 1}{3 + 3x^2} = \frac{1 + 2/x - 1/x^2}{3/x^2 + 3},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1 + 2/x - 1/x^2}{3/x^2 + 3} = \frac{\lim_{x\to\infty}(1 + 2/x - 1/x^2)}{\lim_{x\to\infty}(3/x^2 + 3)} = \frac{1}{3}.$$

32. Divide numerator and denominator by x, giving

$$f(x) = \frac{x^2 + 4}{x + 3} = \frac{x + 4/x}{1 + 3/x},$$

so

$$\lim_{x\to\infty} f(x) = +\infty.$$

33. Divide numerator and denominator by x^3, giving

$$f(x) = \frac{2x^3 - 16x^2}{4x^2 + 3x^3} = \frac{2 - 16/x}{4/x + 3},$$

so

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2 - 16/x}{4/x + 3} = \frac{\lim_{x\to\infty}(2 - 16/x)}{\lim_{x\to\infty}(4/x + 3)} = \frac{2}{3}.$$

34. Divide numerator and denominator by x^5, giving

$$f(x) = \frac{x^4 + 3x}{x^4 + 2x^5} = \frac{1/x + 3/x^4}{1/x + 2},$$

so

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty}(1/x + 3/x^4)}{\lim_{x\to\infty}(1/x + 2)} = \frac{0}{2} = 0.$$

35. Divide numerator and denominator by e^x, giving

$$f(x) = \frac{3e^x + 2}{2e^x + 3} = \frac{3 + 2e^{-x}}{2 + 3e^{-x}},$$

so

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty}(3 + 2e^{-x})}{\lim_{x\to\infty}(2 + 3e^{-x})} = \frac{3}{2}.$$

36. $f(x) = \dfrac{2e^{-x} + 3}{3e^{-x} + 2}$, so $\lim_{x\to\infty} f(x) = \dfrac{\lim_{x\to\infty}(2e^{-x} + 3)}{\lim_{x\to\infty}(3e^{-x} + 2)} = \dfrac{3}{2}.$

37. Because the denominator equals 0 when $x = 4$, so must the numerator. This means $k^2 = 16$ and the choices for k are 4 or -4.

38. Because the denominator equals 0 when $x = 1$, so must the numerator. So $1 - k + 4 = 0$. The only possible value of k is 5.

39. Because the denominator equals 0 when $x = -2$, so must the numerator. So $4 - 8 + k = 0$ and the only possible value of k is 4.

40. Division of numerator and denominator by x^2 yields

$$\frac{x^2 + 3x + 5}{4x + 1 + x^k} = \frac{1 + 3/x + 5/x^2}{4/x + 1/x^2 + x^{k-2}}.$$

As $x \to \infty$, the limit of the numerator is 1. The limit of the denominator depends upon k. If $k > 2$, the denominator approaches ∞ as $x \to \infty$, so the limit of the quotient is 0. If $k = 2$, the denominator approaches 1 as $x \to \infty$, so the limit of the quotient is 1. If $k < 2$ the denominator approaches 0^+ as $x \to \infty$, so the limit of the quotient is ∞. Therefore the values of k we are looking for are $k \geq 2$.

41. For the numerator, $\lim_{x\to-\infty}\left(e^{2x}-5\right) = -5$. If $k > 0$, $\lim_{x\to-\infty}\left(e^{kx}+3\right) = 3$, so the quotient has a limit of $-5/3$. If $k = 0$, $\lim_{x\to-\infty}\left(e^{kx}+3\right) = 4$, so the quotient has limit of $-5/4$. If $k < 0$, the limit of the quotient is given by $\lim_{x\to-\infty}(e^{2x}-5)/(e^{kx}+3) = 0$.

42. Division of numerator and denominator by x^3 yields

$$\frac{x^3-6}{x^k+3} = \frac{1-6/x^3}{x^{k-3}+3/x^3}.$$

As $x \to \infty$, the limit of the numerator is 1. The limit of the denominator depends upon k. If $k > 3$, the denominator approaches ∞ as $x \to \infty$, so the limit of the quotient is 0. If $k = 3$, the denominator approaches 1 as $x \to \infty$, so the limit of the quotient is 1. If $k < 3$ the denominator approaches 0^+ as $x \to \infty$, so the limit of the quotient is ∞. Therefore the values of k we are looking for are $k \geq 3$.

43. We divide both the numerator and denominator by 3^{2x}, giving

$$\lim_{x\to\infty}\frac{3^{kx}+6}{3^{2x}+4} = \frac{3^{(k-2)x}+6/3^{2x}}{1+4/3^{2x}}.$$

In the denominator, $\lim_{x\to\infty} 1+4/3^{2x} = 1$. In the numerator, if $k < 2$, we have $\lim_{x\to\infty} 3^{(k-2)x}+6/3^{2x} = 0$, so the quotient has a limit of 0. If $k = 2$, we have $\lim_{x\to\infty} 3^{(k-2)x}+6/3^{2x} = 1$, so the quotient has a limit of 1. If $k > 2$, we have $\lim_{x\to\infty} 3^{(k-2)x}+6/3^{2x} = \infty$, so the quotient has a limit of ∞.

44. In the denominator, we have $\lim_{x\to-\infty} 3^{2x}+4 = 4$. In the numerator, if $k < 0$, we have $\lim_{x\to-\infty} 3^{kx}+6 = \infty$, so the quotient has a limit of ∞. If $k = 0$, we have $\lim_{x\to-\infty} 3^{kx}+6 = 7$, so the quotient has a limit of $7/4$. If $k > 0$, we have $\lim_{x\to-\infty} 3^{kx}+6 = 6$, so the quotient has a limit of $6/4$.

45. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.2$, $\delta \leq 0.1$.
For $\epsilon = 0.1$, $\delta \leq 0.05$.
For $\epsilon = 0.02$, $\delta \leq 0.01$.
For $\epsilon = 0.01$, $\delta \leq 0.005$.
For $\epsilon = 0.002$, $\delta \leq 0.001$.
For $\epsilon = 0.001$, $\delta \leq 0.0005$.

46. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \leq 0.46$.
For $\epsilon = 0.01$, $\delta \leq 0.21$.
For $\epsilon = 0.001$, $\delta < 0.1$. Thus, we can take $\delta \leq 0.09$.

47. The results of Problem 45 suggest that we can choose $\delta = \epsilon/2$. For any $\epsilon > 0$, we want to find the δ such that

$$|f(x)-3| = |-2x+3-3| = |2x| < \epsilon.$$

Then if $|x| < \delta = \epsilon/2$, it follows that $|f(x)-3| = |2x| < \epsilon$. So $\lim_{x\to0}(-2x+3) = 3$.

48. (a) Since $\sin(n\pi) = 0$ for $n = 1, 2, 3, \ldots$ the sequence of x-values

$$\frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$$

works. These x-values $\to 0$ and are zeroes of $f(x)$.

(b) Since $\sin(n\pi/2) = 1$ for $n = 1, 5, 9\ldots$ the sequence of x-values

$$\frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \ldots$$

works.

(c) Since $\sin(n\pi)/2 = -1$ for $n = 3, 7, 11, \ldots$ the sequence of x-values

$$\frac{2}{3\pi}, \frac{2}{7\pi}, \frac{2}{11\pi} \ldots$$

works.

(d) Any two of these sequences of x-values show that if the limit were to exist, then it would have to have two (different) values: 0 and 1, or 0 and -1, or 1 and -1. Hence, the limit can not exist.

49. (a) If $b = 0$, then the property says $\lim_{x\to c} 0 = 0$, which is easy to see is true.

(b) If $|f(x) - L| < \frac{\epsilon}{|b|}$, then multiplying by $|b|$ gives

$$|b||f(x) - L| < \epsilon.$$

Since

$$|b||f(x) - L| = |b(f(x) - L)| = |bf(x) - bL|,$$

we have

$$|bf(x) - bL| < \epsilon.$$

(c) Suppose that $\lim_{x\to c} f(x) = L$. We want to show that $\lim_{x\to c} bf(x) = bL$. If we are to have

$$|bf(x) - bL| < \epsilon,$$

then we will need

$$|f(x) - L| < \frac{\epsilon}{|b|}.$$

We choose δ small enough that

$$|x - c| < \delta \quad \text{implies} \quad |f(x) - L| < \frac{\epsilon}{|b|}.$$

By part (b), this ensures that

$$|bf(x) - bL| < \epsilon,$$

as we wanted.

