# CHAPTER THREE

## Solutions for Section 3.1

### Exercises

**1.** The derivative, $f'(x)$, is defined as
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$
If $f(x) = 7$, then
$$f'(x) = \lim_{h\to 0} \frac{7-7}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

**2.** The definition of the derivative says that
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$
Therefore,
$$f'(x) = \lim_{h\to 0} \frac{[17(x+h)+11] - [17x+11]}{h} = \lim_{h\to 0} \frac{17h}{h} = 17.$$

**3.** $y' = 11x^{10}$.

**4.** $y' = 12x^{11}$.

**5.** $y' = -12x^{-13}$.

**6.** $y' = 11x^{-12}$.

**7.** $y' = 3.2x^{2.2}$.

**8.** $y' = \frac{4}{3}x^{1/3}$.

**9.** $y' = \frac{3}{4}x^{-1/4}$.

**10.** $y' = -\frac{3}{4}x^{-7/4}$.

**11.** $f'(x) = -4x^{-5}$.

**12.** Since $g(t) = \dfrac{1}{t^5} = t^{-5}$, we have $g'(t) = -5t^{-6}$.

**13.** Since $f(z) = -\dfrac{1}{z^{6.1}} = -z^{-6.1}$, we have $f'(z) = -(-6.1)z^{-7.1} = 6.1z^{-7.1}$.

**14.** Since $y = \dfrac{1}{r^{7/2}} = r^{-7/2}$, we have $\dfrac{dy}{dx} = -\dfrac{7}{2}r^{-9/2}$.

**15.** Since $y = \sqrt{x} = x^{1/2}$, we have $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2}$.

**16.** $f'(x) = \frac{1}{4}x^{-3/4}$.

**17.** Since $h(\theta) = \dfrac{1}{\sqrt[3]{\theta}} = \theta^{-1/3}$, we have $h'(\theta) = -\dfrac{1}{3}\theta^{-4/3}$.

**18.** Since $f(x) = \sqrt{\dfrac{1}{x^3}} = \dfrac{1}{x^{3/2}} = x^{-3/2}$, we have $f'(x) = -\dfrac{3}{2}x^{-5/2}$.

**19.** $f'(x) = ex^{e-1}$.

**20.** $y' = 6x^{1/2} - \frac{5}{2}x^{-1/2}$.

**21.** $f'(t) = 6t - 4$.

**22.** $y' = 17 + 12x^{-1/2}$.

**23.** $y' = 2z - \frac{1}{2z^2}$.

**24.** The power rule gives $f'(x) = 20x^3 - \dfrac{2}{x^3}$.

**25.** $h'(w) = 6w^{-4} + \dfrac{3}{2}w^{-1/2}$

**26.** $y' = 18x^2 + 8x - 2$.

**27.** $y' = 15t^4 - \frac{5}{2}t^{-1/2} - \frac{7}{t^2}$.

**28.** $y' = 6t - \frac{6}{t^{3/2}} + \frac{2}{t^3}$.

**29.** Since $y = \sqrt{x}(x+1) = x^{1/2}x + x^{1/2}\cdot 1 = x^{3/2} + x^{1/2}$, we have $\dfrac{dy}{dx} = \dfrac{3}{2}x^{1/2} + \dfrac{1}{2}x^{-1/2}$.

**30.** Since $y = t^{3/2}(2+\sqrt{t}) = 2t^{3/2} + t^{3/2}t^{1/2} = 2t^{3/2} + t^2$, we have $\frac{dy}{dx} = 3t^{1/2} + 2t$.

**31.** Since $h(t) = \dfrac{3}{t} + \dfrac{4}{t^2} = 3t^{-1} + 4t^{-2}$, we have $h'(t) = -3t^{-2} - 8t^{-3}$.

**32.** Since $y = \sqrt{\theta}\left(\sqrt{\theta} + \dfrac{1}{\sqrt{\theta}}\right) = \theta^{1/2}\theta^{1/2} + \dfrac{\sqrt{\theta}}{\sqrt{\theta}} = \theta + 1$, we have $\frac{dy}{dx} = 1$.

**33.** $y = x + \frac{1}{x}$, so $y' = 1 - \frac{1}{x^2}$.

**34.** $f(z) = \dfrac{z}{3} + \dfrac{1}{3}z^{-1} = \dfrac{1}{3}\left(z + z^{-1}\right)$, so $f'(z) = \dfrac{1}{3}\left(1 - z^{-2}\right) = \dfrac{1}{3}\left(\dfrac{z^2-1}{z^2}\right)$.

**35.** $f(t) = \dfrac{1}{t^2} + \dfrac{1}{t} - \dfrac{1}{t^4} = t^{-2} + t^{-1} - t^{-4}$
$f'(t) = -2t^{-3} - t^{-2} + 4t^{-5}$.

**36.** $y = \frac{\theta}{\sqrt{\theta}} - \frac{1}{\sqrt{\theta}} = \sqrt{\theta} - \frac{1}{\sqrt{\theta}}$
$y' = \frac{1}{2\sqrt{\theta}} + \frac{1}{2\theta^{3/2}}$.

**37.** $j'(x) = \dfrac{3x^2}{a} + \dfrac{2ax}{b} - c$

**38.** Since $f(x) = \dfrac{ax+b}{x} = \dfrac{ax}{x} + \dfrac{b}{x} = a + bx^{-1}$, we have $f'(x) = -bx^{-2}$.

**39.** Since $h(x) = \dfrac{ax+b}{c} = \dfrac{a}{c}x + \dfrac{b}{c}$, we have $h'(x) = \dfrac{a}{c}$.

**40.** Since $g(t) = \dfrac{\sqrt{t}(1+t)}{t^2} = \dfrac{t^{1/2}\cdot 1 + t^{1/2}t}{t^2} = \dfrac{t^{1/2}}{t^2} + \dfrac{t^{3/2}}{t^2} = t^{-3/2} + t^{-1/2}$, we have $g'(t) = -\dfrac{3}{2}t^{-5/2} - \dfrac{1}{2}t^{-3/2}$.

**41.** Since $4/3$, $\pi$, and $b$ are all constants, we have

$$\frac{dV}{dr} = \frac{4}{3}\pi(2r)b = \frac{8}{3}\pi rb.$$

**42.** Since $w$ is a constant times $q$, we have $dw/dq = 3ab^2$.

**43.** Since $a$, $b$, and $c$ are all constants, we have

$$\frac{dy}{dx} = a(2x) + b(1) + 0 = 2ax + b.$$

**44.** Since $a$ and $b$ are constants, we have

$$\frac{dP}{dt} = 0 + b\frac{1}{2}t^{-1/2} = \frac{b}{2\sqrt{t}}.$$

## Problems

**45.** So far, we can only take the derivative of powers of $x$ and the sums of constant multiples of powers of $x$. Since we cannot write $\sqrt{x+3}$ in this form, we cannot yet take its derivative.

**46.** The $x$ is in the exponent and we haven't learned how to handle that yet.

**47.** $g'(x) = \pi x^{(\pi-1)} + \pi x^{-(\pi+1)}$, by the power and sum rules.

**48.** $y' = 6x$. (power rule and sum rule)

**49.** We cannot write $\frac{1}{3x^2+4}$ as the sum of powers of $x$ multiplied by constants.

**50.** $y' = -2/3z^3$. (power rule and sum rule)

**51.** $f'(t) = 6t^2 - 8t + 3$ and $f''(t) = 12t - 8$.

**52.**

$$f'(x) = -8 + 2\sqrt{2}x$$
$$f'(r) = -8 + 2\sqrt{2}r = 4$$
$$r = \frac{12}{2\sqrt{2}} = 3\sqrt{2}.$$

**53.** Differentiating gives

$$f'(x) = 6x^2 - 4x \quad \text{so} \quad f'(1) = 6 - 4 = 2.$$

Thus the equation of the tangent line is $(y-1) = 2(x-1)$ or $y = 2x - 1$.

**54.** (a) We have $f(2) = 8$, so a point on the tangent line is $(2, 8)$. Since $f'(x) = 3x^2$, the slope of the tangent is given by

$$m = f'(2) = 3(2)^2 = 12.$$

Thus, the equation is

$$y - 8 = 12(x - 2) \quad \text{or} \quad y = 12x - 16.$$

(b) See Figure 3.1. The tangent line lies below the function $f(x) = x^3$, so estimates made using the tangent line are underestimates.

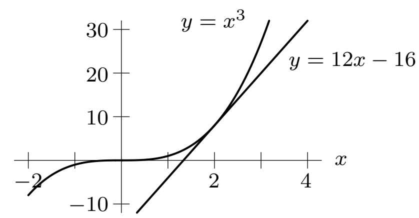



**Figure 3.1**

**55.**

$$\begin{aligned} f'(x) = 12x^2 + 12x - 23 &\geq 1 \\ 12x^2 + 12x - 24 &\geq 0 \\ 12(x^2 + x - 2) &\geq 0 \\ 12(x+2)(x-1) &\geq 0. \end{aligned}$$

Hence $x \geq 1$ or $x \leq -2$.

**56.** The slopes of the tangent lines to $y = x^2 - 2x + 4$ are given by $y' = 2x - 2$. A line through the origin has equation $y = mx$. So, at the tangent point, $x^2 - 2x + 4 = mx$ where $m = y' = 2x - 2$.

$$\begin{aligned} x^2 - 2x + 4 &= (2x - 2)x \\ x^2 - 2x + 4 &= 2x^2 - 2x \\ -x^2 + 4 &= 0 \\ -(x+2)(x-2) &= 0 \\ x &= 2, -2. \end{aligned}$$

Thus, the points of tangency are $(2, 4)$ and $(-2, 12)$. The lines through these points and the origin are $y = 2x$ and $y = -6x$, respectively. Graphically, this can be seen in Figure 3.2:

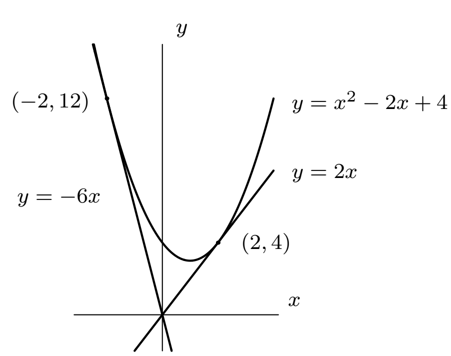



**Figure 3.2**

**57.** Decreasing means $f'(x) < 0$:
$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3),$$
so $f'(x) < 0$ when $x < 3$ and $x \neq 0$. Concave up means $f''(x) > 0$:
$$f''(x) = 12x^2 - 24x = 12x(x-2)$$
so $f''(x) > 0$ when
$$12x(x-2) > 0$$
$$x < 0 \quad \text{or} \quad x > 2.$$
So, both conditions hold for $x < 0$ or $2 < x < 3$.

**58.** The graph increases when $dy/dx > 0$:
$$\frac{dy}{dx} = 5x^4 - 5 > 0$$
$$5(x^4 - 1) > 0 \quad \text{so} \quad x^4 > 1 \quad \text{so} \quad x > 1 \text{ or } x < -1.$$
The graph is concave up when $d^2y/dx^2 > 0$:
$$\frac{d^2y}{dx^2} = 20x^3 > 0 \quad \text{so} \quad x > 0.$$
We need values of $x$ where $\{x > 1 \text{ or } x < -1\}$ AND $\{x > 0\}$, which implies $x > 1$. Thus, both conditions hold for all values of $x$ larger than 1.

**59.** Since $f(x) = x^3 - 6x^2 - 15x + 20$, we have $f'(x) = 3x^2 - 12x - 15$. To find the points at which $f'(x) = 0$, we solve
$$3x^2 - 12x - 15 = 0$$
$$3(x^2 - 4x - 5) = 0$$
$$3(x+1)(x-5) = 0.$$
We see that $f'(x) = 0$ at $x = -1$ and at $x = 5$. The graph of $f(x)$ in Figure 3.3 appears to be horizontal at $x = -1$ and at $x = 5$, confirming what we found analytically.

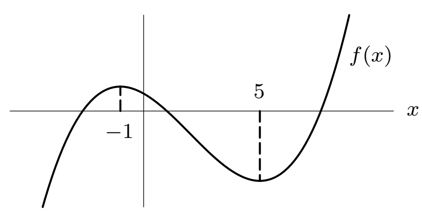



**Figure 3.3**

**60.** (a) Since the power of $x$ will go down by one every time you take a derivative (until the exponent is zero after which the derivative will be zero), we can see immediately that $f^{(8)}(x) = 0$.
(b) $f^{(7)}(x) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot x^0 = 5040$.

**61.** Since $f(t) = 700 - 3t^2$, we have $f(5) = 700 - 3(25) = 625$ cm. Since $f'(t) = -6t$, we have $f'(5) = -30$ cm/year. In the year 2000, the sand dune was 625 cm high and it was eroding at a rate of 30 centimeters per year.

**62.** (a) Velocity $v(t) = \frac{dy}{dt} = \frac{d}{dt}(1250 - 16t^2) = -32t$.
Since $t \geq 0$, the ball's velocity is negative. This is reasonable, since its height $y$ is decreasing.
(b) Acceleration $a(t) = \frac{dv}{dt} = \frac{d}{dt}(-32t) = -32$.
So its acceleration is the negative constant $-32$.
(c) The ball hits the ground when its height $y = 0$. This gives
$$1250 - 16t^2 = 0$$
$$t = \pm 8.84 \text{ seconds}$$
We discard $t = -8.84$ because time $t$ is nonnegative. So the ball hits the ground $8.84$ seconds after its release, at which time its velocity is
$$v(8.84) = -32(8.84) = -282.88 \text{ feet/sec} = -192.84 \text{ mph}.$$

**63.** (a) The average velocity between $t = 0$ and $t = 2$ is given by

$$\text{Average velocity} = \frac{f(2) - f(0)}{2 - 0} = \frac{-4.9(2^2) + 25(2) + 3 - 3}{2 - 0} = \frac{33.4 - 3}{2} = 15.2 \text{ m/sec.}$$

(b) Since $f'(t) = -9.8t + 25$, we have

$$\text{Instantaneous velocity} = f'(2) = -9.8(2) + 25 = 5.4 \text{ m/sec.}$$

(c) Acceleration is given $f''(t) = -9.8$. The acceleration at $t = 2$ (and all other times) is the acceleration due to gravity, which is $-9.8$ m/sec$^2$.

(d) We can use a graph of height against time to estimate the maximum height of the tomato. See Figure 3.4. Alternately, we can find the answer analytically. The maximum height occurs when the velocity is zero and $v(t) = -9.8t + 25 = 0$ when $t = 2.6$ sec. At this time the tomato is at a height of $f(2.6) = 34.9$. The maximum height is 34.9 meters.

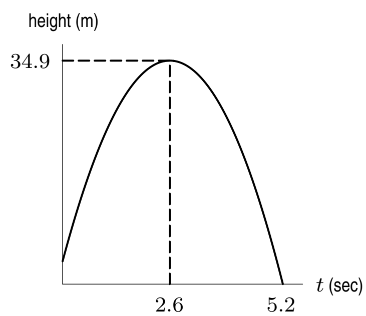



**Figure 3.4**

(e) We see in Figure 3.4 that the tomato hits ground at about $t = 5.2$ seconds. Alternately, we can find the answer analytically. The tomato hits the ground when

$$f(t) = -4.9t^2 + 25t + 3 = 0.$$

We solve for $t$ using the quadratic formula:

$$t = \frac{-25 \pm \sqrt{(25)^2 - 4(-4.9)(3)}}{2(-4.9)}$$
$$t = \frac{-25 \pm \sqrt{683.8}}{-9.8}$$
$$t = -0.12 \quad \text{and} \quad t = 5.2.$$

We use the positive values, so the tomato hits the ground at $t = 5.2$ seconds.

**64.** $\dfrac{dF}{dr} = -\dfrac{2GMm}{r^3}$.

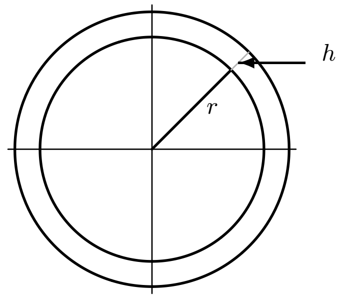
**65.** (a) $T = 2\pi\sqrt{\dfrac{l}{g}} = \dfrac{2\pi}{\sqrt{g}}\left(l^{\frac{1}{2}}\right)$, so $\dfrac{dT}{dl} = \dfrac{2\pi}{\sqrt{g}}\left(\dfrac{1}{2}l^{-\frac{1}{2}}\right) = \dfrac{\pi}{\sqrt{gl}}$.

(b) Since $\dfrac{dT}{dl}$ is positive, the period $T$ increases as the length $l$ increases.

**66.** (a) $A = \pi r^2$
$\frac{dA}{dr} = 2\pi r$.

(b) This is the formula for the circumference of a circle.

(c) $A'(r) \approx \frac{A(r+h) - A(r)}{h}$ for small $h$. When $h > 0$, the numerator of the difference quotient denotes the area of the region contained between the inner circle (radius $r$) and the outer circle (radius $r + h$). See figure below. As $h$ approaches 0, this area can be approximated by the product of the circumference of the inner circle and the "width" of the region, i.e., $h$. Dividing this by the denominator, $h$, we get $A'$ = the circumference of the circle with radius $r$.




We can also think about the derivative of $A$ as the rate of change of area for a small change in radius. If the radius increases by a tiny amount, the area will increase by a thin ring whose area is simply the circumference at that radius times the small amount. To get the rate of change, we divide by the small amount and obtain the circumference.