50. Suppose $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} g(x) = L_2$. Then we need to show that

$$\lim_{x\to c} (f(x) + g(x)) = L_1 + L_2.$$

Let $\epsilon > 0$ be given. We need to show that we can choose $\delta > 0$ so that whenever $|x - c| < \delta$, we will have $|(f(x) + g(x)) - (L_1 + L_2)| < \epsilon$. First choose $\delta_1 > 0$ so that $|x - c| < \delta_1$ implies $|f(x) - L_1| < \frac{\epsilon}{2}$; we can do this since $\lim_{x\to c} f(x) = L_1$. Similarly, choose $\delta_2 > 0$ so that $|x - c| < \delta_2$ implies $|g(x) - L_2| < \frac{\epsilon}{2}$. Now, set δ equal to the smaller of δ_1 and δ_2. Thus $|x - c| < \delta$ will make both $|x - c| < \delta_1$ and $|x - c| < \delta_2$. Then, for $|x - c| < \delta$, we have

$$\begin{aligned} |f(x) + g(x) - (L_1 + L_2)| &= |(f(x) - L_1) + (g(x) - L_2)| \\ &\le |(f(x) - L_1)| + |(g(x) - L_2)| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This proves $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$, which is the result we wanted to prove.

51. (a) We need to show that for any given $\epsilon > 0$, there is a $\delta > 0$ so that $|x - c| < \delta$ implies $|f(x)g(x)| < \epsilon$. If $\epsilon > 0$ is given, choose δ_1 so that when $|x - c| < \delta_1$, we have $|f(x)| < \sqrt{\epsilon}$. This can be done since $\lim_{x\to 0} f(x) = 0$. Similarly, choose δ_2 so that when $|x - c| < \delta_2$, we have $|g(x)| < \sqrt{\epsilon}$. Then, if we take δ to be the smaller of δ_1 and δ_2, we'll have that $|x - c| < \delta$ implies both $|f(x)| < \sqrt{\epsilon}$ and $|g(x)| < \sqrt{\epsilon}$. So when $|x - c| < \delta$, we have $|f(x)g(x)| = |f(x)|\,|g(x)| < \sqrt{\epsilon}\cdot\sqrt{\epsilon} = \epsilon$. Thus $\lim_{x\to c} f(x)g(x) = 0$.

(b) $(f(x) - L_1)(g(x) - L_2) + L_1 g(x) + L_2 f(x) - L_1 L_2$
$= f(x)g(x) - L_1 g(x) - L_2 f(x) + L_1 L_2 + L_1 g(x) + L_2 f(x) - L_1 L_2 = f(x)g(x)$.

(c) $\lim_{x\to c} (f(x) - L_1) = \lim_{x\to c} f(x) - \lim_{x\to c} L_1 = L_1 - L_1 = 0$, using the second limit property. Similarly, $\lim_{x\to c} (g(x) - L_2) = 0$.

(d) Since $\lim_{x\to c} (f(x) - L_1) = \lim_{x\to c} (g(x) - L_2) = 0$, we have that $\lim_{x\to c} (f(x) - L_1)(g(x) - L_2) = 0$ by part (a).

(e) From part (b), we have

$$\begin{aligned} \lim_{x\to c} f(x)g(x) &= \lim_{x\to c} ((f(x) - L_1)(g(x) - L_2) + L_1 g(x) + L_2 f(x) - L_1 L_2) \\ &= \lim_{x\to c} (f(x) - L_1)(g(x) - L_2) + \lim_{x\to c} L_1 g(x) + \lim_{x\to c} L_2 f(x) + \lim_{x\to c} (-L_1 L_2) \\ &\quad \text{(using limit property 2)} \\ &= 0 + L_1 \lim_{x\to c} g(x) + L_2 \lim_{x\to c} f(x) - L_1 L_2 \\ &\quad \text{(using limit property 1 and part (d))} \\ &= L_1 L_2 + L_2 L_1 - L_1 L_2 = L_1 L_2. \end{aligned}$$

52. We will show $f(x) = x$ is continuous at $x = c$. Since $f(c) = c$, we need to show that

$$\lim_{x\to c} f(x) = c$$

that is, since $f(x) = x$, we need to show

$$\lim_{x\to c} x = c.$$

Pick any $\epsilon > 0$, then take $\delta = \epsilon$. Thus,

$$|f(x) - c| = |x - c| < \epsilon \quad \text{for all} \quad |x - c| < \delta = \epsilon.$$

53. Since $f(x) = x$ is continuous, Theorem 1.3 on page 54 shows that products of the form $f(x) \cdot f(x) = x^2$ and $f(x) \cdot x^2 = x^3$, etc., are continuous. By a similar argument, x^n is continuous for any $n > 0$.

54. If c is in the interval, we know $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. Then,

$$\begin{aligned}\lim_{x\to c}(f(x) + g(x)) &= \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \quad \text{by limit property 2}\\ &= f(c) + g(c), \quad \text{so } f + g \text{ is continuous at } x = c.\end{aligned}$$

Also,

$$\begin{aligned}\lim_{x\to c}(f(x)g(x)) &= \lim_{x\to c} f(x) \lim_{x\to c} g(x) \quad \text{by limit property 3}\\ &= f(c)g(c) \quad \text{so } fg \text{ is continuous at } x = c.\end{aligned}$$

Finally,

$$\begin{aligned}\lim_{x\to c}\frac{f(x)}{g(x)} &= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \quad \text{by limit property 4}\\ &= \frac{f(c)}{g(c)}, \quad \text{so } \frac{f}{g} \text{ is continuous at } x = c.\end{aligned}$$

Solutions for Chapter 1 Review

Exercises

1. (a) The domain of f is the set of values of x for which the function is defined. Since the function is defined by the graph and the graph goes from $x = 0$ to $x = 7$, the domain of f is $[0, 7]$.
(b) The range of f is the set of values of y attainable over the domain. Looking at the graph, we can see that y gets as high as 5 and as low as -2, so the range is $[-2, 5]$.
(c) Only at $x = 5$ does $f(x) = 0$. So 5 is the only zero of $f(x)$.
(d) Looking at the graph, we can see that $f(x)$ is decreasing on $(1, 7)$.
(e) The graph indicates that $f(x)$ is concave up at $x = 6$.
(f) The value $f(4)$ is the y-value that corresponds to $x = 4$. From the graph, we can see that $f(4)$ is approximately 1.
(g) This function is not invertible, since it fails the horizontal-line test. A horizontal line at $y = 3$ would cut the graph of $f(x)$ in two places, instead of the required one.

2. Taking logs of both sides

$$\begin{aligned}\log 10 &= \log 4^x = x\log 4\\ x &= \frac{\log 10}{\log 4} = \frac{1}{\log 4} \approx 1.66.\end{aligned}$$

3. Taking logs of both sides

$$\begin{aligned}\log\frac{25}{2} &= \log 5^x = x\log 5\\ x &= \frac{\log(\frac{25}{2})}{\log 5} \approx 1.57.\end{aligned}$$

4.

$$\frac{2}{11} = \frac{7^x}{5^x}$$
$$\frac{2}{11} = \left(\frac{7}{5}\right)^x$$

Taking logs of both sides

$$\log\frac{2}{11} = \log\left(\frac{7}{5}\right)^x$$
$$\log\frac{2}{11} = x\log\left(\frac{7}{5}\right)$$
$$x = \frac{\log(2/11)}{\log(7/5)} \approx -5.07.$$

5. To solve for x, we first divide both sides by 5 and then take the natural logarithm of both sides.

$$\frac{7}{5} = e^{0.2x}$$
$$\ln(7/5) = 0.2x$$
$$x = \frac{\ln(7/5)}{0.2} \approx 1.68.$$

6. The amplitude is 5. The period is 6π. See Figure 1.73.

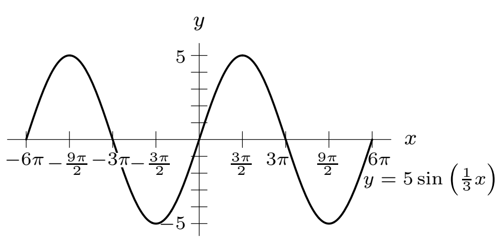


Figure 1.73

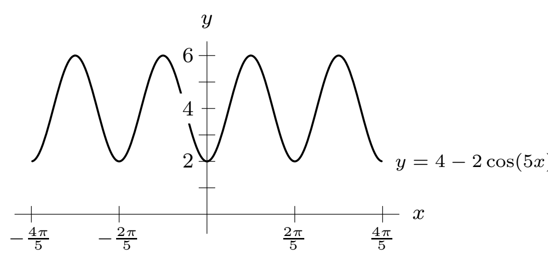


Figure 1.74

7. The amplitude is 2. The period is $2\pi/5$. See Figure 1.74.

8. This is a line with slope $-3/7$ and y-intercept 3, so a possible formula is

$$y = -\frac{3}{7}x + 3.$$

9. Starting with the general exponential equation $y = Ae^{kx}$, we first find that for $(0, 1)$ to be on the graph, we must have $A = 1$. Then to make $(3, 4)$ lie on the graph, we require

$$4 = e^{3k}$$
$$\ln 4 = 3k$$
$$k = \frac{\ln 4}{3} \approx 0.4621.$$

Thus the equation is

$$y = e^{0.4621x}.$$

Alternatively, we can use the form $y = a^x$, in which case we find $y = (1.5874)^x$.

10. This looks like an exponential function. The y-intercept is 3 and we use the form $y = 3e^{kt}$. We substitute the point $(5, 9)$ to solve for k:

$$\begin{aligned} 9 &= 3e^{k5} \\ 3 &= e^{5k} \\ \ln 3 &= 5k \\ k &= 0.2197. \end{aligned}$$

A possible formula is

$$y = 3e^{0.2197t}.$$

Alternatively, we can use the form $y = 3a^t$, in which case we find $y = 3(1.2457)^t$.

11. $y = -kx(x+5) = -k(x^2 + 5x)$, where $k > 0$ is any constant.

12. Since this function has a y-intercept at $(0, 2)$, we expect it to have the form $y = 2e^{kx}$. Again, we find k by forcing the other point to lie on the graph:

$$\begin{aligned} 1 &= 2e^{2k} \\ \frac{1}{2} &= e^{2k} \\ \ln\left(\frac{1}{2}\right) &= 2k \\ k &= \frac{\ln(\frac{1}{2})}{2} \approx -0.34657. \end{aligned}$$

This value is negative, which makes sense since the graph shows exponential decay. The final equation, then, is

$$y = 2e^{-0.34657x}.$$

Alternatively, we can use the form $y = 2a^x$, in which case we find $y = 2(0.707)^x$.

13. $z = 1 - \cos\theta$

14. $y = k(x+2)(x+1)(x-1) = k(x^3 + 2x^2 - x - 2)$, where $k > 0$ is any constant.

15. $x = ky(y-4) = k(y^2 - 4y)$, where $k > 0$ is any constant.

16. $y = 5\sin\left(\frac{\pi t}{20}\right)$

17. This looks like a fourth degree polynomial with roots at -5 and -1 and a double root at 3. The leading coefficient is negative, and so a possible formula is

$$y = -(x+5)(x+1)(x-3)^2.$$

18. This looks like a rational function. There are vertical asymptotes at $x = -2$ and $x = 2$ and so one possibility for the denominator is $x^2 - 4$. There is a horizontal asymptote at $y = 3$ and so the numerator might be $3x^2$. In addition, $y(0) = 0$ which is the case with the numerator of $3x^2$. A possible formula is

$$y = \frac{3x^2}{x^2 - 4}.$$

19. There are many solutions for a graph like this one. The simplest is $y = 1 - e^{-x}$, which gives the graph of $y = e^x$, flipped over the x-axis and moved up by 1. The resulting graph passes through the origin and approaches $y = 1$ as an upper bound, the two features of the given graph.

20. This can be represented by a sine function of amplitude 3 and period 18. Thus,

$$f(x) = 3\sin\left(\frac{\pi}{9}x\right).$$

21. This graph has period 5, amplitude 1 and no vertical shift or horizontal shift from $\sin x$, so it is given by

$$f(x) = \sin\left(\frac{2\pi}{5}x\right).$$

22. $f(x) = \ln x, \quad g(x) = x^3$. (Another possibility: $f(x) = 3x, \quad g(x) = \ln x$.)

23. $f(x) = x^3, \quad g(x) = \ln x$.

24. There is no break in the graph of $f(x)$ although it does have a 'corner' at $x = 0$, so $f(x)$ is continuous.

25. The graph of $g(x)$ is shown in Figure 1.75. It has a break at $x = 0$, so $g(x)$ is not continuous on $[-1, 1]$.

26. The graph has no breaks and is therefore continuous. See Figure 1.76.

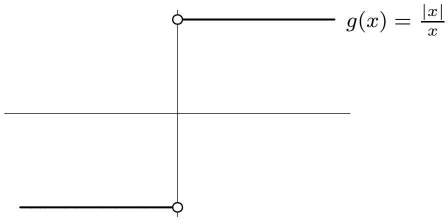


Figure 1.75

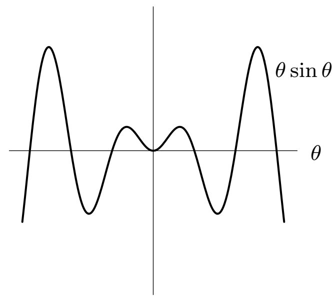


Figure 1.76

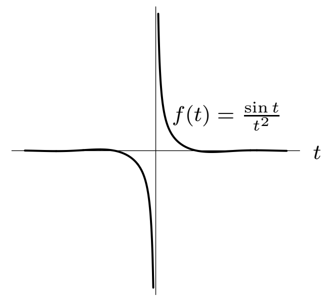


Figure 1.77

27. The graph appears to have a vertical asymptote at $t = 0$, so $f(t)$ is not continuous on $[-1, 1]$. See Figure 1.77.

28. From Table 1.15, it appears the limit is 0. This is confirmed by Figure 1.78. An appropriate window is $-0.015 < x < 0.015$, $-0.01 < y < 0.01$.

Table 1.15

x	$f(x)$	x	$f(x)$
0.1	0.0666	−0.0001	−0.0001
0.01	0.0067	−0.001	−0.0007
0.001	0.0007	−0.01	−0.0067
0.0001	0	−0.1	−0.0666

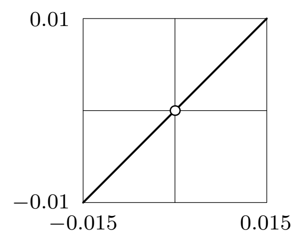


Figure 1.78

29. From Table 1.16, it appears the limit is 0. This is confirmed by Figure 1.79. An appropriate window is $-0.0029 < x < 0.0029$, $-0.01 < y < 0.01$.

Table 1.16

x	$f(x)$	x	$f(x)$
0.1	0.3365	−0.0001	−0.0004
0.01	0.0337	−0.001	−0.0034
0.001	0.0034	−0.01	−0.0337
0.0001	0.0004	−0.1	−0.3365

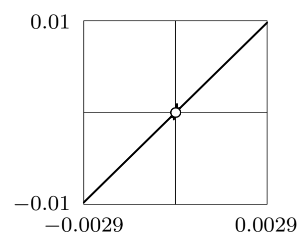


Figure 1.79

30. From Table 1.17, it appears the limit is 0. Figure 1.80 confirms this. An appropriate window is $1.570 < x < 1.5715$, $-0.01 < y < 0.01$.