**67.** $V = \frac{4}{3}\pi r^3$. Differentiating gives $\frac{dV}{dr} = 4\pi r^2 =$ surface area of a sphere.

The difference quotient $\frac{V(r+h)-V(r)}{h}$ is the volume between two spheres divided by the change in radius. Furthermore, when $h$ is very small, the difference between volumes, $V(r+h) - V(r)$, is like a coating of paint of depth $h$ applied to the surface of the sphere. The volume of the paint is about $h \cdot$ (Surface Area) for small $h$: dividing by $h$ gives back the surface area.

Thinking about the derivative as the rate of change of the function for a small change in the variable gives another way of seeing the result. If you increase the radius of a sphere a small amount, the volume increases by a very thin layer whose volume is the surface area at that radius multiplied by that small amount.

**68.** If $f(x) = x^n$, then $f'(x) = nx^{n-1}$. This means $f'(1) = n \cdot 1^{n-1} = n \cdot 1 = n$, because any power of 1 equals 1.

**69.** Since $f(x) = ax^n$, $f'(x) = anx^{n-1}$. We know that $f'(2) = (an)2^{n-1} = 3$, and $f'(4) = (an)4^{n-1} = 24$. Therefore,

$$\begin{aligned}\frac{f'(4)}{f'(2)} &= \frac{24}{3}\\ \frac{(an)4^{n-1}}{(an)2^{n-1}} = \left(\frac{4}{2}\right)^{n-1} &= 8\\ 2^{n-1} &= 8, \text{ and thus } n = 4.\end{aligned}$$

Substituting $n = 4$ into the expression for $f'(2)$, we get $3 = a(4)(8)$, or $a = 3/32$.

**70.** Yes. To see why, we substitute $y = x^n$ into the equation $13x\dfrac{dy}{dx} = y$. We first calculate $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^n) = nx^{n-1}$. The differential equation becomes

$$13x(nx^{n-1}) = x^n$$

But $13x(nx^{n-1}) = 13n(x \cdot x^{n-1}) = 13nx^n$, so we have

$$13n(x^n) = x^n$$

This equality must hold for all $x$, so we get $13n = 1$, so $n = 1/13$. Thus, $y = x^{1/13}$ is a solution.

**71.** (a)

$$\begin{aligned}\frac{d(x^{-1})}{dx} &= \lim_{h\to 0}\frac{(x+h)^{-1} - x^{-1}}{h} = \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{x+h} - \frac{1}{x}\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{x-(x+h)}{x(x+h)}\right] = \lim_{h\to 0}\frac{1}{h}\left[\frac{-h}{x(x+h)}\right]\\ &= \lim_{h\to 0}\frac{-1}{x(x+h)} = \frac{-1}{x^2} = -1x^{-2}.\end{aligned}$$

$$\begin{aligned}\frac{d(x^{-3})}{dx} &= \lim_{h\to 0}\frac{(x+h)^{-3} - x^{-3}}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{(x+h)^3} - \frac{1}{x^3}\right]\end{aligned}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^3-(x+h)^3}{x^3(x+h)^3}\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^3-(x^3+3hx^2+3h^2x+h^3)}{x^3(x+h)^3}\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-3hx^2-3xh^2-h^3}{x^3(x+h)^3}\right]$$
$$= \lim_{h\to 0} \frac{-3x^2-3xh-h^2}{x^3(x+h)^3}$$
$$= \frac{-3x^2}{x^6} = -3x^{-4}.$$

(b) For clarity, let $n=-k$, where $k$ is a positive integer. So $x^n = x^{-k}$.

$$\frac{d(x^{-k})}{dx} = \lim_{h\to 0} \frac{(x+h)^{-k}-x^{-k}}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^k}-\frac{1}{x^k}\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^k-(x+h)^k}{x^k(x+h)^k}\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^k-x^k-khx^{k-1}-\overbrace{\ldots-h^k}^{\text{terms involving } h^2 \text{ and higher powers of } h}}{x^k(x+h)^k}\right]$$
$$= \frac{-kx^{k-1}}{x^k(x)^k} = \frac{-k}{x^{k+1}} = -kx^{-(k+1)} = -kx^{-k-1}.$$

## Solutions for Section 3.2

### Exercises

**1.** $f'(x) = 2e^x + 2x.$

**2.** $y' = 10t + 4e^t.$

**3.** $y' = (\ln 5)5^x.$

**4.** $f'(x) = 12e^x + (\ln 11)11^x.$

**5.** $y' = 10x + (\ln 2)2^x.$

**6.** $f'(x) = (\ln 2)2^x + 2(\ln 3)3^x.$

**7.** $\frac{dy}{dx} = 4(\ln 10)10^x - 3x^2.$

**8.** $\frac{dy}{dx} = 3 - 2(\ln 4)4^x.$

**9.** Since $y = 2^x + \frac{2}{x^3} = 2^x + 2x^{-3}$, we have $\frac{dy}{dx} = (\ln 2)2^x - 6x^{-4}.$

**10.** $\frac{dy}{dx} = \frac{1}{3}(\ln 3)3^x - \frac{33}{2}(x^{-\frac{3}{2}}).$

**11.** $z' = (\ln 4)e^x.$

**12.** $z' = (\ln 4)^2 4^x.$

**13.** $f'(t) = (\ln(\ln 3))(\ln 3)^t.$

**14.** $\frac{dy}{dx} = 5 \cdot 5^t \ln 5 + 6 \cdot 6^t \ln 6$

**15.** $h'(z) = (\ln(\ln 2))(\ln 2)^z.$

**16.** $f'(x) = ex^{e-1}.$

**17.** $f'(x) = 3x^2 + 3^x \ln 3$

**18.** $\dfrac{dy}{dx} = \pi^x \ln \pi$

**19.** $f'(x) = (\ln \pi)\pi^x$.

**20.** This is the sum of an exponential function and a power function, so $f'(x) = \ln(\pi)\pi^x + \pi x^{\pi-1}$.

**21.** Since $e$ and $k$ are constants, $e^k$ is constant, so we have $f'(x) = (\ln k)k^x$.

**22.** $f(x) = e^{1+x} = e^1 \cdot e^x$. Then, since $e^1$ is just a constant,
$f'(x) = e \cdot e^x = e^{1+x}$.

**23.** $f(t) = e^t \cdot e^2$. Then, since $e^2$ is just a constant, $f'(t) = \frac{d}{dt}(e^t e^2) = e^2 \frac{d}{dt} e^t = e^2 e^t = e^{t+2}$.

**24.** $y = e^\theta e^{-1} \quad y' = \dfrac{d}{d\theta}(e^\theta e^{-1}) = e^{-1}\dfrac{d}{d\theta}e^\theta = e^\theta e^{-1} = e^{\theta-1}$.

**25.** $y'(x) = a^x \ln a + ax^{a-1}$.

**26.** $f'(x) = \pi^2 x^{(\pi^2-1)} + (\pi^2)^x \ln(\pi^2)$

**27.** $y' = 2x + (\ln 2)2^x$.

**28.** $y' = \frac{1}{2}x^{-\frac{1}{2}} - \ln\frac{1}{2}(\frac{1}{2})^x = \frac{1}{2\sqrt{x}} + \ln 2(\frac{1}{2})^x$.

**29.** We can take the derivative of the sum $x^2 + 2^x$, but not the product.

**30.** Once again, this is a product of two functions, $2^x$ and $\frac{1}{x}$, each of which we can take the derivative of; but we don't know how to take the derivative of the product.

**31.** Since $y = e^5 e^x$, $y' = e^5 e^x = e^{x+5}$.

**32.** $y = e^{5x} = (e^5)^x$, so $y' = \ln(e^5) \cdot (e^5)^x = 5e^{5x}$.

**33.** The exponent is $x^2$, and we haven't learned what to do about that yet.

**34.** $f'(z) = (\ln\sqrt{4})(\sqrt{4})^z = (\ln 2)2^z$.

**35.** We can't use our rules if the exponent is $\sqrt{\theta}$.

## Problems

**36.** Since $P = 1 \cdot (1.05)^t$, $\frac{dP}{dt} = \ln(1.05)1.05^t$. When $t = 10$,

$$\frac{dP}{dt} = (\ln 1.05)(1.05)^{10} \approx \$0.07947/\text{year} \approx 7.95¢/\text{year}.$$

**37.**

$$\frac{dP}{dt} = 35{,}000 \cdot (\ln 0.98)(0.98^t).$$

At $t = 23$, this is $35{,}000(\ln 0.98)(0.98^{23}) \approx -444.3$ people/year. (Note: the negative sign indicates that the population is decreasing.)

**38.** We have $f(t) = 5.3(1.018)^t$ so $f'(t) = 5.3(\ln 1.018)(1.018)^t = 0.095(1.018)^t$. Therefore

$$f(0) = 5.3 \text{ billion people}$$

and

$$f'(0) = 0.095 \text{ billion people per year.}$$

In 1990, the population of the world was 5.3 billion people and was increasing at a rate of 0.095 billion people per year.

We also have

$$f(30) = 5.3(1.018)^{30} = 9.1 \text{ billion people,}$$

and

$$f'(30) = 0.095(1.018)^{30} = 0.16 \text{ billion people per year.}$$

In the year 2020, this model predicts that the population of the world will be 9.1 billion people and will be increasing at a rate of 0.16 billion people per year.

**39.** $\dfrac{dV}{dt} = 75(1.35)^t \ln 1.35 \approx 22.5(1.35)^t$.

**40.** (a) $V(4) = 25(0.85)^4 = 25(0.522) = 13{,}050$. Thus the value of the car after 4 years is \$13,050.

(b) We have a function of the form $f(t) = Ca^t$. We know that such functions have a derivative of the form $(C \ln a) \cdot a^t$. Thus, $V'(t) = 25(0.85)^t \cdot \ln 0.85 = -4.063(0.85)^t$. The units would be the change in value (in thousands of dollars) with respect to time (in years), or thousands of dollars/year.

(c) $V'(4) = -4.063(0.85)^4 = -4.063(0.522) = -2.121$. This means that at the end of the fourth year, the value of the car is decreasing by \$2121 per year.

(d) $V(t)$ is a positive decreasing function, so that the value of the automobile is positive and decreasing. $V'(t)$ is a negative function whose magnitude is decreasing, meaning the value of the automobile is always dropping, but the yearly loss of value is less as time goes on. The graphs of $V(t)$ and $V'(t)$ confirm that the value of the car decreases with time. What they do not take into account are the *costs* associated with owning the vehicle. At some time, $t$, it is likely that the yearly costs of owning the vehicle will outweigh its value. At that time, it may no longer be worthwhile to keep the car.

**41.** (a) $f(x) = 1 - e^x$ crosses the $x$-axis where $0 = 1 - e^x$, which happens when $e^x = 1$, so $x = 0$. Since $f'(x) = -e^x$, $f'(0) = -e^0 = -1$.

(b) $y = -x$

(c) The negative of the reciprocal of $-1$ is 1, so the equation of the normal line is $y = x$.

**42.** Since $y = 2^x$, $y' = (\ln 2)2^x$. At $(0, 1)$, the tangent line has slope $\ln 2$ so its equation is $y = (\ln 2)x + 1$. At $c$, $y = 0$, so $0 = (\ln 2)c + 1$, thus $c = -\frac{1}{\ln 2}$.

**43.**

$$g(x) = ax^2 + bx + c \qquad f(x) = e^x$$
$$g'(x) = 2ax + b \qquad f'(x) = e^x$$
$$g''(x) = 2a \qquad f''(x) = e^x$$

So, using $g''(0) = f''(0)$, etc., we have $2a = 1$, $b = 1$, and $c = 1$, and thus $g(x) = \frac{1}{2}x^2 + x + 1$, as shown in Figure 3.5.

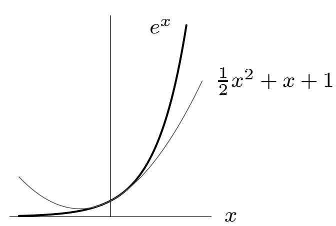
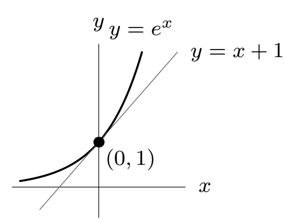



**Figure 3.5**

The two functions do look very much alike near $x = 0$. They both increase for large values of $x$, but $e^x$ increases much more quickly. For very negative values of $x$, the quadratic goes to $\infty$ whereas the exponential goes to 0. By choosing a function whose first few derivatives agreed with the exponential when $x = 0$, we got a function which looks like the exponential for $x$-values near 0.

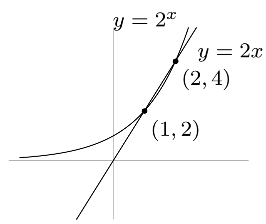
**44.** The first and second derivatives of $e^x$ are $e^x$. Thus, the graph of $y = e^x$ is concave up. The tangent line at $x = 0$ has slope $e^0 = 1$ and equation $y = x + 1$. A graph that is always concave up is always above any of its tangent lines. Thus $e^x \geq x + 1$ for all $x$, as shown in the following figure.



**45.** The equation $2^x = 2x$ has solutions $x = 1$ and $x = 2$. (Check this by substituting these values into the equation). The graph below suggests that these are the only solutions, but how can we be sure?

Let's look at the slope of the curve $f(x) = 2^x$, which is $f'(x) = (\ln 2)2^x \approx (0.693)2^x$, and the slope of the line $g(x) = 2x$ which is 2. At $x = 1$, the slope of $f(x)$ is less than 2; at $x = 2$, the slope of $f(x)$ is more than 2. Since the slope of $f(x)$ is always increasing, there can be no other point of intersection. (If there were another point of intersection, the graph $f$ would have to "turn around".)

Here's another way of seeing this. Suppose $g(x)$ represents the position of a car going a steady 2 mph, while $f(x)$ represents a car which starts ahead of $g$ (because the graph of $f$ is above $g$) and is initially going slower than $g$. The car $f$ is first overtaken by $g$. All the while, however, $f$ is speeding up until eventually it overtakes $g$ again. Notice that the two cars will only meet twice (corresponding to the two intersections of the curve): once when $g$ overtakes $f$ and once when $f$ overtakes $g$.




**46.** For $x = 0$, we have $y = a^0 = 1$ and $y = 1 + 0 = 1$, so both curves go through the point $(0, 1)$ for all values of $a$.

Differentiating gives

$$\left.\frac{d(a^x)}{dx}\right|_{x=0} = a^x \ln a|_{x=0} = a^0 \ln a = \ln a$$

$$\left.\frac{d(1+x)}{dx}\right|_{x=0} = 1.$$

The graphs are tangent at $x = 0$ if

$$\ln a = 1 \qquad \text{so} \qquad a = e.$$

## Solutions for Section 3.3

### Exercises

**1.** By the product rule, $f'(x) = 2x(x^3 + 5) + x^2(3x^2) = 2x^4 + 3x^4 + 10x = 5x^4 + 10x$. Alternatively, $f'(x) = (x^5 + 5x^2)' = 5x^4 + 10x$. The two answers should, and do, match.

**2.** Using the product rule,

$$f'(x) = (\ln 2)2^x 3^x + (\ln 3)2^x 3^x = (\ln 2 + \ln 3)(2^x \cdot 3^x) = \ln(2 \cdot 3)(2 \cdot 3)^x = (\ln 6)6^x$$

or, since $2^x \cdot 3^x = (2 \cdot 3)^x = 6^x$,

$$f'(x) = (6^x)' = (\ln 6)(6^x).$$

The two answers should, and do, match.

**3.** $f'(x) = x \cdot e^x + e^x \cdot 1 = e^x(x + 1)$.

**4.** $y' = 2^x + x(\ln 2)2^x = 2^x(1 + x \ln 2)$.

**5.** $y' = \frac{1}{2\sqrt{x}}2^x + \sqrt{x}(\ln 2)2^x$.

**6.** $\dfrac{dy}{dt} = 2te^t + (t^2 + 3)e^t = e^t(t^2 + 2t + 3)$.

**7.** $f'(x) = (x^2 - x^{\frac{1}{2}}) \cdot 3^x(\ln 3) + 3^x \left(2x - \frac{1}{2}x^{-\frac{1}{2}}\right) = 3^x \left[(\ln 3)(x^2 - x^{\frac{1}{2}}) + \left(2x - \frac{1}{2\sqrt{x}}\right)\right]$.