Table 1.17

x	$f(x)$
1.6708	−1.2242
1.5808	−0.1250
1.5718	−0.0125
1.5709	−0.0013
1.5707	0.0012
1.5698	0.0125
1.5608	0.1249
1.4708	1.2241

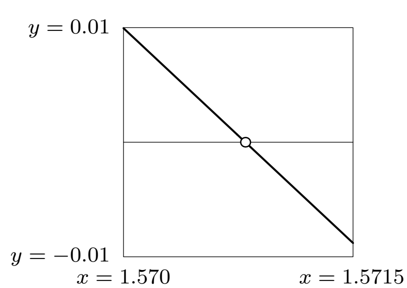


Figure 1.80

31. From Table 1.18, it appears the limit is $1/2$. Figure 1.81 confirms this. An appropriate window is $1.92 < x < 2.07$, $0.49 < y < 0.51$.

Table 1.18

x	$f(x)$
2.1	0.5127
2.01	0.5013
2.001	0.5001
2.0001	0.5000
1.9999	0.5000
1.999	0.4999
1.99	0.4988
1.9	0.4877

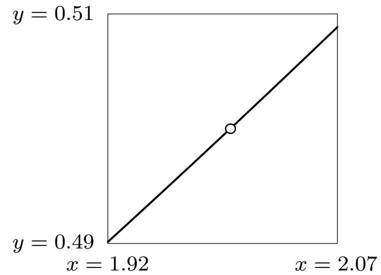


Figure 1.81

32. $$f(x) = \frac{x^3|2x-6|}{x-3} = \begin{cases} \dfrac{x^3(2x-6)}{x-3} = 2x^3, & x > 3 \\ \dfrac{x^3(-2x+6)}{x-3} = -2x^3, & x < 3 \end{cases}$$

Figure 1.82 confirms that $\lim_{x\to 3^+} f(x) = 54$ while $\lim_{x\to 3^-} f(x) = -54$; thus $\lim_{x\to 3} f(x)$ does not exist.

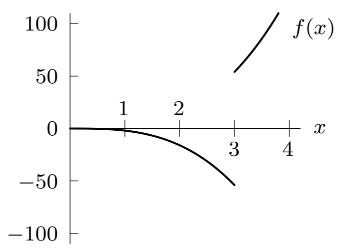


Figure 1.82

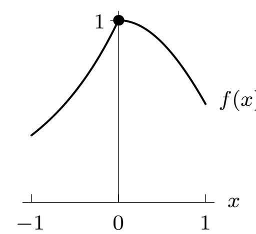


Figure 1.83

33. $$f(x) = \begin{cases} e^x & -1 < x < 0 \\ 1 & x = 0 \\ \cos x & 0 < x < 1 \end{cases}$$

Figure 1.83 confirms that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x = e^0 = 1$, and that $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \cos x = \cos 0 = 1$, so $\lim_{x\to 0} f(x) = 1$.

Problems

34. (a) More fertilizer increases the yield until about 40 lbs.; then it is too much and ruins crops, lowering yield.
(b) The vertical intercept is at $Y = 200$. If there is no fertilizer, then the yield is 200 bushels.
(c) The horizontal intercept is at $a = 80$. If you use 80 lbs. of fertilizer, then you will grow no apples at all.
(d) The range is the set of values of Y attainable over the domain $0 \le a \le 80$. Looking at the graph, we can see that Y goes as high as 550 and as low as 0. So the range is $0 \le Y \le 550$.
(e) Looking at the graph, we can see that Y is decreasing at $a = 60$.
(f) Looking at the graph, we can see that Y is concave down everywhere, so it is certainly concave down at $a = 40$.

35. (a) The height of the rock decreases as time passes, so the graph falls as you move from left to right. One possibility is shown in Figure 1.84.

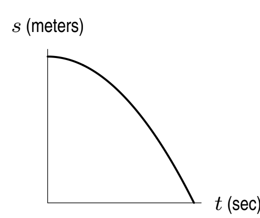


Figure 1.84

(b) The statement $f(7) = 12$ tells us that 7 seconds after the rock is dropped, it is 12 meters above the ground.
(c) The vertical intercept is the value of s when $t = 0$; that is, the height from which the rock is dropped. The horizontal intercept is the value of t when $s = 0$; that is, the time it takes for the rock to hit the ground.

36. Given the doubling time of 2 hours, $200 = 100e^{k(2)}$, we can solve for the growth rate k using the equation:

$$\begin{aligned} 2P_0 &= P_0 e^{2k} \\ \ln 2 &= 2k \\ k &= \frac{\ln 2}{2}. \end{aligned}$$

Using the growth rate, we wish to solve for the time t in the formula

$$P = 100e^{\frac{\ln 2}{2}t}$$

where $P = 3{,}200$, so

$$\begin{aligned} 3{,}200 &= 100e^{\frac{\ln 2}{2}t} \\ t &= 10 \text{ hours.} \end{aligned}$$

37. (a) We find the slope m and intercept b in the linear equation $S = b + mt$. To find the slope m, we use

$$m = \frac{\Delta S}{\Delta t} = \frac{66 - 113}{50 - 0} = -0.94.$$

When $t = 0$, we have $S = 113$, so the intercept b is 113. The linear formula is

$$S = 113 - 0.94t.$$

(b) We use the formula $S = 113 - 0.94t$. When $S = 20$, we have $20 = 113 - 0.94t$ and so $t = 98.9$. If this linear model were correct, the average male sperm count would drop below the fertility level during the year 2038.

38. To find a half-life, we want to find at what t value $Q = \frac{1}{2}Q_0$. Plugging this into the equation of the decay of plutonium-240, we have

$$\begin{aligned} \frac{1}{2} &= e^{-0.00011t} \\ t &= \frac{\ln(1/2)}{-0.00011} \approx 6{,}301 \text{ years.} \end{aligned}$$

The only difference in the case of plutonium-242 is that the constant -0.00011 in the exponent is now -0.0000018. Thus, following the same procedure, the solution for t is

$$t = \frac{\ln(1/2)}{-0.0000018} \approx 385{,}081 \text{ years.}$$

39. We can solve for the growth rate k of the bacteria using the formula $P = P_0 e^{kt}$:

$$\begin{aligned} 1500 &= 500e^{k(2)} \\ k &= \frac{\ln(1500/500)}{2}. \end{aligned}$$

Knowing the growth rate, we can find the population P at time $t = 6$:

$$\begin{aligned} P &= 500e^{(\frac{\ln 3}{2})6} \\ &\approx 13{,}500 \text{ bacteria.} \end{aligned}$$

40. Assuming the US population grows exponentially, we have

$$\begin{aligned} 248.7 &= 226.5e^{10k} \\ k &= \frac{\ln(1.098)}{10} = 0.00935. \end{aligned}$$

We want to find the time t in which

$$\begin{aligned} 300 &= 226.5e^{0.00935t} \\ t &= \frac{\ln(1.324)}{0.00935} = 30 \text{ years.} \end{aligned}$$

Thus, the population will go over 300 million around the year 2010.

41. Since we are told that the rate of decay is *continuous*, we use the function $Q(t) = Q_0e^{rt}$ to model the decay, where $Q(t)$ is the amount of strontium-90 which remains at time t, and Q_0 is the original amount. Then

$$Q(t) = Q_0e^{-0.0247t}.$$

So after 100 years,

$$Q(100) = Q_0e^{-0.0247\cdot 100}$$

and

$$\frac{Q(100)}{Q_0} = e^{-2.47} \approx 0.0846$$

so about 8.46% of the strontium-90 remains.

42. If r was the average yearly inflation rate, in decimals, then $\frac{1}{4}(1+r)^3 = 2{,}400{,}000$, so $r = 211.53$, i.e. $r = 21{,}153\%$.

43. We will let

$$\begin{aligned} T &= \text{amount of fuel for take-off,} \\ L &= \text{amount of fuel for landing,} \\ P &= \text{amount of fuel per mile in the air,} \\ m &= \text{the length of the trip in miles.} \end{aligned}$$

Then Q, the total amount of fuel needed, is given by

$$Q(m) = T + L + Pm.$$

44. The period T_E of the earth is (by definition!) one year or about 365.24 days. Since the semimajor axis of the earth is 150 million km, we can use Kepler's Law to derive the constant of proportionality, k.

$$T_E = k(S_E)^{\frac{3}{2}}$$

where S_E is the earth's semimajor axis, or 150 million km.

$$365.24 = k(150)^{\frac{3}{2}}$$

$$k = \frac{365.24}{(150)^{\frac{3}{2}}} \approx 0.198.$$

Now that we know the constant of proportionality, we can use it to derive the periods of Mercury and Pluto. For Mercury,

$$T_M = (0.198)(58)^{\frac{3}{2}} \approx 87.818 \text{ days.}$$

For Pluto,

$$T_P = (0.198)(6000)^{\frac{3}{2}} \approx 92,400 \text{ days,}$$

or (converting Pluto's period to years),

$$\frac{(0.198)(6000)^{\frac{3}{2}}}{365.24} \approx 253 \text{ years.}$$

45. (a) Let the height of the can be h. Then

$$V = \pi r^2 h.$$

The surface area consists of the area of the ends (each is πr^2) and the curved sides (area $2\pi rh$), so

$$S = 2\pi r^2 + 2\pi rh.$$

Solving for h from the formula for V, we have

$$h = \frac{V}{\pi r^2}.$$

Substituting into the formula for S, we get

$$S = 2\pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}.$$

(b) For large r, the $2V/r$ term becomes negligible, meaning $S \approx 2\pi r^2$, and thus $S \to \infty$ as $r \to \infty$.
(c) The graph is in Figure 1.85.

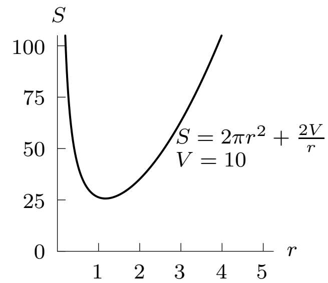


Figure 1.85

46. (a) The line given by $(0, 2)$ and $(1, 1)$ has slope $m = \frac{2-1}{-1} = -1$ and y-intercept 2, so its equation is

$$y = -x + 2.$$

The points of intersection of this line with the parabola $y = x^2$ are given by

$$\begin{aligned} x^2 &= -x + 2 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0. \end{aligned}$$

The solution $x = 1$ corresponds to the point we are already given, so the other solution, $x = -2$, gives the x-coordinate of C. When we substitute back into either equation to get y, we get the coordinates for C, $(-2, 4)$.

(b) The line given by $(0, b)$ and $(1, 1)$ has slope $m = \frac{b-1}{-1} = 1 - b$, and y-intercept at $(0, b)$, so we can write the equation for the line as we did in part (a):

$$y = (1 - b)x + b.$$

We then solve for the points of intersection with $y = x^2$ the same way:

$$\begin{aligned} x^2 &= (1 - b)x + b \\ x^2 - (1 - b)x - b &= 0 \\ x^2 + (b - 1)x - b &= 0 \\ (x + b)(x - 1) &= 0 \end{aligned}$$

Again, we have the solution at the given point $(1, 1)$, and a new solution at $x = -b$, corresponding to the other point of intersection C. Substituting back into either equation, we can find the y-coordinate for C is b^2, and thus C is given by $(-b, b^2)$. This result agrees with the particular case of part (a) where $b = 2$.

47. Depth $= 7 + 1.5 \sin \left(\dfrac{\pi}{3} t\right)$

48. Over the one-year period, the average value is about 75° and the amplitude of the variation is about $\frac{90-60}{2} = 15^\circ$. The function assumes its minimum value right at the beginning of the year, so we want a negative cosine function. Thus, for t in years, we have the function

$$f(t) = 75 - 15 \cos \left(\frac{2\pi}{12} t\right).$$

(Many other answers are possible, depending on how you read the chart.)

49. (a) Yes, f is invertible, since f is increasing everywhere.
(b) $f^{-1}(400)$ is the year in which 400 million motor vehicles were registered in the world. From the picture, we see that $f^{-1}(400)$ is around 1979.
(c) Since the graph of f^{-1} is the reflection of the graph of f over the line $y = x$, we get Figure 1.86.

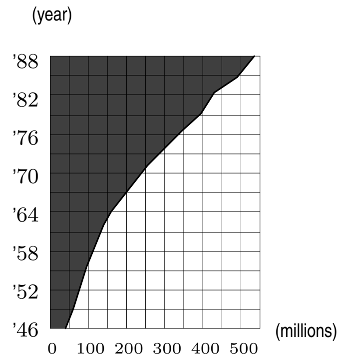


Figure 1.86: Graph of f^{-1}

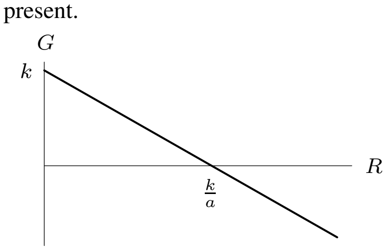
50. (a) is $g(x)$ since it is linear. (b) is $f(x)$ since it has decreasing slope; the slope starts out about 1 and then decreases to about $\frac{1}{10}$. (c) is $h(x)$ since it has increasing slope; the slope starts out about $\frac{1}{10}$ and then increases to about 1.

51. (a) The period is 2π.
(b) After π, the values of $\cos 2\theta$ repeat, but the values of $2\sin\theta$ do not (in fact, they repeat but flipped over the x-axis). After another π, that is after a total of 2π, the values of $\cos 2\theta$ repeat *again*, and now the values of $2\sin\theta$ repeat also, so the function $2\sin\theta + 3\cos 2\theta$ repeats at that point.

52. (a) The rate R is the difference of the rate at which the glucose is being injected, which is given to be constant, and the rate at which the glucose is being broken down, which is given to be proportional to the amount of glucose present. Thus we have the formula

$$R = k - aG$$

where k is the rate that the glucose is being injected, a is the constant relating the rate that it is broken down to the amount present, and G is the amount present.
(b)



53. (a) $r(p) = kp(A - p)$, where $k > 0$ is a constant.
(b) $p = A/2$.

54. (a) The domain is $(0, 4000)$, the range is $(0, 10^8)$.
(b) The domain is $(0, 3000)$, the range is $(0, 10^7)$.
(c) The domain is $(0, 0.2)$, the range is $(0, 0.04)$.

55. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \leq 0.1$
For $\epsilon = 0.05$, $\delta \leq 0.05$.
For $\epsilon = 0.0007$, $\delta \leq 0.00007$.

56. By tracing on a calculator or solving equations, we find the following values of δ:
For $\epsilon = 0.1$, $\delta \leq 0.45$.
For $\epsilon = 0.001$, $\delta \leq 0.0447$.
For $\epsilon = 0.00001$, $\delta \leq 0.00447$.

CAS Challenge Problems

57. (a) A CAS gives $f(x) = (x - a)(x + a)(x + b)(x - c)$.
(b) The graph of $f(x)$ crosses the x-axis at $x = a$, $x = -a$, $x = -b$, $x = c$; it crosses the y-axis at a^2bc. Since the coefficient of x^4 (namely 1) is positive, the graph of f looks like that shown in Figure 1.87.

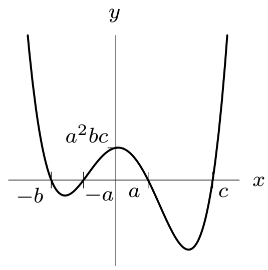


Figure 1.87: Graph of $f(x) = (x-a)(x+a)(x+b)(x-c)$

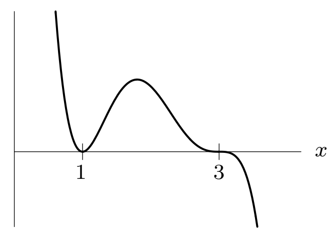


Figure 1.88: Graph of $f(x) = -(x-1)^2(x-3)^3$

58. (a) A CAS gives $f(x) = -(x-1)^2(x-3)^3$.