**8.** It is easier to do this by multiplying it out first, rather than using the product rule first: $z = s^4 - s, \quad z' = 4s^3 - 1.$

**9.** $f'(y) = (\ln 4)4^y(2 - y^2) + 4^y(-2y) = 4^y((\ln 4)(2 - y^2) - 2y).$

**10.** $y' = (3t^2 - 14t)e^t + (t^3 - 7t^2 + 1)e^t = (t^3 - 4t^2 - 14t + 1)e^t.$

**11.** $f'(x) = \dfrac{e^x \cdot 1 - x \cdot e^x}{(e^x)^2} = \dfrac{e^x(1 - x)}{(e^x)^2} = \dfrac{1 - x}{e^x}.$

**12.** $g'(x) = \dfrac{50xe^x - 25x^2e^x}{e^{2x}} = \dfrac{50x - 25x^2}{e^x}.$

**13.** $\dfrac{dy}{dx} = \dfrac{1 \cdot 2^t - (t + 1)(\ln 2)2^t}{(2^t)^2} = \dfrac{2^t(1 - (t + 1)\ln 2)}{(2^t)^2} = \dfrac{1 - (t + 1)\ln 2}{2^t}$

**14.** $g'(w) = \dfrac{3.2w^{2.2}(5^w) - (\ln 5)(w^{3.2})5^w}{5^{2w}} = \dfrac{3.2w^{2.2} - w^{3.2}(\ln 5)}{5^w}.$

**15.** $q'(r) = \dfrac{3(5r + 2) - 3r(5)}{(5r + 2)^2} = \dfrac{15r + 6 - 15r}{(5r + 2)^2} = \dfrac{6}{(5r + 2)^2}$

**16.** $g'(t) = \dfrac{(t + 4) - (t - 4)}{(t + 4)^2} = \dfrac{8}{(t + 4)^2}.$

**17.** $\dfrac{dz}{dt} = \dfrac{3(5t + 2) - (3t + 1)5}{(5t + 2)^2} = \dfrac{15t + 6 - 15t - 5}{(5t + 2)^2} = \dfrac{1}{(5t + 2)^2}.$

**18.** $z' = \dfrac{(2t + 5)(t + 3) - (t^2 + 5t + 2)}{(t + 3)^2} = \dfrac{t^2 + 6t + 13}{(t + 3)^2}.$

**19.** Using the quotient rule gives $\dfrac{dz}{dt} = \dfrac{(2t + 3)(t + 1) - (t^2 + 3t + 1)}{(t + 1)^2}$ or $\dfrac{dz}{dt} = \dfrac{t^2 + 2t + 2}{(t + 1)^2}.$

**20.** Divide and then differentiate

$$\begin{aligned} f(x) &= x + \frac{3}{x} \\ f'(x) &= 1 - \frac{3}{x^2}. \end{aligned}$$

**21.** $w = y^2 - 6y + 7. \quad w' = 2y - 6, y \neq 0.$

**22.** $y' = \dfrac{\frac{1}{2\sqrt{t}}(t^2 + 1) - \sqrt{t}(2t)}{(t^2 + 1)^2}.$

**23.** $\dfrac{d}{dz}\left(\dfrac{z^2 + 1}{\sqrt{z}}\right) = \dfrac{d}{dz}(z^{\frac{3}{2}} + z^{-\frac{1}{2}}) = \dfrac{3}{2}z^{\frac{1}{2}} - \dfrac{1}{2}z^{-\frac{3}{2}} = \dfrac{\sqrt{z}}{2}(3 - z^{-2}).$

**24.** $g'(t) = -4(3 + \sqrt{t})^{-2}\left(\dfrac{1}{2}t^{-1/2}\right) = \dfrac{-2}{\sqrt{t}(3 + \sqrt{t})^2}$

**25.** $h'(r) = \dfrac{d}{dr}\left(\dfrac{r^2}{2r + 1}\right) = \dfrac{(2r)(2r + 1) - 2r^2}{(2r + 1)^2} = \dfrac{2r(r + 1)}{(2r + 1)^2}.$

**26.** Notice that you can cancel a $z$ out of the numerator and denominator to get

$$f(z) = \frac{3z}{5z + 7}, \qquad z \neq 0$$

Then

$$\begin{aligned} f'(z) &= \frac{(5z + 7)3 - 3z(5)}{(5z + 7)^2} \\ &= \frac{15z + 21 - 15z}{(5z + 7)^2} \\ &= \frac{21}{(5z + 7)^2}, z \neq 0. \end{aligned}$$

[If you used the quotient rule correctly without canceling the $z$ out first, your answer should simplify to this one, but it is usually a good idea to simplify as much as possible before differentiating.]

**27.** $w'(x) = \dfrac{17e^x(2^x) - (\ln 2)(17e^x)2^x}{2^{2x}} = \dfrac{17e^x(2^x)(1 - \ln 2)}{2^{2x}} = \dfrac{17e^x(1 - \ln 2)}{2^x}.$

**28.** $h'(p) = \dfrac{2p(3+2p^2) - 4p(1+p^2)}{(3+2p^2)^2} = \dfrac{6p+4p^3-4p-4p^3}{(3+2p^2)^2} = \dfrac{2p}{(3+2p^2)^2}.$

**29.**

$$\begin{aligned} f'(x) &= \frac{(2+3x+4x^2)(1) - (1+x)(3+8x)}{(2+3x+4x^2)^2} \\ &= \frac{2+3x+4x^2-3-11x-8x^2}{(2+3x+4x^2)^2} \\ &= \frac{-4x^2-8x-1}{(2+3x+4x^2)^2}. \end{aligned}$$

**30.** We use the quotient rule. We have

$$f'(x) = \frac{(cx+k)(a) - (ax+b)(c)}{(cx+k)^2} = \frac{acx+ak-acx-bc}{(cx+k)^2} = \frac{ak-bc}{(cx+k)^2}.$$

## Problems

**31.** Using the product rule, we know that $h'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$. We use slope to compute the derivatives. Since $f(x)$ is linear on the interval $0 < x < 2$, we compute the slope of the line to see that $f'(x) = 2$ on this interval. Similarly, we compute the slope on the interval $2 < x < 4$ to see that $f'(x) = -2$ on the interval $2 < x < 4$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist.

Similarly, $g(x)$ is linear on the interval shown, and we see that the slope of $g(x)$ on this interval is $-1$ so we have $g'(x) = -1$ on this interval.

(a) We have $h'(1) = f'(1)\cdot g(1) + f(1)\cdot g'(1) = 2\cdot 3 + 2(-1) = 6 - 2 = 4$.
(b) We have $h'(2) = f'(2)\cdot g(2) + f(2)\cdot g'(2)$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, $h'(2)$ does not exist.
(c) We have $h'(3) = f'(3)\cdot g(3) + f(3)\cdot g'(3) = (-2)1 + 2(-1) = -2 - 2 = -4$.

**32.** Using the quotient rule, we know that $k'(x) = (f'(x)\cdot g(x) - f(x)\cdot g'(x))/(g(x))^2$. We use slope to compute the derivatives. Since $f(x)$ is linear on the interval $0 < x < 2$, we compute the slope of the line to see that $f'(x) = 2$ on this interval. Similarly, we compute the slope on the interval $2 < x < 4$ to see that $f'(x) = -2$ on the interval $2 < x < 4$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist.

Similarly, $g(x)$ is linear on the interval shown, and we see that the slope of $g(x)$ on this interval is $-1$ so we have $g'(x) = -1$ on this interval.

(a) We have

$$k'(1) = \frac{f'(1)\cdot g(1) - f(1)\cdot g'(1)}{(g(1))^2} = \frac{2\cdot 3 - 2(-1)}{3^2} = \frac{6+2}{9} = \frac{8}{9}.$$

(b) We have $k'(2) = (f'(2)\cdot g(2) - f(2)\cdot g'(2))/(g(2)^2)$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, $k'(2)$ does not exist.
(c) We have

$$k'(3) = \frac{f'(3)\cdot g(3) - f(3)\cdot g'(3)}{(g(3))^2} = \frac{(-2)1 - 2(-1)}{1^2} = \frac{-2+2}{1} = 0.$$

**33.** Using the quotient rule, we know that $j'(x) = (g'(x)\cdot f(x) - g(x)\cdot f'(x))/(f(x))^2$. We use slope to compute the derivatives. Since $f(x)$ is linear on the interval $0 < x < 2$, we compute the slope of the line to see that $f'(x) = 2$ on this interval. Similarly, we compute the slope on the interval $2 < x < 4$ to see that $f'(x) = -2$ on the interval $2 < x < 4$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist.

Similarly, $g(x)$ is linear on the interval shown, and we see that the slope of $g(x)$ on this interval is $-1$ so we have $g'(x) = -1$ on this interval.

(a) We have

$$j'(1) = \frac{g'(1)\cdot f(1) - g(1)\cdot f'(1)}{(f(1))^2} = \frac{(-1)2 - 3\cdot 2}{2^2} = \frac{-2-6}{4} = \frac{-8}{4} = -2.$$

(b) We have $j'(2) = (g'(2)\cdot f(2) - g(2)\cdot f'(2))/(f(2)^2)$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, $j'(2)$ does not exist.
(c) We have

$$j'(3) = \frac{g'(3)\cdot f(3) - g(3)\cdot f'(3)}{(f(3))^2} = \frac{(-1)2 - 1(-2)}{2^2} = \frac{-2+2}{4} = 0.$$

**34.** From the graphs, we estimate $f(1) \approx -0.4$, $f'(1) \approx 0.5$, $g(1) \approx 2$, and $g'(1) \approx 1$. By the product rule,

$$h'(1) = f'(1) \cdot g(1) + f(1) \cdot g'(1) \approx (0.5)2 + (-0.4)1 = 0.6.$$

**35.** From the graphs, we estimate $f(1) \approx -0.4$, $f'(1) \approx 0.5$, $g(1) \approx 2$, and $g'(1) \approx 1$. By the quotient rule,

$$k'(1) = \frac{f'(1) \cdot g(1) - f(1) \cdot g'(1)}{(g(1))^2} \approx \frac{(0.5)2 - (-0.4)1}{2^2} = 0.35.$$

**36.** From the graphs, we estimate $f(2) \approx 0.3$, $f'(2) \approx 1.1$, $g(2) \approx 1.6$, and $g'(2) \approx -0.5$. By the product rule,

$$h'(2) = f'(2) \cdot g(2) + f(2) \cdot g'(2) \approx 1.1(1.6) + 0.3(-0.5) = 1.61.$$

**37.** From the graphs, we estimate $f(2) \approx 0.3$, $f'(2) \approx 1.1$, $g(2) \approx 1.6$, and $g'(2) \approx -0.5$. By the quotient rule,

$$k'(2) = \frac{f'(2) \cdot g(2) - f(2) \cdot g'(2)}{(g(2))^2} \approx \frac{1.1(1.6) - 0.3(-0.5)}{(1.6)^2} = 0.75.$$

**38.** From the graphs, we estimate $f(1) \approx -0.4$, $f'(1) \approx 0.5$, $g(1) \approx 2$, and $g'(1) \approx 1$. By the quotient rule,

$$l'(1) = \frac{g'(1) \cdot f(1) - g(1) \cdot f'(1)}{(f(1))^2} \approx \frac{1(-0.4) - 2(0.5)}{(-0.4)^2} = -8.75.$$

**39.** From the graphs, we estimate $f(2) \approx 0.3$, $f'(2) \approx 1.1$, $g(2) \approx 1.6$, and $g'(2) \approx -0.5$. By the quotient rule,

$$l'(2) = \frac{g'(2) \cdot f(2) - g(2) \cdot f'(2)}{(f(2))^2} \approx \frac{(-0.5)0.3 - 1.6(1.1)}{(0.3)^2} = -21.22.$$

**40.**

$$\begin{aligned} f'(x) &= 3(2x-5) + 2(3x+8) = 12x + 1 \\ f''(x) &= 12. \end{aligned}$$

**41.**

$$\begin{aligned} f(t) &= \frac{1}{e^t} \\ f'(t) &= \frac{e^t \cdot 0 - e^t \cdot 1}{(e^t)^2} \\ &= \frac{-1}{e^t} = -e^{-t}. \end{aligned}$$

**42.** $f(x) = e^x \cdot e^x$
$f'(x) = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}$.

**43.**

$$\begin{aligned} f(x) &= e^x e^{2x} \\ f'(x) &= e^x (e^{2x})' + (e^x)' e^{2x} \\ &= 2e^x e^{2x} + e^x e^{2x} \text{ (from Problem 42)} \\ &= 3e^{3x}. \end{aligned}$$

**44.** We have

$$\begin{aligned} f'(x) &= e^x + xe^x \\ f''(x) &= e^x + e^x + xe^x = (2+x)e^x. \end{aligned}$$

Since $f(x)$ is concave up when $f''(x) > 0$, we see that $f(x)$ is concave up when $x > -2$.

**45.** Using the quotient rule, we have

$$\begin{aligned}
g'(x) &= \frac{0-1(2x)}{(x^2+1)^2} = \frac{-2x}{(x^2+1)^2} \\
g''(x) &= \frac{-2(x^2+1)^2+2x(4x^3+4x)}{(x^2+1)^4} \\
&= \frac{-2(x^2+1)^2+8x^2(x^2+1)}{(x^2+1)^4} \\
&= \frac{-2(x^2+1)+8x^2}{(x^2+1)^3} \\
&= \frac{2(3x^2-1)}{(x^2+1)^3}.
\end{aligned}$$

Since $(x^2+1)^3 > 0$ for all $x$, we have $g''(x) < 0$ if $(3x^2-1) < 0$, or when

$$\begin{aligned}
3x^2 &< 1 \\
-\frac{1}{\sqrt{3}} < x &< \frac{1}{\sqrt{3}}.
\end{aligned}$$

**46.** Since $f(0) = -5/1 = -5$, the tangent line passes through the point $(0,-5)$, so its vertical intercept is $-5$. To find the slope of the tangent line, we find the derivative of $f(x)$ using the quotient rule:

$$f'(x) = \frac{(x+1)\cdot 2-(2x-5)\cdot 1}{(x+1)^2} = \frac{7}{(x+1)^2}.$$

At $x = 0$, the slope of the tangent line is $m = f'(0) = 7$. The equation of the tangent line is $y = 7x - 5$.

**47.** (a) Although the answer you would get by using the quotient rule is equivalent, the answer looks simpler in this case if you just use the product rule:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{e^x}{x}\right) &= \frac{d}{dx}\left(e^x\cdot\frac{1}{x}\right) = \frac{e^x}{x} - \frac{e^x}{x^2} \\
\frac{d}{dx}\left(\frac{e^x}{x^2}\right) &= \frac{d}{dx}\left(e^x\cdot\frac{1}{x^2}\right) = \frac{e^x}{x^2} - \frac{2e^x}{x^3} \\
\frac{d}{dx}\left(\frac{e^x}{x^3}\right) &= \frac{d}{dx}\left(e^x\cdot\frac{1}{x^3}\right) = \frac{e^x}{x^3} - \frac{3e^x}{x^4}.
\end{aligned}$$

(b) $\dfrac{d}{dx}\dfrac{e^x}{x^n} = \dfrac{e^x}{x^n} - \dfrac{ne^x}{x^{n+1}}.$

**48.**

$$\begin{aligned}
\frac{d(x^2)}{dx} &= \frac{d}{dx}(x\cdot x) \\
&= x\frac{d(x)}{dx} + x\frac{d(x)}{dx} \\
&= 2x.
\end{aligned}$$

$$\begin{aligned}
\frac{d(x^3)}{dx} &= \frac{d}{dx}(x^2\cdot x) \\
&= x^2\frac{d(x)}{dx} + x\frac{d(x^2)}{dx} \\
&= x^2\frac{d(x)}{dx} + x\left[x\frac{d(x)}{dx} + x\frac{d(x)}{dx}\right] \\
&= x^2\frac{d(x)}{dx} + x^2\frac{d(x)}{dx} + x^2\frac{d(x)}{dx} \\
&= 3x^2.
\end{aligned}$$

**49.** Since

$$x^{1/2}\cdot x^{1/2} = x,$$

we differentiate to obtain

$$\frac{d}{dx}(x^{1/2})\cdot x^{1/2} + x^{1/2}\cdot\frac{d}{dx}(x^{1/2}) = 1.$$

Now solve for $d(x^{1/2})/dx$:

$$\begin{aligned} 2x^{1/2}\frac{d}{dx}(x^{1/2}) &= 1 \\ \frac{d}{dx}(x^{1/2}) &= \frac{1}{2x^{1/2}}. \end{aligned}$$

**50.** (a) We have $h'(2) = f'(2) + g'(2) = 5 - 2 = 3$.
(b) We have $h'(2) = f'(2)g(2) + f(2)g'(2) = 5(4) + 3(-2) = 14$.
(c) We have $h'(2) = \dfrac{f'(2)g(2) - f(2)g'(2)}{(g(2))^2} = \dfrac{5(4) - 3(-2)}{4^2} = \dfrac{26}{16} = \dfrac{13}{8}$.

**51.** (a) $G'(z) = F'(z)H(z) + H'(z)F(z)$, so
$G'(3) = F'(3)H(3) + H'(3)F(3) = 4 \cdot 1 + 3 \cdot 5 = 19$.
(b) $G'(w) = \dfrac{F'(w)H(w) - H'(w)F(w)}{[H(w)]^2}$, so $G'(3) = \dfrac{4(1) - 3(5)}{1^2} = -11$.

**52.** $f'(x) = 10x^9e^x + x^{10}e^x$ is of the form $g'h + h'g$, where

$$g(x) = x^{10},\ g'(x) = 10x^9$$

and

$$h(x) = e^x,\ h'(x) = e^x.$$

Therefore, using the product rule, let $f = g \cdot h$, with $g(x) = x^{10}$ and $h(x) = e^x$. Thus

$$f(x) = x^{10}e^x.$$

**53.** (a) $f(140) = 15{,}000$ says that 15,000 skateboards are sold when the cost is \$140 per board.
$f'(140) = -100$ means that if the price is increased from \$140, roughly speaking, every dollar of increase will decrease the total sales by 100 boards.
(b) $\dfrac{dR}{dp} = \dfrac{d}{dp}(p \cdot q) = \dfrac{d}{dp}(p \cdot f(p)) = f(p) + pf'(p)$.
So,

$$\begin{aligned} \left.\frac{dR}{dp}\right|_{p=140} &= f(140) + 140f'(140) \\ &= 15{,}000 + 140(-100) = 1000. \end{aligned}$$

(c) From (b) we see that $\left.\dfrac{dR}{dp}\right|_{p=140} = 1000 > 0$. This means that the revenue will increase by about \$1000 if the price is raised by \$1.