(b) For large $|x|$, the graph of $f(x)$ looks like the graph of $y = -x^5$, so $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to \infty$. The answer to part (a) shows that f has a double root at $x = 1$, so near $x = 1$, the graph of f looks like a parabola touching the x-axis at $x = 1$. Similarly, f has a triple root at $x = 3$. Near $x = 3$, the graph of f looks like the graph of $y = x^3$, flipped over the x-axis and shifted to the right by 3, so that the "seat" is at $x = 3$. See Figure 1.88.

59. (a) As $x \to \infty$, the term e^{6x} dominates and tends to ∞. Thus, $f(x) \to \infty$ as $x \to \infty$.
As $x \to -\infty$, the terms of the form e^{kx}, where $k = 6, 5, 4, 3, 2, 1$, all tend to zero. Thus, $f(x) \to 16$ as $x \to -\infty$.

(b) A CAS gives

$$f(x) = (e^x + 1)(e^{2x} - 2)(e^x - 2)(e^{2x} + 2e^x + 4).$$

Since e^x is always positive, the factors $(e^x + 1)$ and $(e^{2x} + 2e^x + 4)$ are never zero. The other factors each lead to a zero, so there are two zeros.

(c) The zeros are given by

$$\begin{aligned} e^{2x} &= 2 \quad \text{so} \quad x = \frac{\ln 2}{2} \\ e^x &= 2 \quad \text{so} \quad x = \ln 2. \end{aligned}$$

Thus, one zero is twice the size of the other.

60. (a) Since $f(x) = x^2 - x$,

$$f(f(x)) = (f(x))^2 - f(x) = (x^2 - x)^2 - (x^2 - x) = x - 2\,x^3 + x^4.$$

Using the CAS to define the function $f(x)$, and then asking it to expand $f(f(f(x)))$, we get

$$f(f(f(x))) = -x + x^2 + 2\,x^3 - 5\,x^4 + 2\,x^5 + 4\,x^6 - 4\,x^7 + x^8.$$

(b) The degree of $f(f(x))$ (that is, f composed with itself 2 times) is $4 = 2^2$. The degree of $f(f(f(x)))$ (that is, f composed with itself 3 times), is $8 = 2^3$. Each time you substitute f into itself, the degree is multiplied by 2, because you are substituting in a degree 2 polynomial. So we expect the degree of $f(f(f(f(f(f(x))))))$ (that is, f composed with itself 6 times) to be $64 = 2^6$.

61. (a) A CAS or division gives

$$f(x) = \frac{x^3 - 30}{x - 3} = x^2 + 3x + 9 - \frac{3}{x - 3},$$

so $p(x) = x^2 + 3x + 9$, and $r(x) = -3$, and $q(x) = x - 3$.

(b) The vertical asymptote is $x = 3$. Near $x = 3$, the values of $p(x)$ are much smaller than the values of $r(x)/q(x)$. Thus

$$f(x) \approx \frac{-3}{x - 3} \qquad \text{for } x \text{ near } 3.$$

(c) For large x, the values of $p(x)$ are much larger than the value of $r(x)/q(x)$. Thus

$$f(x) \approx x^2 + 3x + 9 \qquad \text{as } x \to \infty, x \to -\infty.$$

(d) Figure 1.89 shows $f(x)$ and $y = -3/(x - 3)$ for x near 3. Figure 1.90 shows $f(x)$ and $y = x^2 + 3x + 9$ for $-20 \le x \le 20$. Note that in each case the graphs of f and the approximating function are close.

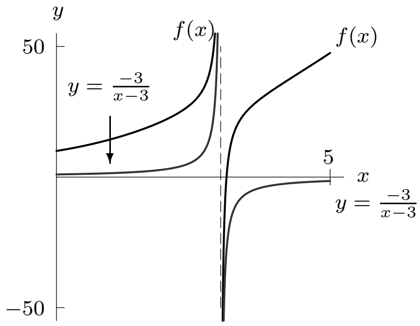


Figure 1.89: Close-up view of $f(x)$ and $y = -3/(x-3)$

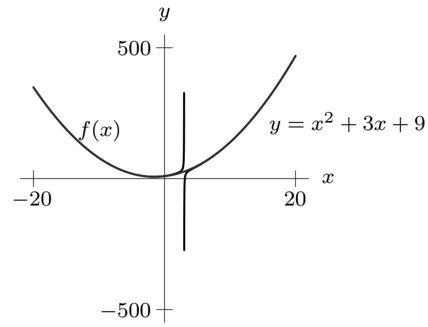


Figure 1.90: Far-away view of $f(x)$ and $y = x^2 + 3x + 9$

62. Using the trigonometric expansion capabilities of your CAS, you get something like

$$\sin(5x) = 5\cos^4(x)\ \sin(x) - 10\cos^2(x)\sin^3(x) + \sin^5(x).$$

Answers may vary. To get rid of the powers of cosine, use the identity $\cos^2(x) = 1 - \sin^2(x)$. This gives

$$\sin(5x) = 5\ \sin(x)\left(1 - \sin^2(x)\right)^2 - 10\sin^3(x)\ \left(1 - \sin^2(x)\right) + \sin^5(x).$$

Finally, using the CAS to simplify,

$$\sin(5x) = 5\ \sin(x) - 20\sin^3(x) + 16\sin^5(x).$$

63. Using the trigonometric expansion capabilities of your computer algebra system, you get something like

$$\cos(4x) = \cos^4(x) - 6\cos^2(x)\sin^2(x) + \sin^4(x).$$

Answers may vary.

(a) To get rid of the powers of cosine, use the identity $\cos^2(x) = 1 - \sin^2(x)$. This gives

$$\cos(4x) = \cos^4(x) - 6\cos^2(x)\ \left(1 - \cos^2(x)\right) + \left(1 - \cos^2(x)\right)^2.$$

Finally, using the CAS to simplify,

$$\cos(4x) = 1 - 8\cos^2(x) + 8\cos^4(x).$$

(b) This time we use $\sin^2(x) = 1 - \cos^2(x)$ to get rid of powers of sine. We get

$$\cos(4x) = \left(1 - \sin^2(x)\right)^2 - 6\sin^2(x)\ \left(1 - \sin^2(x)\right) + \sin^4(x) = 1 - 8\sin^2(x) + 8\sin^4(x).$$

CHECK YOUR UNDERSTANDING

1. False. A line can be put through any two points in the plane. However, if the line is vertical, it is not the graph of a function.

2. True. The graph of $y = 10^x$ is moved horizontally by h units if we replace x by $x - h$ for some number h. Writing $100 = 10^2$, we have $f(x) = 100(10^x) = 10^2 \cdot 10^x = 10^{x+2}$. The graph of $f(x) = 10^{x+2}$ is the graph of $g(x) = 10^x$ shifted two units to the left.

3. True, as seen from the graph.

4. False, since $\log(x-1) = 0$ if $x - 1 = 10$, so $x = 11$.

5. True. The highest degree term in a polynomial determines how the polynomial behaves when x is very large in the positive or negative direction. When n is odd, x^n is positive when x is large and positive but negative when x is large and negative. Thus if a polynomial $p(x)$ has odd degree, it will be positive for some values of x and negative for other values of x. Since every polynomial is continuous, the Intermediate Value Theorem then guarantees that $p(x) = 0$ for some value of x.

6. False. The y-intercept is $y = 2 + 3e^{-0} = 5$.

7. True, since, as $t \to \infty$, we know $e^{-4t} \to 0$, so $y = 5 - 3e^{-4t} \to 5$.

8. True. Suppose we start at $x = x_1$ and increase x by 1 unit to $x_1 + 1$. If $y = b + mx$, the corresponding values of y are $b + mx_1$ and $b + m(x_1 + 1)$. Thus y increases by

$$\Delta y = b + m(x_1 + 1) - (b + mx_1) = m.$$

9. False. Suppose $y = 5^x$. Then increasing x by 1 increases y by a factor of 5. However increasing x by 2 increases y by a factor of 25, not 10, since

$$y = 5^{x+2} = 5^x \cdot 5^2 = 25 \cdot 5^x.$$

(Other examples are possible.)

10. True. Suppose $y = Ab^x$ and we start at the point (x_1, y_1), so $y_1 = Ab^{x_1}$. Then increasing x_1 by 1 gives $x_1 + 1$, so the new y-value, y_2, is given by

$$y_2 = Ab^{x_1+1} = Ab^{x_1}b = (Ab^{x_1})b,$$

so

$$y_2 = by_1.$$

Thus, y has increased by a factor of b, so $b = 3$, and the function is $y = A3^x$.

However, if x_1 is increased by 2, giving $x_1 + 2$, then the new y-value, y_3, is given by

$$y_3 = A3^{x_1+2} = A3^{x_1}3^2 = 9A3^{x_1} = 9y_1.$$

Thus, y has increased by a factor of 9.

11. False, since $\cos\theta$ is decreasing and $\sin\theta$ is increasing.

12. False. The period is $2\pi/(0.05\pi) = 40$

13. True. The period is $2\pi/(200\pi) = 1/100$ seconds. Thus, the function executes 100 cycles in 1 second.

14. False. If $\theta = \pi/2, 3\pi/2, 5\pi/2\ldots$, then $\theta - \pi/2 = 0, \pi, 2\pi\ldots$, and the tangent is defined (it is zero) at these values.

15. False. For example, $\sin(0) \neq \sin((2\pi)^2)$, since $\sin(0) = 0$ but $\sin((2\pi)^2) = 0.98$.

16. True. Since $\sin(\theta + 2\pi) = \sin\theta$ for all θ, we have $g(\theta + 2\pi) = e^{\sin(\theta+2\pi)} = e^{\sin\theta} = g(\theta)$ for all θ.

17. False. A counterexample is given by $f(x) = \sin x$, which has period 2π, and $g(x) = x^2$. The graph of $f(g(x)) = \sin(x^2)$ in Figure 1.91 is not periodic with period 2π.

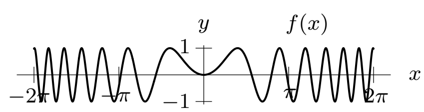


Figure 1.91

18. True. If $g(x)$ has period k, then $g(x + k) = g(x)$. Thus we have

$$f(g(x + k)) = f(g(x))$$

which shows that $f(g(x))$ is periodic with period k.

19. False. For $x < 0$, as x increases, x^2 decreases, so e^{-x^2} increases.

20. False. The inverse function is $y = 10^x$.

21. True. If f is increasing then its reflection about the line $y = x$ is also increasing. An example is shown in Figure 1.92. The statement is true.

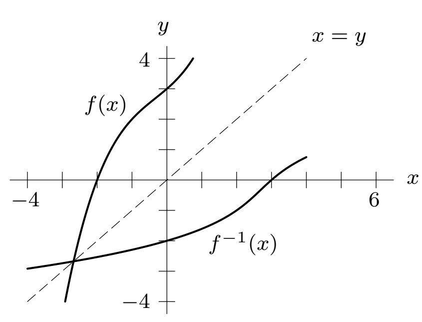


Figure 1.92

22. True, since $|\sin(-x)| = |-\sin x| = \sin x$.

23. True. If $f(x)$ is even, we have $f(x) = f(-x)$ for all x. For example, $f(-2) = f(2)$. This means that the graph of $f(x)$ intersects the horizontal line $y = f(2)$ at two points, $x = 2$ and $x = -2$. Thus, f has no inverse function.

24. False. For example, $f(x) = x$ and $g(x) = x^3$ are both odd. Their inverses are $f^{-1}(x) = x$ and $g^{-1}(x) = x^{1/3}$.

25. True. If $b > 1$, then $ab^x \to 0$ as $x \to -\infty$. If $0 < b < 1$, then $ab^x \to 0$ as $x \to \infty$. In either case, the function $y = a + ab^x$ has $y = a$ as the horizontal asymptote.

26. False, since $ax + b = 0$ if $x = -b/a$. Thus $y = \ln(ax + b)$ has a vertical asymptote at $x = -b/a$.

27. True, since $e^{-kt} \to 0$ as $t \to \infty$, so $y \to 20$ as $t \to \infty$.

28. True. We have $g(-x) = g(x)$ since g is even, and therefore $f(g(-x)) = f(g(x))$.

29. False. A counterexample is given by $f(x) = x^2$ and $g(x) = x + 1$. The function $f(g(x)) = (x + 1)^2$ is not even because $f(g(1)) = 4$ and $f(g(-1)) = 0 \neq 4$.

30. False. All we know is that if h is close enough to zero then $f(h)$ will be as close as we please to L. We do not know how close would be close enough to zero for $f(h)$ to be closer to L than is $f(0.01)$. It might be that we have to get a lot closer than 0.0001. It is even possible that $f(0.01) = L$ but $f(0.0001) \neq L$ so $f(h)$ could never get closer to L than $f(0.01)$.

31. Let $f(x) = \begin{cases} 1 & x \leq 2 \\ x & x > 2 \end{cases}$. Then $f(x)$ is continuous at every point in $[0, 3]$ except at $x = 2$. Other answers are possible.

32. Let $f(x) = \begin{cases} x & x \leq 3 \\ 2x & x > 3 \end{cases}$. Then $f(x)$ is increasing for all x but $f(x)$ is not continuous at $x = 3$. Other answers are possible.

33. Let $f(x) = \dfrac{1}{x + 7\pi}$. Other answers are possible.

34. Let $f(x) = \dfrac{1}{(x-1)(x-2)(x-3)\cdots(x-16)(x-17)}$. This function has an asymptote corresponding to every factor in the denominator. Other answers are possible.

35. The function $f(x) = \dfrac{x-1}{x-2}$ has $y = 1$ as the horizontal asymptote and $x = 2$ as the vertical asymptote. These lines cross at the point $(2, 1)$. Other answers are possible.

36. We have

$$g(x) = f(x + 2)$$

because the graph of g is obtained by moving the graph of f to the left by 2 units. We also have

$$g(x) = f(x) + 3$$

because the graph of g is obtained by moving the graph of f up by 3 units. Thus, we have $f(x+2) = f(x)+3$. The graph of f climbs 3 units whenever x increases by 2. The simplest choice for f is a linear function of slope $3/2$, for example $f(x) = 1.5x$, so $g(x) = 1.5x + 3$.

37. Let $f(x) = x$ and $g(x) = -2x$. Then $f(x) + g(x) = -x$, which is decreasing. Note f is increasing since it has positive slope, and g is decreasing since it has negative slope.

38. This is impossible. If $a < b$, then $f(a) < f(b)$, since f is increasing, and $g(a) > g(b)$, since g is decreasing, so $-g(a) < -g(b)$. Therefore, if $a < b$, then $f(a) - g(a) < f(b) - g(b)$, which means that $f(x) + g(x)$ is increasing.

39. Let $f(x) = e^x$ and let $g(x) = e^{-2x}$. Note f is increasing since it is an exponential growth function, and g is decreasing since it is an exponential decay function. Then $f(x)g(x) = e^{-x}$, which is decreasing.

40. This is impossible. As x increases, $g(x)$ decreases. As $g(x)$ decreases, so does $f(g(x))$ because f is increasing (an increasing function increases as its variable increases, so it decreases as its variable decreases).

41. False. For example, $f(x) = x/(x^2 + 1)$ has no vertical asymptote since the denominator is never 0.

42. False. For example, let $f(x) = \log x$. Then $f(x)$ is increasing on $[1, 2]$, but $f(x)$ is concave down. (Other examples are possible.)

43. False. For example, let $y = x + 1$. Then the points $(1, 2)$ and $(2, 3)$ are on the line. However the ratios

$$\frac{2}{1} = 2 \qquad \text{and} \qquad \frac{3}{2} = 1.5$$

are different. The ratio y/x is constant for linear functions of the form $y = mx$, but not in general. (Other examples are possible.)