**54.** We want $dR/dr_1$. Solving for $R$:

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{r_2 + r_1}{r_1 r_2}, \text{ which gives } R = \frac{r_1 r_2}{r_2 + r_1}.$$

So, thinking of $r_2$ as a constant and using the quotient rule,

$$\frac{dR}{dr_1} = \frac{r_2(r_2 + r_1) - r_1 r_2(1)}{(r_2 + r_1)^2} = \frac{r_2^2}{(r_1 + r_2)^2}.$$

**55.** (a) If the museum sells the painting and invests the proceeds $P(t)$ at time $t$, then $t$ years have elapsed since 2000, and the time span up to 2020 is $20 - t$. This is how long the proceeds $P(t)$ are earning interest in the bank. Each year the money is in the bank it earns 5% interest, which means the amount in the bank is multiplied by a factor of 1.05. So, at the end of $(20 - t)$ years, the balance is given by

$$B(t) = P(t)(1 + 0.05)^{20-t} = P(t)(1.05)^{20-t}.$$

(b)
$$B(t) = P(t)(1.05)^{20}(1.05)^{-t} = (1.05)^{20}\frac{P(t)}{(1.05)^t}.$$

(c) By the quotient rule,
$$B'(t) = (1.05)^{20}\left[\frac{P'(t)(1.05)^t - P(t)(1.05)^t \ln 1.05}{(1.05)^{2t}}\right].$$

So,
$$\begin{aligned} B'(10) &= (1.05)^{20}\left[\frac{5000(1.05)^{10} - 150{,}000(1.05)^{10}\ln 1.05}{(1.05)^{20}}\right] \\ &= (1.05)^{10}(5000 - 150{,}000 \ln 1.05) \\ &\approx -3776.63. \end{aligned}$$

**56.** Note first that $f(v)$ is in $\frac{\text{liters}}{\text{km}}$, and $v$ is in $\frac{\text{km}}{\text{hour}}$.

(a) $g(v) = \frac{1}{f(v)}$. (This is in $\frac{\text{km}}{\text{liter}}$.) Differentiating gives
$$g'(v) = \frac{-f'(v)}{(f(v))^2}.$$

So,
$$g(80) = \frac{1}{0.05} = 20\,\frac{\text{km}}{\text{liter}}.$$
$$g'(80) = \frac{-0.0005}{(0.05)^2} = -\frac{1}{5}\,\frac{\text{km}}{\text{liter}} \text{ for each } 1\,\frac{\text{km}}{\text{hr}} \text{ increase in speed.}$$

(b) $h(v) = v \cdot f(v)$. (This is in $\frac{\text{km}}{\text{hour}} \cdot \frac{\text{liters}}{\text{km}} = \frac{\text{liters}}{\text{hour}}$.) Differentiating gives
$$h'(v) = f(v) + v \cdot f'(v),$$

so
$$h(80) = 80(0.05) = 4\,\frac{\text{liters}}{\text{hr}}.$$
$$h'(80) = 0.05 + 80(0.0005) = 0.09\,\frac{\text{liters}}{\text{hr}} \text{ for each } 1\,\frac{\text{km}}{\text{hr}} \text{ increase in speed.}$$

(c) Part (a) tells us that at 80 km/hr, the car can go 20 km on 1 liter. Since the first derivative evaluated at this velocity is negative, this implies that as velocity increases, fuel efficiency decreases, i.e., at higher velocities the car will not go as far on 1 liter of gas. Part (b) tells us that at 80 km/hr, the car uses 4 liters in an hour. Since the first derivative evaluated at this velocity is positive, this means that at higher velocities, the car will use more gas per hour.

**57.** Assume for $g(x) \neq f(x)$, $g'(x) = g(x)$ and $g(0) = 1$. Then for
$$h(x) = \frac{g(x)}{e^x}$$
$$h'(x) = \frac{g'(x)e^x - g(x)e^x}{(e^x)^2} = \frac{e^x(g'(x) - g(x))}{(e^x)^2} = \frac{g'(x) - g(x)}{e^x}.$$

But, since $g(x) = g'(x)$, $h'(x) = 0$, so $h(x)$ is constant. Thus, the ratio of $g(x)$ to $e^x$ is constant. Since $\frac{g(0)}{e^0} = \frac{1}{1} = 1$, $\frac{g(x)}{e^x}$ must equal 1 for all $x$. Thus $g(x) = e^x = f(x)$ for all $x$, so $f$ and $g$ are the same function.

**58.** (a) $f'(x) = (x-2) + (x-1)$.

(b) Think of $f$ as the product of two factors, with the first as $(x-1)(x-2)$. (The reason for this is that we have already differentiated $(x-1)(x-2)$).
$$f(x) = [(x-1)(x-2)](x-3).$$
Now $f'(x) = [(x-1)(x-2)]'(x-3) + [(x-1)(x-2)](x-3)'$
Using the result of a):
$$\begin{aligned} f'(x) &= [(x-2) + (x-1)](x-3) + [(x-1)(x-2)] \cdot 1 \\ &= (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2). \end{aligned}$$

(c) Because we have already differentiated $(x-1)(x-2)(x-3)$, rewrite $f$ as the product of two factors, the first being $(x-1)(x-2)(x-3)$:

$$f(x) = [(x-1)(x-2)(x-3)](x-4)$$

Now $f'(x) = [(x-1)(x-2)(x-3)]'(x-4) + [(x-1)(x-2)(x-3)](x-4)'$.

$$\begin{aligned} f'(x) &= [(x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)](x-4) \\ &\quad + [(x-1)(x-2)(x-3)] \cdot 1 \\ &= (x-2)(x-3)(x-4) + (x-1)(x-3)(x-4) \\ &\quad + (x-1)(x-2)(x-4) + (x-1)(x-2)(x-3). \end{aligned}$$

From the solutions above, we can observe that when $f$ is a product, its derivative is obtained by differentiating each factor in turn (leaving the other factors alone), and adding the results.

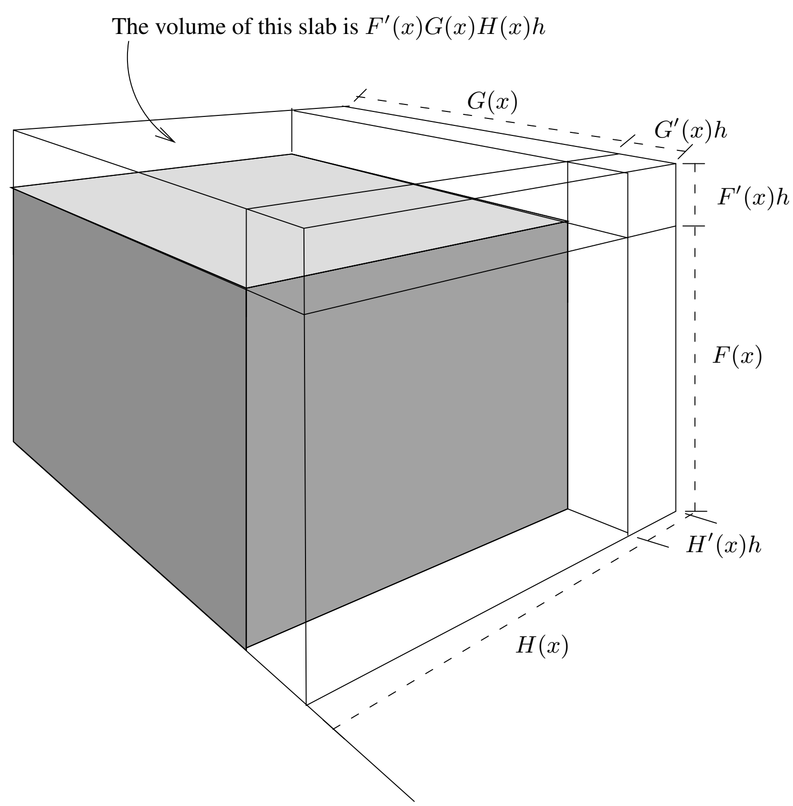
**59.** From the answer to Problem 58, we find that

$$\begin{aligned} f'(x) &= (x-r_1)(x-r_2)\cdots(x-r_{n-1}) \cdot 1 \\ &\quad + (x-r_1)(x-r_2)\cdots(x-r_{n-2}) \cdot 1 \cdot (x-r_n) \\ &\quad + (x-r_1)(x-r_2)\cdots(x-r_{n-3}) \cdot 1 \cdot (x-r_{n-1})(x-r_n) \\ &\quad + \cdots + 1 \cdot (x-r_2)(x-r_3)\cdots(x-r_n) \\ &= f(x)\left(\frac{1}{x-r_1} + \frac{1}{x-r_2} + \cdots + \frac{1}{x-r_n}\right). \end{aligned}$$

**60.** (a) We can approximate $\frac{d}{dx}[F(x)G(x)H(x)]$ using the large rectangular solids by which our original cube is increased:

Volume of whole $-$ volume of original solid $=$ change in volume.

$F(x+h)G(x+h)H(x+h) - F(x)G(x)H(x)$ = change in volume.



As in the book, we will ignore the smaller regions which are added (the long, thin rectangular boxes and the small cube in the corner.) This can be justified by recognizing that as $h \to 0$, these volumes will shrink much faster than the volumes of the big slabs and will therefore be insignificant. (Note that these smaller regions have an $h^2$ or $h^3$ in the formulas of their volumes.) Then we can approximate the change in volume above by:

$$\begin{aligned} F(x+h)G(x+h)H(x+h) - F(x)G(x)H(x) \approx\ & F'(x)G(x)H(x)h \quad \text{(top slab)} \\ & + F(x)G'(x)H(x)h \quad \text{(front slab)} \\ & + F(x)G(x)H'(x)h \quad \text{(other slab).} \end{aligned}$$

Dividing by $h$ gives

$$\begin{aligned} &\frac{F(x+h)G(x+h)H(x+h) - F(x)G(x)H(x)}{h} \\ &\approx F'(x)G(x)H(x) + F(x)G'(x)H(x) + F(x)G(x)H'(x). \end{aligned}$$

Letting $h \to 0$

$$(FGH)' = F'GH + FG'H + FGH'.$$

(b) Verifying,

$$\begin{aligned} \frac{d}{dx}[(F(x)\cdot G(x))\cdot H(x)] &= (F\cdot G)'(H) + (F\cdot G)(H)' \\ &= [F'G + FG']H + FGH' \\ &= F'GH + FG'H + FGH' \end{aligned}$$

as before.

(c) From the answer to (b), we observe that the derivative of a product is obtained by differentiating each factor in turn (leaving the other factors alone), and adding the results. So, in general,

$$(f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_n)' = f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdots f_n + \cdots + f_1 \cdots f_{n-1} f_n'.$$

**61.** (a) Since $x = a$ is a double zero of a polynomial $P(x)$, we can write $P(x) = (x-a)^2 Q(x)$, so $P(a) = 0$. Using the product rule, we have

$$P'(x) = 2(x-a)Q(x) + (x-a)^2 Q'(x).$$

Substituting in $x = a$, we see $P'(a) = 0$ also.

(b) Since $P(a) = 0$, we know $x = a$ is a zero of $P$, so that $x - a$ is a factor of $P$ and we can write

$$P(x) = (x-a)Q(x),$$

where $Q$ is some polynomial. Differentiating this expression for $P$ using the product rule, we get

$$P'(x) = Q(x) + (x-a)Q'(x).$$

Since we are told that $P'(a) = 0$, we have

$$P'(a) = Q(a) + (a-a)Q'(a) = 0$$

and so $Q(a) = 0$. Therefore $x = a$ is a zero of $Q$, so again we can write

$$Q(x) = (x-a)R(x),$$

where $R$ is some other polynomial. As a result,

$$P(x) = (x-a)Q(x) = (x-a)^2 R(x),$$

so that $x = a$ is a double zero of $P$.

## Solutions for Section 3.4

### Exercises

**1.** $f'(x) = 99(x+1)^{98} \cdot 1 = 99(x+1)^{98}$.

**2.** $w' = 100(t^2+1)^{99}(2t) = 200t(t^2+1)^{99}$.

**3.** $w' = 100(t^3+1)^{99}(3t^2) = 300t^2(t^3+1)^{99}$.

**4.** $\dfrac{d}{dx}\left((4x^2+1)^7\right) = 7(4x^2+1)^6 \dfrac{d}{dx}(4x^2+1) = 7(4x^2+1)^6 \cdot 8x = 56x(4x^2+1)^6$.

**5.** $f'(x) = \dfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = \dfrac{-x}{\sqrt{1-x^2}}.$

**6.** $\dfrac{d}{dx}(\sqrt{e^x+1}) = \dfrac{d}{dx}(e^x+1)^{1/2} = \dfrac{1}{2}(e^x+1)^{-1/2}\dfrac{d}{dx}(e^x+1) = \dfrac{e^x}{2\sqrt{e^x+1}}.$

**7.** $w' = 100(\sqrt{t}+1)^{99}\left(\frac{1}{2\sqrt{t}}\right) = \frac{50}{\sqrt{t}}(\sqrt{t}+1)^{99}.$

**8.** $h'(w) = 5(w^4-2w)^4(4w^3-2)$

**9.** We can write $w(r) = (r^4+1)^{1/2}$, so

$w'(r) = \dfrac{1}{2}(r^4+1)^{-1/2}(4r^3) = \dfrac{2r^3}{\sqrt{r^4+1}}.$

**10.** $k'(x) = 4(x^3+e^x)^3(3x^2+e^x).$

**11.** $f'(x) = 2e^{2x}[x^2+5^x] + e^{2x}[2x+(\ln 5)5^x] = e^{2x}[2x^2+2x+(\ln 5+2)5^x].$

**12.** $f'(t) = e^{3t}\cdot 3 = 3e^{3t}.$

**13.** $g(x) = \pi e^{\pi x}.$

**14.** $f(\theta) = (2^{-1})^\theta = (\frac{1}{2})^\theta$ so $f'(\theta) = (\ln\frac{1}{2})2^{-\theta}.$

**15.** $y' = (\ln\pi)\pi^{(x+2)}.$

**16.** $g'(x) = 2(\ln 3)3^{(2x+7)}.$

**17.** $f'(t) = 1\cdot e^{5-2t} + te^{5-2t}(-2) = e^{5-2t}(1-2t).$

**18.** $p'(t) = 4e^{4t+2}.$

**19.** Using the product rule gives $v'(t) = 2te^{-ct} - ce^{-ct}t^2 = (2t-ct^2)e^{-ct}.$

**20.** $\dfrac{d}{dt}e^{(1+3t)^2} = e^{(1+3t)^2}\dfrac{d}{dt}(1+3t)^2 = e^{(1+3t)^2}\cdot 2(1+3t)\cdot 3 = 6(1+3t)e^{(1+3t)^2}.$

**21.** $y' = \frac{3}{2}e^{\frac{3}{2}w}.$

**22.** $y' = -4e^{-4t}.$

**23.** $y' = \dfrac{3s^2}{2\sqrt{s^3+1}}.$

**24.** $w' = \dfrac{1}{2\sqrt{s}}e^{\sqrt{s}}.$

**25.** $y' = 1\cdot e^{-t^2} + te^{-t^2}(-2t)$

**26.** $f'(z) = \dfrac{1}{2\sqrt{z}}e^{-z} - \sqrt{z}e^{-z}.$

**27.** $z'(x) = \dfrac{(\ln 2)2^x}{3\sqrt[3]{(2^x+5)^2}}.$

**28.** $z' = 5\cdot\ln 2\cdot 2^{5t-3}.$

**29.** $w' = \dfrac{3}{2}\sqrt{x^2\cdot 5^x}[2x(5^x)+(\ln 5)(x^2)(5^x)] = \dfrac{3}{2}x^2\sqrt{5^{3x}}(2+x\ln 5).$

**30.** $f(y) = \left[10^{(5-y)}\right]^{\frac{1}{2}} = 10^{\frac{5}{2}-\frac{1}{2}y}$

$f'(y) = (\ln 10)\left(10^{\frac{5}{2}-\frac{1}{2}y}\right)\left(-\dfrac{1}{2}\right) = -\dfrac{1}{2}(\ln 10)(10^{\frac{5}{2}-\frac{1}{2}y}).$

**31.** We can write this as $f(z) = \sqrt{z}e^{-z}$, in which case it is the same as problem 26. So $f'(z) = \dfrac{1}{2\sqrt{z}}e^{-z} - \sqrt{z}e^{-z}.$

**32.** $y' = \dfrac{\frac{2^z}{2\sqrt{z}} - (\sqrt{z})(\ln 2)(2^z)}{2^{2z}} = \dfrac{1-2z\ln 2}{2^{z+1}\sqrt{z}}.$

**33.** $y' = 2\left(\dfrac{x^2+2}{3}\right)\left(\dfrac{2x}{3}\right) = \dfrac{4}{9}x\left(x^2+2\right)$

**34.** We can write $h(x) = \left(\dfrac{x^2+9}{x+3}\right)^{1/2}$, so

$$h'(x) = \frac{1}{2}\left(\frac{x^2+9}{x+3}\right)^{-1/2}\left[\frac{2x(x+3)-(x^2+9)}{(x+3)^2}\right] = \frac{1}{2}\sqrt{\frac{x+3}{x^2+9}}\left[\frac{x^2+6x-9}{(x+3)^2}\right].$$

**35.** $\frac{dy}{dx} = \frac{2e^{2x}(x^2+1) - e^{2x}(2x)}{(x^2+1)^2} = \frac{2e^{2x}(x^2+1-x)}{(x^2+1)^2}$

**36.** $y' = \frac{-(3e^{3x}+2x)}{(e^{3x}+x^2)^2}$.

**37.** $h'(z) = \frac{-8b^4 z}{(a+z^2)^5}$

**38.** $f'(z) = -2(e^z+1)^{-3} \cdot e^z = \frac{-2e^z}{(e^z+1)^3}$.

**39.** $w' = (2t+3)(1-e^{-2t}) + (t^2+3t)(2e^{-2t})$.

**40.** $h'(x) = (\ln 2)(3e^{3x})2^{e^{3x}} = 3e^{3x}2^{e^{3x}} \ln 2$.

**41.** $f'(x) = 6(e^{5x})(5) + (e^{-x^2})(-2x) = 30e^{5x} - 2xe^{-x^2}$.

**42.** $f'(x) = e^{-(x-1)^2} \cdot (-2)(x-1)$.

**43.**

$$\begin{aligned} f'(w) &= (e^{w^2})(10w) + (5w^2+3)(e^{w^2})(2w) \\ &= 2we^{w^2}(5+5w^2+3) \\ &= 2we^{w^2}(5w^2+8). \end{aligned}$$