44. True. For example, $f(x) = (0.5)^x$ is an exponential function which decreases. (Other examples are possible.)

45. False. For example, if $y = 4x + 1$ (so $m = 4$) and $x = 1$, then $y = 5$. Increasing x by 2 units gives 3, so $y = 4(3) + 1 = 13$. Thus, y has increased by 8 units, not $4 + 2 = 6$. (Other examples are possible.)

46. False. For example, let $f(x) = \begin{cases} 1 & x \leq 3 \\ 2 & x > 3 \end{cases}$, then $f(x)$ is defined at $x = 3$ but it is not continuous at $x = 3$. (Other examples are possible.)

47. False. A counterexample is graphed in Figure 1.93, in which $f(5) < 0$.

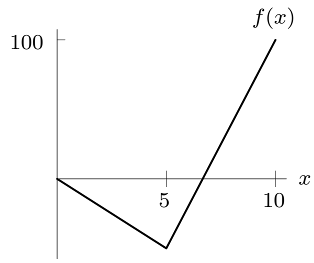


Figure 1.93

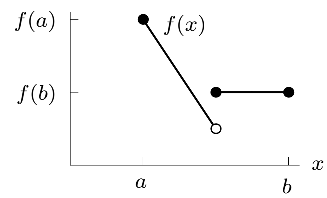


Figure 1.94

48. False. A counterexample is graphed in Figure 1.94.

49. True. The constant function $f(x) = 0$ is the only function that is both even and odd. This follows, since if f is both even and odd, then, for all x, $f(-x) = f(x)$ (if f is even) and $f(-x) = -f(x)$ (if f is odd). Thus, for all x, $f(x) = -f(x)$ i.e. $f(x) = 0$, for all x. So $f(x) = 0$ is both even and odd and is the only such function.

50. True, by Property 3 of limits in Theorem 1.2, since $\lim_{x\to 3} x = 3$.

51. False. If $\lim_{x\to 3} g(x)$ does not exist, then $\lim_{x\to 3} f(x)g(x)$ may not even exist. For example, let $f(x) = 2x + 1$ and define g by:

$$g(x) = \begin{cases} 1/(x-3) & \text{if } x \neq 3 \\ 4 & \text{if } x = 3 \end{cases}$$

Then $\lim_{x\to 3} f(x) = 7$ and $g(3) = 4$, but $\lim_{x\to 3} f(x)g(x) \neq 28$, since $\lim_{x\to 3}(2x + 1)/(x - 3)$ does not exist.

52. True, by Property 2 of limits in Theorem 1.2.

53. True, by Properties 2 and 3 of limits in Theorem 1.2.

$$\lim_{x\to 3} g(x) = \lim_{x\to 3}(f(x) + g(x) + (-1)f(x)) = \lim_{x\to 3}(f(x) + g(x)) + (-1)\lim_{x\to 3} f(x) = 12 + (-1)7 = 5.$$

54. False. For example, define f as follows:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \neq 2.99 \\ 1000 & \text{if } x = 2.99. \end{cases}$$

Then $f(2.9) = 2(2.9) + 1 = 6.8$, whereas $f(2.99) = 1000$.

55. False. For example, define f as follows:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \neq 3.01 \\ -1000 & \text{if } x = 3.01. \end{cases}$$

Then $f(3.1) = 2(3.1) + 1 = 7.2$, whereas $f(3.01) = -1000$.

56. True. Suppose instead that $\lim_{x\to 3} g(x)$ does not exist but $\lim_{x\to 3}(f(x)g(x))$ did exist. Since $\lim_{x\to 3} f(x)$ exists and is not zero, then $\lim_{x\to 3}((f(x)g(x))/f(x))$ exists, by Property 4 of limits in Theorem 1.2. Furthermore, $f(x) \neq 0$ for all x in some interval about 3, so $(f(x)g(x))/f(x) = g(x)$ for all x in that interval. Thus $\lim_{x\to 3} g(x)$ exists. This contradicts our assumption that $\lim_{x\to 3} g(x)$ does not exist.

57. False. For some functions we need to pick smaller values of δ. For example, if $f(x) = x^{1/3} + 2$ and $c = 0$ and $L = 2$, then $f(x)$ is within 10^{-3} of 2 if $|x^{1/3}| < 10^{-3}$. This only happens if x is within $(10^{-3})^3 = 10^{-9}$ of 0. If $x = 10^{-3}$ then $x^{1/3} = (10^{-3})^{1/3} = 10^{-1}$, which is too large.

58. False. The definition of a limit guarantees that, for any positive ϵ, there is a δ. This statement, which guarantees an ϵ for a specific $\delta = 10^{-3}$, is not equivalent to $\lim_{x\to c} f(x) = L$. For example, consider a function with a vertical asymptote within 10^{-3} of 0, such as $c = 0$, $L = 0$, $f(x) = x/(x - 10^{-4})$.

59. True. This is equivalent to the definition of a limit.

60. False. Although x may be far from c, the value of $f(x)$ could be close to L. For example, suppose $f(x) = L$, the constant function.

61. False. The definition of the limit says that if x is within δ of c, then $f(x)$ is within ϵ of L, not the other way round.

62. (a) This statement follows: if we interchange the roles of f and g in the original statement, we get this statement.
(b) This statement is true, but it does not follow directly from the original statement, which says nothing about the case $g(a) = 0$.
(c) This follows, since if $g(a) \neq 0$ the original statement would imply f/g is continuous at $x = a$, but we are told it is not.
(d) This does not follow. Given that f is continuous at $x = a$ and $g(a) \neq 0$, then the original statement says g continuous implies f/g continuous, not the other way around. In fact, statement (d) is not true: if $f(x) = 0$ for all x, then g could be any discontinuous, non-zero function, and f/g would be zero, and therefore continuous. Thus the conditions of the statement would be satisfied, but not the conclusion.

PROJECTS FOR CHAPTER ONE

1. Notice that whenever x increases by 0.5, $f(x)$ increases by 1, indicating that $f(x)$ is linear. By inspection, we see that $f(x) = 2x$.

Similarly, $g(x)$ decreases by 1 each time x increases by 0.5. We know, therefore, that $g(x)$ is a linear function with slope $\frac{-1}{0.5} = -2$. The y-intercept is 10, so $g(x) = 10 - 2x$.

$h(x)$ is an even function which is always positive. Comparing the values of x and $h(x)$, it appears that $h(x) = x^2$.

$F(x)$ is an odd function that seems to vary between -1 and 1. We guess that $F(x) = \sin x$ and check with a calculator.

$G(x)$ is also an odd function that varies between -1 and 1. Notice that $G(x) = F(2x)$, and thus $G(x) = \sin 2x$.

Notice also that $H(x)$ is exactly 2 more than $F(x)$ for all x, so $H(x) = 2 + \sin x$.

2. **(a)** Compounding daily (continuously),

$$\begin{aligned} P &= P_0 e^{rt} \\ &= \$450{,}000 e^{(0.06)(213)} \\ &\approx \$1.5977 \cdot 10^{11}. \end{aligned}$$

This amounts to approximately \$160 billion.

(b) Compounding yearly,

$$\begin{aligned} A &= \$450{,}000\,(1 + 0.06)^{213} \\ &= \$450{,}000(1.06)^{213} \approx \$450{,}000(245{,}555.29) \\ &\approx \$1.10499882 \cdot 10^{11}. \end{aligned}$$

This is only about \$110.5 billion.

(c) We first wish to find the interest that will accrue during 1990. For 1990, the principal is $\$1.105 \cdot 10^{11}$. At 6% annual interest, during 1990 the money will earn

$$0.06 \cdot \$1.105 \cdot 10^{11} = \$6.63 \cdot 10^9.$$

The number of seconds in a year is

$$\left(365 \frac{\text{days}}{\text{year}}\right)\left(24 \frac{\text{hours}}{\text{day}}\right)\left(60 \frac{\text{mins}}{\text{hour}}\right)\left(60 \frac{\text{secs}}{\text{min}}\right) = 31{,}536{,}000 \text{ sec}.$$

Thus, over 1990, interest is accumulating at the rate of

$$\frac{\$6.63 \cdot 10^9}{31{,}536{,}000 \text{ sec}} \approx \$210.24 \text{ /sec}.$$