**44.** The power and chain rules give

$$f'(\theta) = -1(e^\theta + e^{-\theta})^{-2} \cdot \frac{d}{d\theta}(e^\theta + e^{-\theta}) = -(e^\theta + e^{-\theta})^{-2}(e^\theta + e^{-\theta}(-1)) = -\left(\frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}\right).$$

**45.** We write $y = (e^{-3t^2}+5)^{1/2}$, so

$$\begin{aligned} \frac{dy}{dt} = \frac{1}{2}(e^{-3t^2}+5)^{-1/2} \cdot \frac{d}{dt}(e^{-3t^2}+5) &= \frac{1}{2}(e^{-3t^2}+5)^{-1/2} \cdot e^{-3t^2} \cdot \frac{d}{dt}(-3t^2) \\ &= \frac{1}{2}(e^{-3t^2}+5)^{-1/2} \cdot e^{-3t^2} \cdot (-6t) = -\frac{3te^{-3t^2}}{\sqrt{e^{-3t^2}+5}}. \end{aligned}$$

**46.** Using the product and chain rules, we have

$$\begin{aligned} \frac{dz}{dt} &= 9(te^{3t}+e^{5t})^8 \cdot \frac{d}{dt}(te^{3t}+e^{5t}) = 9(te^{3t}+e^{5t})^8(1 \cdot e^{3t} + t \cdot e^{3t} \cdot 3 + e^{5t} \cdot 5) \\ &= 9(te^{3t}+e^{5t})^8(e^{3t}+3te^{3t}+5e^{5t}). \end{aligned}$$

**47.** $f'(y) = e^{e^{(y^2)}} \left[(e^{y^2})(2y)\right] = 2ye^{[e^{(y^2)}+y^2]}$.

**48.** $f'(t) = 2(e^{-2e^{2t}})(-2e^{2t})2 = -8(e^{-2e^{2t}+2t})$.

**49.** Since $a$ and $b$ are constants, we have $f'(x) = 3(ax^2+b)^2(2ax) = 6ax(ax^2+b)^2$.

**50.** Since $a$ and $b$ are constants, we have $f'(t) = ae^{bt}(b) = abe^{bt}$.

**51.** We use the product rule. We have

$$f'(x) = (ax)(e^{-bx}(-b)) + (a)(e^{-bx}) = -abxe^{-bx} + ae^{-bx}.$$

**52.** Using the product and chain rules, we have

$$\begin{aligned} g'(\alpha) &= e^{\alpha e^{-2\alpha}} \cdot \frac{d}{dx}(\alpha e^{-2\alpha}) = e^{\alpha e^{-2\alpha}}(1 \cdot e^{-2\alpha} + \alpha e^{-2\alpha}(-2)) \\ &= e^{\alpha e^{-2\alpha}}(e^{-2\alpha} - 2\alpha e^{-2\alpha}) \\ &= (1-2\alpha)e^{-2\alpha}e^{\alpha e^{-2\alpha}}. \end{aligned}$$

## Problems

**53.** Using the chain rule, we know that $h'(x) = f'(g(x)) \cdot g'(x)$. We use slope to compute the derivatives. Since $f(x)$ is linear on the interval $0 < x < 2$, we compute the slope of the line to see that $f'(x) = 2$ on this interval. Similarly, we compute the slope on the interval $2 < x < 4$ to see that $f'(x) = -2$ on the interval $2 < x < 4$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist.

Similarly, $g(x)$ is linear on the interval shown, and we see that the slope of $g(x)$ on this interval is $-1$ so we have $g'(x) = -1$ on this interval.

(a) We have $h'(1) = f'(g(1)) \cdot g'(1) = (f'(3))(-1) = (-2)(-1) = 2$.
(b) We have $h'(2) = f'(g(2)) \cdot g'(2) = (f'(2))(-1)$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, $h'(2)$ does not exist.
(c) We have $h'(3) = f'(g(3)) \cdot g'(3) = (f'(1))(-1) = 2(-1) = -2$.

**54.** Using the chain rule, we know that $u'(x) = g'(f(x)) \cdot f'(x)$. We use slope to compute the derivatives. Since $f(x)$ is linear on the interval $0 < x < 2$, we compute the slope of the line to see that $f'(x) = 2$ on this interval. Similarly, we compute the slope on the interval $2 < x < 4$ to see that $f'(x) = -2$ on the interval $2 < x < 4$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist.

Similarly, $g(x)$ is linear on the interval shown, and we see that the slope of $g(x)$ on this interval is $-1$ so we have $g'(x) = -1$ on this interval.

(a) We have $u'(1) = g'(f(1)) \cdot f'(1) = (g'(2))2 = (-1)2 = -2$.
(b) We have $u'(2) = g'(f(2)) \cdot f'(2)$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, $u'(2)$ does not exist.
(c) We have $u'(3) = g'(f(3)) \cdot f'(3) = (g'(2))(-2) = (-1)(-2) = 2$.

**55.** Using the chain rule, we know that $v'(x) = f'(f(x)) \cdot f'(x)$. We use slope to compute the derivatives. Since $f(x)$ is linear on the interval $0 < x < 2$, we compute the slope of the line to see that $f'(x) = 2$ on this interval. Similarly, we compute the slope on the interval $2 < x < 4$ to see that $f'(x) = -2$ on the interval $2 < x < 4$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist.

(a) We have $v'(1) = f'(f(1)) \cdot f'(1) = f'(2) \cdot 2$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, $v'(1)$ does not exist.
(b) We have $v'(2) = f'(f(2)) \cdot f'(2)$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, $v'(2)$ does not exist.
(c) We have $v'(3) = f'(f(3)) \cdot f'(3) = (f'(2))(-2)$. Since $f(x)$ has a corner at $x = 2$, we know that $f'(2)$ does not exist. Therefore, $v'(3)$ does not exist.

**56.** Using the chain rule, we know that $w'(x) = g'(g(x)) \cdot g'(x)$. We use slope to compute the derivatives. Since $g(x)$ is linear on the interval shown, with slope equal to $-1$, we have $g'(x) = -1$ on this interval.

(a) We have $w'(1) = g'(g(1)) \cdot g'(1) = (g'(3))(-1) = (-1)(-1) = 1$.
(b) We have $w'(2) = g'(g(2)) \cdot g'(2) = (g'(2))(-1) = (-1)(-1) = 1$.
(c) We have $w'(3) = g'(g(3)) \cdot g'(3) = (g'(1))(-1) = (-1)(-1) = 1$.

**57.** The chain rule gives

$$\frac{d}{dx} f(g(x))\bigg|_{x=30} = f'(g(30))g'(30) = f'(55)g'(30) = (1)(\frac{1}{2}) = \frac{1}{2}.$$

**58.** The chain rule gives

$$\frac{d}{dx} f(g(x))\bigg|_{x=70} = f'(g(70))g'(70) = f'(60)g'(70) = (1)(0) = 0.$$

**59.** The chain rule gives

$$\frac{d}{dx} g(f(x))\bigg|_{x=30} = g'(f(30))f'(30) = g'(20)f'(30) = (1/2)(-2) = -1.$$

**60.** The chain rule gives

$$\frac{d}{dx} g(f(x))\bigg|_{x=70} = g'(f(70))f'(70) = g'(30)f'(70) = (1)(\frac{1}{2}) = \frac{1}{2}.$$

**61.** We have $f(2) = (2-1)^3 = 1$, so $(2, 1)$ is a point on the tangent line. Since $f'(x) = 3(x-1)^2$, the slope of the tangent line is

$$m = f'(2) = 3(2-1)^2 = 3.$$

The equation of the line is

$$y - 1 = 3(x-2) \quad \text{or} \quad y = 3x - 5.$$

**62.**

$$f(x) = 6e^{5x} + e^{-x^2} \qquad f'(x) = 30e^{5x} - 2xe^{-x^2}$$
$$f(1) = 6e^5 + e^{-1} \qquad f'(1) = 30e^5 - 2(1)e^{-1}$$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (6e^5 + e^{-1}) &= (30e^5 - 2e^{-1})(x-1) \\ y - (6e^5 + e^{-1}) &= (30e^5 - 2e^{-1})x - (30e^5 - 2e^{-1}) \\ y &= (30e^5 - 2e^{-1})x - 30e^5 + 2e^{-1} + 6e^5 + e^{-1} \\ &\approx 4451.66x - 3560.81. \end{aligned}$$

**63.** The graph is concave down when $f''(x) < 0$.

$$\begin{aligned} f'(x) &= e^{-x^2}(-2x) \\ f''(x) &= \left[e^{-x^2}(-2x)\right](-2x) + e^{-x^2}(-2) \\ &= \frac{4x^2}{e^{x^2}} - \frac{2}{e^{x^2}} \\ &= \frac{4x^2 - 2}{e^{x^2}} < 0 \end{aligned}$$

The graph is concave down when $4x^2 < 2$. This occurs when $x^2 < \frac{1}{2}$, or $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

**64.** We rewrite $e^{-x} = 1/e^x$ so that we can use the quotient rule, then

$$\begin{aligned} f(x) &= \frac{x}{e^x}, \\ f'(x) &= \frac{1 \cdot e^x - x \cdot e^x}{(e^x)^2} = \frac{(1-x)e^x}{(e^x)^2} = \frac{1-x}{e^x}, \\ f''(x) &= \frac{-1 \cdot e^x - (1-x)e^x}{(e^x)^2} = \frac{-e^x - e^x + xe^x}{(e^x)^2} = \frac{(-2+x)e^x}{(e^x)^2} = \frac{x-2}{e^x}. \end{aligned}$$

Since $e^{-x} > 0$, for all $x$, we have $f''(x) < 0$ if $x - 2 < 0$, that is, $x < 2$.

**65.**

$$\begin{aligned} f'(x) &= [10(2x+1)^9(2)][(3x-1)^7] + [(2x+1)^{10}][7(3x-1)^6(3)] \\ &= (2x+1)^9(3x-1)^6[20(3x-1) + 21(2x+1)] \\ &= [(2x+1)^9(3x-1)^6](102x+1) \\ f''(x) &= [9(2x+1)^8(2)(3x-1)^6 + (2x+1)^9(6)(3x-1)^5(3)](102x+1) \\ &\quad + (2x+1)^9(3x-1)^6(102). \end{aligned}$$

**66.** (a) The rate of change of the population is $P'(t)$. If $P'(t)$ is proportional to $P(t)$, we have

$$P'(t) = kP(t).$$

(b) If $P(t) = Ae^{kt}$, then $P'(t) = kAe^{kt} = kP(t)$.

**67.** (a) With $\mu$ and $\sigma$ constant, differentiating $m(t) = e^{\mu t+\sigma^2 t^2/2}$ with respect to $t$ gives

$$m'(t) = e^{ut+\sigma^2 t^2/2} \cdot \left(\mu + \frac{2\sigma^2 t}{2}\right) = e^{\mu t+\sigma^2 t^2/2}(\mu + \sigma^2 t).$$

Thus,

$$\text{Mean } = m'(0) = e^0(\mu + 0) = \mu.$$

(b) Differentiating $m'(t) = e^{\mu t+\sigma^2 t^2/2}(\mu + \sigma^2 t)$, we have

$$m''(t) = e^{\mu t+\sigma^2 t^2/2}(\mu + \sigma^2 t)^2 + e^{\mu t+\sigma^2 t^2/2}\sigma^2.$$

Thus

$$\text{Variance } = m''(0) - (m'(0))^2 = e^0\mu^2 + e^0\sigma^2 - \mu^2 = \sigma^2.$$

**68.** (a) If

$$p(x) = k(2x),$$

then

$$p'(x) = k'(2x) \cdot 2.$$

When $x = \frac{1}{2}$,

$$p'\left(\frac{1}{2}\right) = k'\left(2 \cdot \frac{1}{2}\right)(2) = 2 \cdot 2 = 4.$$

(b) If

$$q(x) = k(x + 1),$$

then

$$q'(x) = k'(x + 1) \cdot 1.$$

When $x = 0$,

$$q'(0) = k'(0 + 1)(1) = 2 \cdot 1 = 2.$$

(c) If

$$r(x) = k\left(\frac{1}{4}x\right),$$

then

$$r'(x) = k'\left(\frac{1}{4}x\right) \cdot \frac{1}{4}.$$

When $x = 4$,

$$r'(4) = k'\left(\frac{1}{4}4\right)\frac{1}{4} = 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

**69.** Yes. To see why, simply plug $x = \sqrt[3]{2t+5}$ into the expression $3x^2\dfrac{dx}{dt}$ and evaluate it. To do this, first we calculate $\dfrac{dx}{dt}$. By the chain rule,

$$\frac{dx}{dt} = \frac{d}{dt}(2t+5)^{\frac{1}{3}} = \frac{2}{3}(2t+5)^{-\frac{2}{3}} = \frac{2}{3}[(2t+5)^{\frac{1}{3}}]^{-2}.$$

But since $x = (2t+5)^{\frac{1}{3}}$, we have (by substitution)

$$\frac{dx}{dt} = \frac{2}{3}x^{-2}.$$

It follows that $3x^2\dfrac{dx}{dt} = 3x^2\left(\dfrac{2}{3}x^{-2}\right) = 2$.

**70.** We see that $m'(x)$ is nearly of the form $f'(g(x)) \cdot g'(x)$ where

$$f(g) = e^g \quad \text{and} \quad g(x) = x^6,$$

but $g'(x)$ is off by a multiple of 6. Therefore, using the chain rule, let

$$m(x) = \frac{f(g(x))}{6} = \frac{e^{(x^6)}}{6}.$$

**71.** (a) $H(x) = F(G(x))$
$H(4) = F(G(4)) = F(2) = 1$
(b) $H(x) = F(G(x))$
$H'(x) = F'(G(x)) \cdot G'(x)$
$H'(4) = F'(G(4)) \cdot G'(4) = F'(2) \cdot 6 = 5 \cdot 6 = 30$
(c) $H(x) = G(F(x))$
$H(4) = G(F(4)) = G(3) = 4$
(d) $H(x) = G(F(x))$
$H'(x) = G'(F(x)) \cdot F'(x)$
$H'(4) = G'(F(4)) \cdot F'(4) = G'(3) \cdot 7 = 8 \cdot 7 = 56$
(e) $H(x) = \frac{F(x)}{G(x)}$
$H'(x) = \frac{G(x) \cdot F'(x) - F(x) \cdot G'(x)}{[G(x)]^2}$
$H'(4) = \frac{G(4) \cdot F'(4) - F(4) \cdot G'(4)}{[G(4)]^2} = \frac{2 \cdot 7 - 3 \cdot 6}{2^2} = \frac{14-18}{4} = \frac{-4}{4} = -1$

**72.** (a) Differentiating $g(x) = \sqrt{f(x)} = (f(x))^{1/2}$, we have

$$g'(x) = \frac{1}{2}(f(x))^{-1/2} \cdot f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$$
$$g'(1) = \frac{f'(1)}{2\sqrt{f(1)}} = \frac{3}{2\sqrt{4}} = \frac{3}{4}.$$

(b) Differentiating $h(x) = f(\sqrt{x})$, we have

$$h'(x) = f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$
$$h'(1) = f'(\sqrt{1}) \cdot \frac{1}{2\sqrt{1}} = \frac{f'(1)}{2} = \frac{3}{2}.$$

**73.** We have $h(0) = f(g(0)) = f(d) = d$. From the chain rule, $h'(0) = f'(g(0))g'(0)$. From the graph of $g$, we see that $g'(0) = 0$, so $h'(0) = f'(g(0)) \cdot 0 = 0$.

**74.** We have $h(-c) = f(g(-c)) = f(-b) = 0$. From the chain rule,

$$h'(-c) = f'(g(-c))g'(-c).$$

Since $g$ is increasing at $x = -c$, we know that $g'(-c) > 0$. We have

$$f'(g(-c)) = f'(-b),$$

and since $f$ is decreasing at $x = -b$, we have $f'(g(-c)) < 0$. Thus,

$$h'(-c) = \underbrace{f'(g(-c))}_{-} \cdot \underbrace{g'(-c)}_{+} < 0,$$

so $h$ is decreasing at $x = -c$.

**75.** We have

$$h'(a) = f'(g(a))g'(a).$$

From the graph of $g$, we see that $g$ is decreasing at $x = a$, so $g'(a) < 0$. We have

$$f'(g(a)) = f'(b),$$

and from the graph of $f$, we see that $f$ is increasing at $x = b$, so $f'(b) > 0$. Thus,

$$h'(a) = \underbrace{f'(g(a))}_{+} \cdot \underbrace{g'(a)}_{-} < 0,$$

so $h$ is decreasing at $x = a$.

**76.** We have $h(d) = f(g(d)) = f(-d) = d$ so $h(d)$ is positive. From the chain rule,

$$h'(d) = f'(g(d))g'(d).$$

We have

$$f'(g(d)) = f'(-d).$$

From the graph of $f$, we see that $f'(-d) < 0$, and from the graph of $g$, we see that $g'(d) < 0$. This means the sign of $h'(d)$ is the product of two negative numbers, so $h'(d) > 0$.

**77.** On the interval $-d < x < -b$, we see that the value of $g(x)$ increases from $-d$ to 0. On the interval $-d < x < 0$, the value of $f(x)$ decreases from $d$ to $-d$. Thus, the value of $h(x) = f(g(x))$ decreases on the interval $-d < x < -b$ from

$$h(-d) = f(g(-d)) = f(-d) = d \quad \text{to} \quad h(-b) = f(g(-b)) = f(0) - d.$$

Confirming this using derivatives and the chain rule, we see

$$h'(x) = f'(g(x)) \cdot g'(x),$$

and since $g'(x)$ is negative on $-d < x < -b$ and $f'(g(x))$ is positive on this interval, the value of $h(x)$ is decreasing.

**78.** We have $f(0) = 6$ and $f(10) = 6e^{0.013(10)} = 6.833$. The derivative of $f(t)$ is

$$f'(t) = 6e^{0.013t} \cdot 0.013 = 0.078e^{0.013t},$$

and so $f'(0) = 0.078$ and $f'(10) = 0.089$.

These values tell us that in 1999 (at $t = 0$), the population of the world was 6 billion people and the population was growing at a rate of 0.078 billion people per year. In the year 2009 (at $t = 10$), this model predicts that the population of the world will be 6.833 billion people and growing at a rate of 0.089 billion people per year.

**79.** (a) $\dfrac{dB}{dt} = P\left(1 + \dfrac{r}{100}\right)^t \ln\left(1 + \dfrac{r}{100}\right)$. The expression $\dfrac{dB}{dt}$ tells us how fast the amount of money in the bank is changing with respect to time for fixed initial investment $P$ and interest rate $r$.

(b) $\dfrac{dB}{dr} = Pt\left(1 + \dfrac{r}{100}\right)^{t-1} \dfrac{1}{100}$. The expression $\dfrac{dB}{dr}$ indicates how fast the amount of money changes with respect to the interest rate $r$ , assuming fixed initial investment $P$ and time $t$.

**80.** (a)

$$\begin{aligned}\frac{dm}{dv} &= \frac{d}{dv}\left[m_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2}\right] \\ &= m_0\left(-\frac{1}{2}\right)\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right) \\ &= \frac{m_0 v}{c^2}\frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)^3}}.\end{aligned}$$

(b) $\dfrac{dm}{dv}$ represents the rate of change of mass with respect to the speed $v$.

**81.** (a) For $t < 0, I = \dfrac{dQ}{dt} = 0$.

For $t > 0, I = \dfrac{dQ}{dt} = -\dfrac{Q_0}{RC}e^{-t/RC}$.

(b) For $t > 0$, $t \to 0$ (that is, as $t \to 0^+$),

$$I = -\frac{Q_0}{RC}e^{-t/RC} \to -\frac{Q_0}{RC}.$$

Since $I = 0$ just to the left of $t = 0$ and $I = -Q_0/RC$ just to the right of $t = 0$, it is not possible to define $I$ at $t = 0$.

(c) $Q$ is not differentiable at $t = 0$ because there is no tangent line at $t = 0$.

**82.** Recall that $v = dx/dt$. We want to find the acceleration, $dv/dt$, when $x = 2$. Differentiating the expression for $v$ with respect to $t$ using the chain rule and substituting for $v$ gives

$$\frac{dv}{dt} = \frac{d}{dx}(x^2 + 3x - 2) \cdot \frac{dx}{dt} = (2x + 3)v = (2x + 3)(x^2 + 3x - 2).$$

Substituting $x = 2$ gives

$$\text{Acceleration} = \left.\frac{dv}{dt}\right|_{x=2} = (2(2) + 3)(2^2 + 3 \cdot 2 - 2) = 56 \text{ cm/sec}^2.$$

**83.** Let $f$ have a zero of multiplicity $m$ at $x = a$ so that

$$f(x) = (x - a)^m h(x), \quad h(a) \neq 0.$$

Differentiating this expression gives

$$f'(x) = (x - a)^m h'(x) + m(x - a)^{(m-1)} h(x)$$

and both terms in the sum are zero when $x = a$ so $f'(a) = 0$. Taking another derivative gives

$$f''(x) = (x - a)^m h''(x) + 2m(x - a)^{(m-1)} h'(x) + m(m - 1)(x - a)^{(m-2)} h(x).$$

Again, each term in the sum contains a factor of $(x - a)$ to some positive power, so at $x = a$ this will evaluate to $0$. Differentiating repeatedly, all derivatives will have positive integer powers of $(x - a)$ until the $m^{\text{th}}$ and will therefore vanish. However,

$$f^{(m)}(a) = m!h(a) \neq 0.$$

**84.** Since $2x$ is the derivative of $x^2 + 1$, the chain rule tells us that

$$\frac{d}{dx} f(x^2 + 1) = 2xf'(x^2 + 1).$$

Thus using the information given in the problem, we have

$$2xf'(x^2 + 1) = \frac{2x}{x^2 + 1},$$

so

$$f'(x^2 + 1) = \frac{1}{x^2 + 1}.$$

Thus, replacing $x^2 + 1$ by $x$, we have

$$f'(x) = \frac{1}{x}.$$

**85.** The problem tells us that

$$\frac{d}{dt} G(a - bt) = H(a - bt).$$

Since $\frac{d}{dt}(a - bt) = -b$, the chain rule tells us that

$$-bG'(a - bt) = H(a - bt),$$

so

$$G'(a - bt) = \left(-\frac{1}{b}\right) H(a - bt).$$

Replacing $a - bt$ by $t$, we have

$$G'(t) = \left(-\frac{1}{b}\right) H(t)$$

.

**86.** By the product rule, $\frac{d}{dt} tf(t) = f(t) + tf'(t)$. Thus, using the information given in the problem, we have

$$f(t) + tf'(t) = 1 + f(t).$$

Subtracting $f(t)$ from both sides gives $tf'(t) = 1$, so $f'(t) = 1/t$.

**87.** By the chain rule,

$$\frac{d}{dx}f(e^x) = f'(e^x)\frac{d}{dx}(e^x) = f'(e^x)e^x,$$

so, using the information given in the problem, we have

$$f'(e^x)e^x = 2e^{2x}.$$

Dividing by $e^x$ we get

$$f'(e^x) = \frac{2e^{2x}}{e^x},$$

so

$$f'(e^x) = 2e^x.$$

Thus, replacing $^x$ by $x$, we have

$$f'(x) = 2x,$$

so

$$f(x) = x^2.$$

# Solutions for Section 3.5

## Exercises

**1.**

**Table 3.1**

| $x$ | $\cos x$ | Difference Quotient | $-\sin x$ |
|---|---|---|---|
| 0 | 1.0 | $-0.0005$ | 0.0 |
| 0.1 | 0.995 | $-0.10033$ | $-0.099833$ |
| 0.2 | 0.98007 | $-0.19916$ | $-0.19867$ |
| 0.3 | 0.95534 | $-0.296$ | $-0.29552$ |
| 0.4 | 0.92106 | $-0.38988$ | $-0.38942$ |
| 0.5 | 0.87758 | $-0.47986$ | $-0.47943$ |
| 0.6 | 0.82534 | $-0.56506$ | $-0.56464$ |

**2.** $r'(\theta) = \cos\theta - \sin\theta$.

**3.** $s'(\theta) = -\sin\theta\sin\theta + \cos\theta\cos\theta = \cos^2\theta - \sin^2\theta = \cos 2\theta$.

**4.** $z' = -4\sin(4\theta)$.

**5.** $f'(x) = \cos(3x)\cdot 3 = 3\cos(3x)$.

**6.** $\dfrac{d}{dx}\sin(2-3x) = \cos(2-3x)\dfrac{d}{dx}(2-3x) = -3\cos(2-3x)$.

**7.** Using the chain rule gives $R'(x) = 3\pi\sin(\pi x)$.

**8.** $g'(\theta) = 2\sin(2\theta)\cos(2\theta)\cdot 2 - \pi = 4\sin(2\theta)\cos(2\theta) - \pi$

**9.** $f'(x) = (2x)(\cos x) + x^2(-\sin x) = 2x\cos x - x^2\sin x$.

**10.** $w' = e^t\cos(e^t)$.

**11.** $f'(x) = (e^{\cos x})(-\sin x) = -\sin x e^{\cos x}$.

**12.** $f'(y) = (\cos y)e^{\sin y}$.

**13.** $z' = e^{\cos\theta} - \theta(\sin\theta)e^{\cos\theta}$.

**14.** Using the chain rule gives $R'(\theta) = 3\cos(3\theta)e^{\sin(3\theta)}$.

**15.** $g'(\theta) = \dfrac{\cos(\tan\theta)}{\cos^2\theta}$

**16.** $w'(x) = \dfrac{2x}{\cos^2(x^2)}$

**17.**
$$\begin{aligned} f(x) &= (1-\cos x)^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}(1-\cos x)^{-\frac{1}{2}}(-(-\sin x)) \\ &= \frac{\sin x}{2\sqrt{1-\cos x}}. \end{aligned}$$

**18.** $f'(x) = [-\sin(\sin x)](\cos x)$.

**19.** $f'(x) = \dfrac{\cos x}{\cos^2(\sin x)}$.

**20.** $k'(x) = \frac{3}{2}\sqrt{\sin(2x)}(2\cos(2x)) = 3\cos(2x)\sqrt{\sin(2x)}$.

**21.** $f'(x) = 2\cdot[\sin(3x)] + 2x[\cos(3x)]\cdot 3 = 2\sin(3x) + 6x\cos(3x)$

**22.** $y' = e^{\theta}\sin(2\theta) + 2e^{\theta}\cos(2\theta)$.

**23.** $f'(x) = (e^{-2x})(-2)(\sin x) + (e^{-2x})(\cos x) = -2\sin x(e^{-2x}) + (e^{-2x})(\cos x) = e^{-2x}[\cos x - 2\sin x]$.

**24.** $z' = \dfrac{\cos t}{2\sqrt{\sin t}}$.

**25.** $y' = 5\sin^4\theta\cos\theta$.

**26.** $g'(z) = \dfrac{e^z}{\cos^2(e^z)}$.

**27.** $z' = \dfrac{-3e^{-3\theta}}{\cos^2(e^{-3\theta})}$.

**28.** $w' = (-\cos\theta)e^{-\sin\theta}$.

**29.** $h'(t) = 1\cdot(\cos t) + t(-\sin t) + \frac{1}{\cos^2 t} = \cos t - t\sin t + \frac{1}{\cos^2 t}$.

**30.** $f'(\alpha) = -\sin\alpha + 3\cos\alpha$

**31.** $k'(\alpha) = (5\sin^4\alpha\cos\alpha)\cos^3\alpha + \sin^5\alpha(3\cos^2\alpha(-\sin\alpha)) = 5\sin^4\alpha\cos^4\alpha - 3\sin^6\alpha\cos^2\alpha$

**32.** $f'(\theta) = 3\theta^2\cos\theta - \theta^3\sin\theta$.

**33.** $y' = -2\cos w\sin w - \sin(w^2)(2w) = -2(\cos w\sin w + w\sin(w^2))$

**34.** $y' = \cos(\cos x + \sin x)(\cos x - \sin x)$

**35.** $y' = 2\cos(2x)\sin(3x) + 3\sin(2x)\cos(3x)$.

**36.** $t'(\theta) = \dfrac{-\sin\theta\sin\theta - \cos\theta\cos\theta}{\sin^2\theta} = -\dfrac{(\sin^2\theta + \cos^2\theta)}{\sin^2\theta} = -\dfrac{1}{\sin^2\theta}$.

**37.** Using the power and quotient rules gives

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(\frac{1-\sin x}{1-\cos x}\right)^{-1/2}\left[\frac{-\cos x(1-\cos x) - (1-\sin x)\sin x}{(1-\cos x)^2}\right] \\ &= \frac{1}{2}\sqrt{\frac{1-\cos x}{1-\sin x}}\left[\frac{-\cos x(1-\cos x) - (1-\sin x)\sin x}{(1-\cos x)^2}\right] \\ &= \frac{1}{2}\sqrt{\frac{1-\cos x}{1-\sin x}}\left[\frac{1-\cos x-\sin x}{(1-\cos x)^2}\right]. \end{aligned}$$

**38.** $\dfrac{d}{dy}\left(\dfrac{y}{\cos y + a}\right) = \dfrac{\cos y + a - y(-\sin y)}{(\cos y + a)^2} = \dfrac{\cos y + a + y\sin y}{(\cos y + a)^2}$.

**39.** The quotient rule gives $G'(x) = \dfrac{2\sin x\cos x(\cos^2 x + 1) + 2\sin x\cos x(\sin^2 x + 1)}{(\cos^2 x + 1)^2}$

or, using $\sin^2 x + \cos^2 x = 1$,

$G'(x) = \dfrac{6\sin x\cos x}{(\cos^2 x + 1)^2}$.

## Problems

**40.** We begin by taking the derivative of $y = \sin(x^4)$ and evaluating at $x = 10$:

$$\frac{dy}{dx} = \cos(x^4) \cdot 4x^3.$$

Evaluating $\cos(10{,}000)$ on a calculator (in radians) we see $\cos(10{,}000) < 0$, so we know that $dy/dx < 0$, and therefore the function is decreasing.
Next, we take the second derivative and evaluate it at $x = 10$, giving $\sin(10{,}000) < 0$:

$$\frac{d^2y}{dx^2} = \underbrace{\cos(x^4) \cdot (12x^2)}_{\text{negative}} + \underbrace{4x^3 \cdot (-\sin(x^4))(4x^3)}_{\substack{\text{positive, but much}\\ \text{larger in magnitude}}}.$$

From this we can see that $d^2y/dx^2 > 0$, thus the graph is concave up.

**41.** The pattern in the table below allows us to generalize and say that the (4n)$^{\text{th}}$ derivative of $\cos x$ is $\cos x$, i.e.,

$$\frac{d^4y}{dx^4} = \frac{d^8y}{dx^8} = \cdots = \frac{d^{4n}y}{dx^{4n}} = \cos x.$$

Thus we can say that $d^{48}y/dx^{48} = \cos x$. From there we differentiate twice more to obtain $d^{50}y/dx^{50} = -\cos x$.

| $n$ | 1 | 2 | 3 | 4 | $\cdots$ | 48 | 49 | 50 |
|---|---|---|---|---|---|---|---|---|
| $n^{\text{th}}$ derivative | $-\sin x$ | $-\cos x$ | $\sin x$ | $\cos x$ | | $\cos x$ | $-\sin x$ | $-\cos x$ |

**42.** We see that $q'(x)$ is of the form

$$\frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2},$$

with $f(x) = e^x$ and $g(x) = \sin x$. Therefore, using the quotient rule, let

$$q(x) = \frac{f(x)}{g(x)} = \frac{e^x}{\sin x}.$$

**43.** Since $F'(x)$ is of the form $\sin u$, we can make an initial guess that

$$F(x) = \cos(4x),$$

then

$$F'(x) = -4\sin(4x)$$

so we're off by a factor of $-4$. To fix this problem, we modify our guess by a factor of $-4$, so the next try is

$$F(x) = -(1/4)\cos(4x),$$

which has

$$F'(x) = \sin(4x).$$

**44.** (a) Differentiating gives

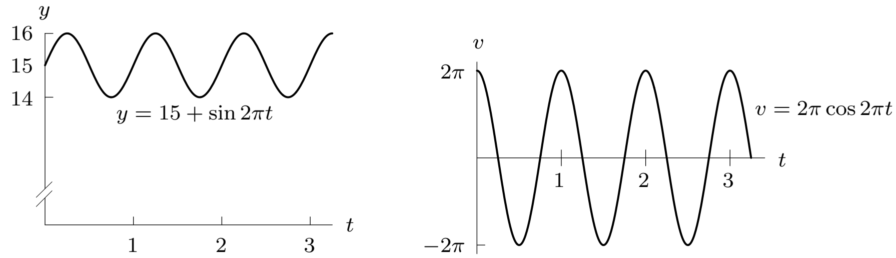
$$\frac{dy}{dt} = -\frac{4.9\pi}{6}\sin\left(\frac{\pi}{6}t\right).$$

The derivative represents the rate of change of the depth of the water in feet/hour.

(b) The derivative, $dy/dt$, is zero where the tangent line to the curve $y$ is horizontal. This occurs when $dy/dt = \sin(\frac{\pi}{6}t) = 0$, or at $t = 6$, 12, 18 and 24 (6 am, noon, 6 pm, and midnight). When $dy/dt = 0$, the depth of the water is no longer changing. Therefore, it has either just finished rising or just finished falling, and we know that the harbor's level is at a maximum or a minimum.

**45.** (a) $v(t) = \dfrac{dy}{dt} = \dfrac{d}{dt}(15 + \sin(2\pi t)) = 2\pi\cos(2\pi t)$.

(b)




**46.** (a) Differentiating, we find

$$\begin{aligned}\text{Rate of change of voltage with time} &= \frac{dV}{dt} = -120\pi \cdot 156\sin(120\pi t)\\ &= -18720\pi\sin(120\pi t) \text{ volts per second.}\end{aligned}$$

(b) The rate of change of voltage with time is zero when $\sin(120\pi t) = 0$. This occurs when $120\pi t$ equals any multiple of $\pi$. For example, $\sin(120\pi t) = 0$ when $120\pi t = \pi$, or at $t = 1/120$ seconds. Since there are an infinite number of multiples of $\pi$, there are many times when the rate of change $dV/dt$ is zero.

(c) The maximum value of the rate of change is $18720\pi = 58810.6$ volts/sec.

**47.** (a) When $\sqrt{\frac{k}{m}}t = \frac{\pi}{2}$ the spring is farthest from the equilibrium position. This occurs at time $t = \frac{\pi}{2}\sqrt{\frac{m}{k}}$

$v = A\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right)$, so the maximum velocity occurs when $t = 0$

$a = -A\frac{k}{m}\sin\left(\sqrt{\frac{k}{m}}t\right)$, so the maximum acceleration occurs when $\sqrt{\frac{k}{m}}t = \frac{3\pi}{2}$, which is at time $t = \frac{3\pi}{2}\sqrt{\frac{m}{k}}$

(b) $T = \frac{2\pi}{\sqrt{k/m}} = 2\pi\sqrt{\frac{m}{k}}$

(c) $\dfrac{dT}{dm} = \dfrac{2\pi}{\sqrt{k}} \cdot \dfrac{1}{2}m^{-\frac{1}{2}} = \dfrac{\pi}{\sqrt{km}}$

Since $\dfrac{dT}{dm} > 0$, an increase in the mass causes the period to increase.

**48.** The tangent lines to $f(x) = \sin x$ have slope $\dfrac{d}{dx}(\sin x) = \cos x$. The tangent line at $x = 0$ has slope $f'(0) = \cos 0 = 1$ and goes through the point $(0, 0)$. Consequently, its equation is $y = g(x) = x$. The approximate value of $\sin(\pi/6)$ given by this equation is $g(\pi/6) = \pi/6 \approx 0.524$.

Similarly, the tangent line at $x = \frac{\pi}{3}$ has slope

$$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

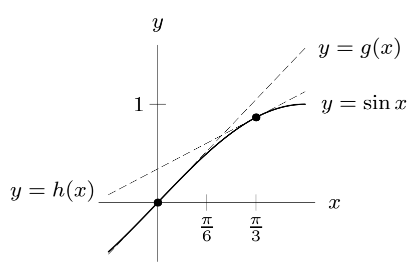
and goes through the point $(\pi/3, \sqrt{3}/2)$. Consequently, its equation is

$$y = h(x) = \frac{1}{2}x + \frac{3\sqrt{3} - \pi}{6}.$$

The approximate value of $\sin(\pi/6)$ given by this equation is then

$$h\left(\frac{\pi}{6}\right) = \frac{6\sqrt{3} - \pi}{12} \approx 0.604.$$

The actual value of $\sin(\pi/6)$ is $\frac{1}{2}$, so the approximation from 0 is better than that from $\pi/3$. This is because the slope of the function changes less between $x = 0$ and $x = \pi/6$ than it does between $x = \pi/6$ and $x = \pi/3$. This is illustrated by the following figure.



**49.** If the graphs of $y = \sin x$ and $y = ke^{-x}$ are tangent, then the $y$-values and the derivatives, $\dfrac{dy}{dx} = \cos x$ and $\dfrac{dy}{dx} = -ke^{-x}$, are equal at that point, so

$$\sin x = ke^{-x} \qquad \text{and} \qquad \cos x = -ke^{-x}.$$

Thus $\sin x = -\cos x$ so $\tan x = -1$. The smallest $x$-value is $x = 3\pi/4$, which leads to the smallest $k$ value

$$k = \frac{\sin(3\pi/4)}{e^{-3\pi/4}} = 7.46.$$

When $x = \dfrac{3\pi}{4}$, we have $y = \sin\left(\dfrac{3\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$ so the point is $\left(\dfrac{3\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$.

**50.** Differentiating with respect to $t$ using the chain rule and substituting for $dx/dt$ gives

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dx}(x \sin x) \cdot \frac{dx}{dt} = (\sin x + x\cos x)x \sin x.$$

**51.** (a) If $f(x) = \sin x$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{(\sin x \cos h + \sin h \cos x) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h} \\ &= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}. \end{aligned}$$

(b) $\frac{\cos h - 1}{h} \to 0$ and $\frac{\sin h}{h} \to 1$, as $h \to 0$. Thus, $f'(x) = \sin x \cdot 0 + \cos x \cdot 1 = \cos x$.

(c) Similarly,

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\ &= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\ &= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\ &= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \\ &= -\sin x. \end{aligned}$$

**52.** (a) Sector OAQ is a sector of a circle with radius $\frac{1}{\cos\theta}$ and angle $\Delta\theta$. Thus its area is the left side of the inequality. Similarly, the area of Sector OBR is the right side of the equality. The area of the triangle OQR is $\frac{1}{2}\Delta \tan\theta$ since it is a triangle with base $\Delta \tan\theta$ (the segment QR) and height 1 (if you turn it sideways, it is easier to see this). Thus, using the given fact about areas (which is also clear from looking at the picture), we have

$$\frac{\Delta\theta}{2\pi} \cdot \pi \left(\frac{1}{\cos\theta}\right)^2 \le \frac{1}{2} \cdot \Delta(\tan\theta) \le \frac{\Delta\theta}{2\pi} \cdot \pi \left(\frac{1}{\cos(\theta + \Delta\theta)}\right)^2.$$

(b) Dividing the inequality through by $\frac{\Delta\theta}{2}$ and canceling the $\pi$'s gives:

$$\left(\frac{1}{\cos\theta}\right)^2 \le \frac{\Delta \tan\theta}{\Delta\theta} \le \left(\frac{1}{\cos(\theta+\Delta\theta)}\right)^2$$

Then as $\Delta\theta \to 0$, the right and left sides both tend toward $\left(\frac{1}{\cos\theta}\right)^2$ while the middle (which is the difference quotient for tangent) tends to $(\tan\theta)'$. Thus, the derivative of tangent is "squeezed" between two values heading toward the same thing and must, itself, also tend to that value. Therefore, $(\tan\theta)' = \left(\frac{1}{\cos\theta}\right)^2$.

(c) Take the identity $\sin^2\theta + \cos^2\theta = 1$ and divide through by $\cos^2\theta$ to get $(\tan\theta)^2 + 1 = (\frac{1}{\cos\theta})^2$. Differentiating with respect to $\theta$ yields:

$$\begin{aligned} 2(\tan\theta)\cdot(\tan\theta)' &= 2\left(\frac{1}{\cos\theta}\right)\cdot\left(\frac{1}{\cos\theta}\right)' \\ 2\left(\frac{\sin\theta}{\cos\theta}\right)\cdot\left(\frac{1}{\cos\theta}\right)^2 &= 2\left(\frac{1}{\cos\theta}\right)\cdot(-1)\left(\frac{1}{\cos\theta}\right)^2(\cos\theta)' \\ 2\frac{\sin\theta}{\cos^3\theta} &= (-1)2\frac{1}{\cos^3\theta}(\cos\theta)' \\ -\sin\theta &= (\cos\theta)'. \end{aligned}$$

(d)

$$\begin{aligned} \frac{d}{d\theta}\left(\sin^2\theta + \cos^2\theta\right) &= \frac{d}{d\theta}(1) \\ 2\sin\theta\cdot(\sin\theta)' + 2\cos\theta\cdot(\cos\theta)' &= 0 \\ 2\sin\theta\cdot(\sin\theta)' + 2\cos\theta\cdot(-\sin\theta) &= 0 \\ (\sin\theta)' - \cos\theta &= 0 \\ (\sin\theta)' &= \cos\theta. \end{aligned}$$

## Solutions for Section 3.6

### Exercises

**1.** $f'(t) = \dfrac{2t}{t^2+1}$.

**2.** $f'(x) = \dfrac{-1}{1-x} = \dfrac{1}{x-1}$.

**3.** Since $\ln(e^{2x}) = 2x$, the derivative $f'(x) = 2$.

**4.** Since $e^{\ln(e^{2x^2+3})} = e^{2x^2+3}$, the derivative $f'(x) = 4xe^{2x^2+3}$.

**5.** $f'(x) = \dfrac{1}{1-e^{-x}}\cdot(-e^{-x})(-1) = \dfrac{e^{-x}}{1-e^{-x}}$.

**6.** $f'(\alpha) = \dfrac{1}{\sin\alpha}\cdot\cos\alpha = \dfrac{\cos\alpha}{\sin\alpha}$.

**7.** $f'(x) = \dfrac{1}{e^x+1}\cdot e^x$.

**8.** $\dfrac{dy}{dx} = \ln x + x\left(\dfrac{1}{x}\right) - 1 = \ln x$

**9.** $j'(x) = \dfrac{ae^{ax}}{(e^{ax}+b)}$

**10.** Using the product and chain rules gives $h'(w) = 3w^2\ln(10w) + w^3\dfrac{10}{10w} = 3w^2\ln(10w) + w^2$.

**11.** $f'(x) = \dfrac{1}{e^{7x}}\cdot(e^{7x})7 = 7$.
(Note also that $\ln(e^{7x}) = 7x$ implies $f'(x) = 7$.)

**12.** Note that $f(x) = e^{\ln x}\cdot e^1 = x\cdot e = ex$. So $f'(x) = e$. (Remember, $e$ is just a constant.) You might also use the chain rule to get:
$f'(x) = e^{(\ln x)+1}\cdot\frac{1}{x}$.
[Are the two answers the same? Of course they are, since

$$e^{(\ln x)+1}\left(\frac{1}{x}\right) = e^{\ln x}\cdot e\left(\frac{1}{x}\right) = xe\left(\frac{1}{x}\right) = e.]$$

**13.** $f'(w) = \dfrac{1}{\cos(w-1)}[-\sin(w-1)] = -\tan(w-1).$

[This could be done easily using the answer from Problem 6 and the chain rule.]

**14.** $f(t) = \ln t$ (because $\ln e^x = x$ *or* because $e^{\ln t} = t$), so $f'(t) = \frac{1}{t}$.

**15.** $f'(y) = \dfrac{2y}{\sqrt{1-y^4}}.$

**16.** $g'(t) = \dfrac{3}{(3t-4)^2+1}.$

**17.** $g(\alpha) = \alpha$, so $g'(\alpha) = 1$.

**18.** $g'(t) = e^{\arctan(3t^2)}\left(\dfrac{1}{1+(3t^2)^2}\right)(6t) = e^{\arctan(3t^2)}\left(\dfrac{6t}{1+9t^4}\right).$

**19.** $g'(t) = \dfrac{-\sin(\ln t)}{t}.$

**20.** $h'(z) = (\ln 2)z^{(\ln 2 - 1)}.$

**21.** $h'(w) = \arcsin w + \dfrac{w}{\sqrt{1-w^2}}.$

**22.** Note that $f(x) = kx$ so, $f'(x) = k$.

**23.** Using the chain rule gives $r'(t) = \dfrac{2}{\sqrt{1-4t^2}}.$

**24.** $j'(x) = -\sin\left(\sin^{-1} x\right)\cdot\left[\dfrac{1}{\sqrt{1-x^2}}\right] = -\dfrac{x}{\sqrt{1-x^2}}$

**25.** $f'(x) = -\sin(\arctan 3x)\left(\dfrac{1}{1+(3x)^2}\right)(3) = \dfrac{-3\sin(\arctan 3x)}{1+9x^2}.$

**26.** Note that $g(x) = \arcsin(\sin \pi x) = \pi x$.
Thus, $g'(x) = \pi$.

**27.** $f'(z) = -1(\ln z)^{-2}\cdot\dfrac{1}{z} = \dfrac{-1}{z(\ln z)^2}.$

**28.** Using the quotient rule gives

$$\begin{aligned} f'(x) &= \frac{1+\ln x - x(\frac{1}{x})}{(1+\ln x)^2} \\ &= \frac{\ln x}{(1+\ln x)^2}. \end{aligned}$$

**29.** $\dfrac{dy}{dx} = 2(\ln x + \ln 2) + 2x\left(\dfrac{1}{x}\right) - 2 = 2(\ln x + \ln 2) = 2\ln(2x)$

**30.** Using the chain rule gives $f'(x) = \dfrac{\cos x - \sin x}{\sin x + \cos x}.$

**31.** $f'(t) = \dfrac{1}{\ln t}\cdot\dfrac{1}{t} = \dfrac{1}{t\ln t}$

**32.** Using the chain rule gives

$$\begin{aligned} T'(u) &= \left[\frac{1}{1+\left(\frac{u}{1+u}\right)^2}\right]\left[\frac{(1+u)-u}{(1+u)^2}\right] \\ &= \frac{(1+u)^2}{(1+u)^2+u^2}\left[\frac{1}{(1+u)^2}\right] \\ &= \frac{1}{1+2u+2u^2}. \end{aligned}$$

**33.** Since $\ln\left[\left(\frac{1-\cos t}{1+\cos t}\right)^4\right] = 4\ln\left[\left(\frac{1-\cos t}{1+\cos t}\right)\right]$ we have

$$\begin{aligned} a'(t) &= 4\left(\frac{1+\cos t}{1-\cos t}\right)\left[\frac{\sin t(1+\cos t)+\sin t(1-\cos t)}{(1+\cos t)^2}\right] \\ &= \left[\frac{1+\cos t}{1-\cos t}\right]\left[\frac{8\sin t}{(1+\cos t)^2}\right] \\ &= \frac{8\sin t}{1-\cos^2 t} \\ &= \frac{8}{\sin t}. \end{aligned}$$

**34.** $f'(x) = -\sin(\arcsin(x+1))(\frac{1}{\sqrt{1-(x+1)^2}}) = \frac{-(x+1)}{\sqrt{1-(x+1)^2}}$.

## Problems

**35.** From the graphs, we estimate $g(1) \approx 2$, $g'(1) \approx 1$, and $f'(2) \approx 0.8$. Thus, by the chain rule,

$$h'(1) = f'(g(1)) \cdot g'(1) \approx f'(2) \cdot g'(1) \approx 0.8 \cdot 1 = 0.8.$$

**36.** From the graphs, we estimate $f(1) \approx -0.4$, $f'(1) \approx 0.5$, and $g'(-0.4) \approx 2$. Thus, by the chain rule,

$$k'(1) = g'(f(1)) \cdot f'(1) \approx g'(-0.4) \cdot 0.5 \approx 2 \cdot 0.5 = 1.$$

**37.** From the graphs, we estimate $g(2) \approx 1.6$, $g'(2) \approx -0.5$, and $f'(1.6) \approx 0.8$. Thus, by the chain rule,

$$h'(2) = f'(g(2)) \cdot g'(2) \approx f'(1.6) \cdot g'(2) \approx 0.8(-0.5) = -0.4.$$

**38.** From the graphs, we estimate $f(2) \approx 0.3$, $f'(2) \approx 1.1$, and $g'(0.3) \approx 1.7$. Thus, by the chain rule,

$$k'(2) = g'(f(2)) \cdot f'(2) \approx g'(0.3) \cdot f'(2) \approx 1.7 \cdot 1.1 \approx 1.9.$$

**39.** Differentiating

$$\begin{aligned} f'(x) &= \frac{1}{x^2+1} \cdot 2x = 2x(x^2+1)^{-1} \\ f''(x) &= 2(x^2+1)^{-1} - 2x(x^2+1)^{-2} \cdot 2x \\ &= \frac{2}{(x^2+1)} - \frac{4x^2}{(x^2+1)^2} = \frac{2x^2+2}{(x^2+1)^2} - \frac{4x^2}{(x^2+1)^2} \\ &= \frac{2(1-x^2)}{(x^2+1)^2}. \end{aligned}$$

Since $(x^2+1)^2 > 0$ for all $x$, we see that $f''(0) > 0$ for $1 - x^2 > 0$ or $x^2 < 1$. That is, $\ln(x^2+1)$ is concave up on the interval $-1 < x < 1$.

**40.** Let

$$g(x) = \arcsin x$$

so

$$\sin[g(x)] = x.$$

Differentiating,

$$\begin{aligned} \cos[g(x)] \cdot g'(x) &= 1 \\ g'(x) &= \frac{1}{\cos[g(x)]} \end{aligned}$$

Using the fact that $\sin^2\theta + \cos^2\theta = 1$, and $\cos[g(x)] \ge 0$, since $-\frac{\pi}{2} \le g(x) \le \frac{\pi}{2}$, we get

$$\cos[g(x)] = \sqrt{1 - (\sin[g(x)])^2}.$$

Therefore,

$$g'(x) = \frac{1}{\sqrt{1-(\sin[g(x)])^2}}$$

Since $\sin[g(x)] = x$, we have

$$g'(x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1.$$

**41.** Let

$$g(x) = \log x.$$

Then

$$10^{g(x)} = x.$$

Differentiating,

$$\begin{aligned}(\ln 10)[10^{g(x)}]g'(x) &= 1\\ g'(x) &= \frac{1}{(\ln 10)[10^{g(x)}]}\\ g'(x) &= \frac{1}{(\ln 10)x}.\end{aligned}$$

**42.** $\text{pH} = 2 = -\log x$ means $\log x = -2$ so $x = 10^{-2}$. Rate of change of pH with hydrogen ion concentration is

$$\frac{d}{dx}\text{pH} = -\frac{d}{dx}(\log x) = \frac{-1}{x(\ln 10)} = -\frac{1}{(10^{-2})\ln 10} = -43.4$$

**43.** (a) For $y = \ln x$, we have $y' = 1/x$, so the slope of the tangent line is $f'(1) = 1/1 = 1$. The equation of the tangent line is $y - 0 = 1(x-1)$, so, on the tangent line, $y = g(x) = x - 1$.

(b) Using a value on the tangent line to approximate $\ln(1,1)$, we have

$$\ln(1.1) \approx g(1.1) = 1.1 - 1 = 0.1.$$

Similarly, $\ln(2)$ is approximated by

$$\ln(2) \approx g(2) = 2 - 1 = 1.$$

(c) From Figure 3.6, we see that $f(1.1)$ and $f(2)$ are below $g(x) = x - 1$. Similarly, $f(0.9)$ and $f(0.5)$ are also below $g(x)$. This is true for any approximation of this function by a tangent line since $f$ is concave down ($f''(x) = -\frac{1}{x^2} < 0$ for all $x > 0$). Thus, for a given $x$-value, the $y$-value given by the function is always below the value given by the tangent line.

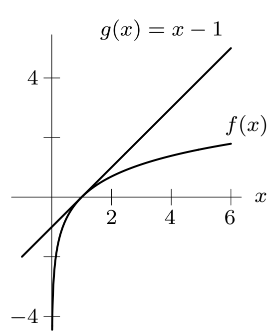



Figure 3.6

**44.** (a) Let $g(x) = ax^2 + bx + c$ be our quadratic and $f(x) = \ln x$. For the best approximation, we want to find a quadratic with the same value as $\ln x$ at $x = 1$ and the same first and second derivatives as $\ln x$ at $x = 1$. $g'(x) = 2ax + b, g''(x) = 2a, f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}$.

$$\begin{aligned} g(1) = a(1)^2 + b(1) + c &\quad f(1) = 0 \\ g'(1) = 2a(1) + b &\quad f'(1) = 1 \\ g''(1) = 2a &\quad f''(1) = -1 \end{aligned}$$

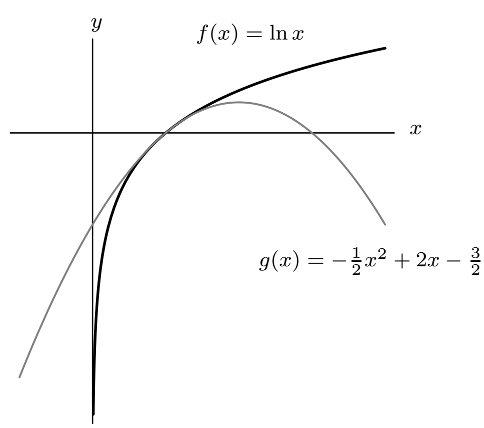
Thus, we obtain the equations

$$\begin{aligned} a + b + c &= 0 \\ 2a + b &= 1 \\ 2a &= -1 \end{aligned}$$

We find $a = -\frac{1}{2}$, $b = 2$ and $c = -\frac{3}{2}$. Thus our approximation is:

$$g(x) = -\frac{1}{2}x^2 + 2x - \frac{3}{2}$$

(b) From the graph below, we notice that around $x = 1$, the value of $f(x) = \ln x$ and the value of $g(x) = -\frac{1}{2}x^2 + 2x - \frac{3}{2}$ are very close.




(c) $g(1.1) = 0.095 \quad g(2) = 0.5$
Compare with $f(1.1) = 0.0953, f(2) = 0.693$.

**45.** (a)

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) \\ &= \frac{1}{1+x^2} + \left(-\frac{1}{x^2+1}\right) \\ &= \frac{1}{1+x^2} - \frac{1}{1+x^2} \\ &= 0 \end{aligned}$$

(b) $f$ is a constant function. Checking at a few values of $x$,

**Table 3.2**

| $x$ | $\arctan x$ | $\arctan x^{-1}$ | $f(x) = \arctan x + \arctan x^{-1}$ |
|---|---|---|---|
| 1 | 0.785392 | 0.7853982 | 1.5707963 |
| 2 | 1.1071487 | 0.4636476 | 1.5707963 |
| 3 | 1.2490458 | 0.3217506 | 1.5707963 |

**46.** The closer you look at the function, the more it begins to look like a line with slope equal to the derivative of the function at $x = 0$. Hence, functions whose derivatives at $x = 0$ are equal will look the same there.

The following functions look like the line $y = x$ since, in all cases, $y' = 1$ at $x = 0$.

| | |
|---|---|
| $y = x$ | $y' = 1$ |
| $y = \sin x$ | $y' = \cos x$ |
| $y = \tan x$ | $y' = \frac{1}{\cos^2 x}$ |
| $y = \ln(x+1)$ | $y' = \frac{1}{x+1}$ |

The following functions look like the line $y = 0$ since, in all cases, $y' = 0$ at $x = 0$.

| | |
|---|---|
| $y = x^2$ | $y' = 2x$ |
| $y = x\sin x$ | $y' = x\cos x + \sin x$ |
| $y = x^3$ | $y' = 3x^2$ |
| $y = \frac{1}{2}\ln(x^2+1)$ | $y' = 2x \cdot \frac{1}{2} \cdot \frac{1}{x^2+1} = \frac{x}{x^2+1}$ |
| $y = 1 - \cos x$ | $y' = \sin x$ |

The following functions look like the line $x = 0$ since, in all cases, as $x \to 0^+$, the slope $y' \to \infty$.

| | |
|---|---|
| $y = \sqrt{x}$ | $y' = \frac{1}{2\sqrt{x}}$ |
| $y = \sqrt{\frac{x}{x+1}}$ | $y' = \frac{(x+1)-x}{(x+1)^2} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{x}{x+1}}} = \frac{1}{2(x+1)^2} \cdot \sqrt{\frac{x+1}{x}}$ |
| $y = \sqrt{2x - x^2}$ | $y' = (2-2x)\frac{1}{2} \cdot \frac{1}{\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{2x-x^2}}$ |

**47.** Since the chain rule gives $h'(x) = n'(m(x))m'(x) = -2$ we must find values $a$ and $x$ such that $a = m(x)$ and $n'(a)m'(x) = -2$.

Calculating slopes from the graph of $n$ gives

$$n'(a) = \begin{cases} 1 & \text{if } 0 < a < 50 \\ 1/2 & \text{if } 50 < a < 100. \end{cases}$$

Calculating slopes from the graph of $m$ gives

$$m'(x) = \begin{cases} -2 & \text{if } 0 < x < 50 \\ 2 & \text{if } 50 < x < 100. \end{cases}$$

The only values of the derivative $n'$ are 1 and $1/2$ and the only values of the derivative $m'$ are 2 and $-2$. In order to have $n'(a)m'(x) = -2$ we must therefore have $n'(a) = 1$ and $m'(x) = -2$. Thus $0 < a < 50$ and $0 < x < 50$.

Now $a = m(x)$ and from the graph of $m$ we see that $0 < m(x) < 50$ for $25 < x < 75$.

The two conditions on $x$ we have found are both satisfied when $25 < x < 50$. Thus $h'(x) = -2$ for all $x$ in the interval $25 < x < 50$. The question asks for just one of these $x$ values, for example $x = 40$.

**48.** Since the chain rule gives $h'(x) = n'(m(x))m'(x) = 2$ we must find values $a$ and $x$ such that $a = m(x)$ and $n'(a)m'(x) = 2$.

Calculating slopes from the graph of $n$ gives

$$n'(a) = \begin{cases} 1 & \text{if } 0 < a < 50 \\ 1/2 & \text{if } 50 < a < 100. \end{cases}$$

Calculating slopes from the graph of $m$ gives

$$m'(x) = \begin{cases} -2 & \text{if } 0 < x < 50 \\ 2 & \text{if } 50 < x < 100. \end{cases}$$

The only values of the derivative $n'$ are 1 and $1/2$ and the only values of the derivative $m'$ are 2 and $-2$. In order to have $n'(a)m'(x) = 2$ we must therefore have $n'(a) = 1$ and $m'(x) = 2$. Thus $0 < a < 50$ and $50 < x < 100$.

Now $a = m(x)$ and from the graph of $m$ we see that $0 < m(x) < 50$ for $25 < x < 75$.

The two conditions on $x$ we have found are both satisfied when $50 < x < 75$. Thus $h'(x) = 2$ for all $x$ in the interval $50 < x < 75$. The question asks for just one of these $x$ values, for example $x = 60$.

**49.** Since the chain rule gives $h'(x) = n'(m(x))m'(x) = 1$ we must find values $a$ and $x$ such that $a = m(x)$ and $n'(a)m'(x) = 1$.

Calculating slopes from the graph of $n$ gives

$$n'(a) = \begin{cases} 1 & \text{if } 0 < a < 50 \\ 1/2 & \text{if } 50 < a < 100. \end{cases}$$

Calculating slopes from the graph of $m$ gives

$$m'(x) = \begin{cases} -2 & \text{if } 0 < x < 50 \\ 2 & \text{if } 50 < x < 100. \end{cases}$$

The only values of the derivative $n'$ are 1 and $1/2$ and the only values of the derivative $m'$ are 2 and $-2$. In order to have $n'(a)m'(x) = 1$ we must therefore have $n'(a) = 1/2$ and $m'(x) = 2$. Thus $50 < a < 100$ and $50 < x < 100$.

Now $a = m(x)$ and from the graph of $m$ we see that $50 < m(x) < 100$ for $0 < x < 25$ or $75 < x < 100$.

The two conditions on $x$ we have found are both satisfied when $75 < x < 100$. Thus $h'(x) = 1$ for all $x$ in the interval $75 < x < 100$. The question asks for just one of these $x$ values, for example $x = 80$.

**50.** Since the chain rule gives $h'(x) = n'(m(x))m'(x) = -1$ we must find values $a$ and $x$ such that $a = m(x)$ and $n'(a)m'(x) = -1$.

Calculating slopes from the graph of $n$ gives

$$n'(a) = \begin{cases} 1 & \text{if } 0 < a < 50 \\ 1/2 & \text{if } 50 < a < 100. \end{cases}$$

Calculating slopes from the graph of $m$ gives

$$m'(x) = \begin{cases} -2 & \text{if } 0 < x < 50 \\ 2 & \text{if } 50 < x < 100. \end{cases}$$

The only values of the derivative $n'$ are 1 and $1/2$ and the only values of the derivative $m'$ are 2 and $-2$. In order to have $n'(a)m'(x) = -1$ we must therefore have $n'(a) = 1/2$ and $m'(x) = -2$. Thus $50 < a < 100$ and $0 < x < 50$.

Now $a = m(x)$ and from the graph of $m$ we see that $50 < m(x) < 100$ for $0 < x < 25$ or $75 < x < 100$.

The two conditions on $x$ we have found are both satisfied when $0 < x < 25$. Thus $h'(x) = -1$ for all $x$ in the interval $0 < x < 25$. The question asks for just one of these $x$ values, for example $x = 10$.

**51.** Since the point $(2, 5)$ is on the curve, we know $f(2) = 5$. The point $(2.1, 5.3)$ is on the tangent line, so

$$\text{Slope tangent } = \frac{5.3 - 5}{2.1 - 2} = \frac{0.3}{0.1} = 3.$$

Thus, $f'(2) = 3$.

By the chain rule

$$h'(2) = 3(f(2))^2 \cdot f'(2) = 3 \cdot 5^2 \cdot 3 = 225.$$

**52.** Since the point $(2, 5)$ is on the curve, we know $f(2) = 5$. The point $(2.1, 5.3)$ is on the tangent line, so

$$\text{Slope tangent } = \frac{5.3 - 5}{2.1 - 2} = \frac{0.3}{0.1} = 3.$$

Thus, $f'(2) = 3$.

By the chain rule

$$k'(2) = -(f(2))^{-2} \cdot f'(2) = -5^{-2} \cdot 3 = -0.12.$$

**53.** Since the point $(2, 5)$ is on the curve, we know $f(2) = 5$. The point $(2.1, 5.3)$ is on the tangent line, so

$$\text{Slope tangent } = \frac{5.3 - 5}{2.1 - 2} = \frac{0.3}{0.1} = 3.$$

Thus, $f'(2) = 3$. Since $g$ is the inverse function of $f$ and $f(2) = 5$, we know $f^{-1}(5) = 2$, so $g(5) = 2$.

Differentiating, we have

$$g'(2) = \frac{1}{f'(g(5))} = \frac{1}{f'(2)} = \frac{1}{3}.$$

**54.** (a) Since $f(x) = x^3$, we have $f'(x) = 3x^2$. Thus, $f'(2) = 3(2)^2 = 12$.
(b) To find $f^{-1}(x)$, we switch $x$s and $y$s and solve for $y$.
Since $y = x^3$, we get $x = y^3$.
Solving for $y$ gives $y = \sqrt[3]{x}$.
Thus, $f^{-1}(x) = \sqrt[3]{x}$.
(c) To find $(f^{-1})'(x)$, we differentiate. Since $f^{-1}(x) = \sqrt[3]{x} = x^{1/3}$, we get

$$(f^{-1})'(x) = \frac{1}{3}x^{-2/3}.$$

Thus,

$$(f^{-1})'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{3 \cdot 4} = \frac{1}{12}.$$

(d) The point $(2, 8)$ is on the graph of $f$. Thus the point $(8, 2)$ is on the graph of $f^{-1}$, so $f^{-1}(8) = 2$. Therefore,

$$(f^{-1})'(8) = \frac{1}{f'(f^{-1}(8))} = \frac{1}{f'(2)} = \frac{1}{12}.$$

**55.** (a) Since $f(x) = 2x^5 + 3x^3 + x$, we differentiate to get $f'(x) = 10x^4 + 9x^2 + 1$.
(b) Because $f'(x)$ is always positive, we know that $f(x)$ is increasing everywhere. Thus, $f(x)$ is a one-to-one function and is invertible.
(c) To find $f(1)$, substitute 1 for $x$ into $f(x)$. We get $f(1) = 2(1)^5 + 3(1)^3 + 1 = 2 + 3 + 1 = 6$.
(d) To find $f'(1)$, substitute 1 for $x$ into $f'(x)$. We get $f'(1) = 10(1)^4 + 9(1)^2 + 1 = 20$.
(e) Since $f(1) = 6$, we have $f^{-1}(6) = 1$, so

$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(1)} = \frac{1}{20}.$$

**56.** Since $g$ is the inverse of $f$, we know that $g(4) = f^{-1}(4) = 3$, so

$$g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(3)} = \frac{1}{6}.$$

**57.** To find $(f^{-1})'(3)$, we first look in the table to find that $3 = f(9)$, so $f^{-1}(3) = 9$. Thus,

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(9)} = \frac{1}{5}.$$

**58.** (a) Knowing $f(2000) = 281$ tells us that the US population was 281 million in the year 2000.
(b) Since $f(2000) = 281$, we have $f^{-1}(281) = 2000$. This tells us that the year in which the US population was 281 million was 2000.
(c) Knowing $f'(2000) = 3.476$ tells us that in the year 2000, the US population was growing at a rate of 3.476 million people per year.
(d) Using parts (b) and (c), we have

$$(f^{-1})'(281) = \frac{1}{f'(f^{-1}(281))} = \frac{1}{f'(2000)} = \frac{1}{3.476} = 0.288.$$

The units of the derivative of $f^{-1}$ are years per million people (the reciprocal of the units of $f'$). The statement $(f^{-1})'(281) = 0.288$ tells us that when the US population was 281 million, it took 0.288 of a year (between 3 and 4 months) for the population to increase by another million.

**59.** Each grid mark on the horizontal axis represents 3 years and each grid mark on the vertical axis represents 50 million vehicles.
(a) Reading from the graph

$$f(21) \approx 200 \text{ million vehicles.}$$

This tells us that 21 years after 1946, in 1967, there were 200 million registered vehicles.
(b) Drawing a tangent line to the curve at $t = 21$, we have

$$\text{Slope } = f'(21) \approx \frac{90}{6} = 15 \text{ million vehicles/year.}$$

Thus, 21 years after 1946, in 1967, the number of registered vehicles was increasing at 15 million vehicles per year.

(c) From the graph or part (a)
$$f^{-1}(200) = 21 \text{ years.}$$
Thus, there were 200 million cars registered when $t = 21$,that is, in 1967.

(d) We have
$$(f^{-1})'(200) = \frac{1}{f'(f^{-1}(200))} = \frac{1}{f'(21)} = \frac{1}{15} = 0.0667 \text{ years/million.}$$
Thus, when 200 million vehicles were already registered, it took 0.0667 year, or about 24 days, for another million to be registered.

**60.** We have $(f^{-1})'(8) = 1/f'(f^{-1}(8))$. From the graph we see $f^{-1}(8) = 4$. Thus $(f^{-1})'(8) = \dfrac{1}{f'(4)} = \dfrac{1}{3.0}$.

**61.** We must have
$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(10)} = \frac{1}{8}.$$

**62.** All three values equal 1.

(a) We have $f^{-1}(A) = a$, so $(f^{-1})'(A) = \frac{1}{f'(f^{-1}(A))} = \frac{1}{f'(a)}$. Thus $f'(a)(f^{-1})'(A) = 1$.

(b) We have $f^{-1}(B) = b$, so $(f^{-1})'(B) = \frac{1}{f'(f^{-1}(B))} = \frac{1}{f'(b)}$. Thus $f'(b)(f^{-1})'(B) = 1$.

(c) We have $f^{-1}(C) = c$, so $(f^{-1})'(C) = \frac{1}{f'(f^{-1}(C))} = \frac{1}{f'(c)}$. Thus $f'(c)(f^{-1})'(C) = 1$.

**63.** A continuous invertible function $f(x)$ cannot be increasing on one interval and decreasing on another because it would fail the horizontal line test. The same is true of the inverse function $f^{-1}(x)$. Either $f^{-1}(x)$ is increasing and $(f^{-1})'(x) \geq 0$ for all $x$, or $f^{-1}(x)$ is decreasing and $(f^{-1})'(x) \leq 0$ for all $x$. We can not have both $(f^{-1})'(10) = 8$ and $(f^{-1})'(20) = -6$.

**64.** (a) The definition of the derivative of $\ln(1+x)$ at $x = 0$ is
$$\lim_{h\to 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0} \frac{\ln(1+h)}{h} = \left.\frac{1}{1+x}\right|_{x=0} = 1.$$

(b) The rules of logarithms give
$$\lim_{h\to 0} \frac{\ln(1+h)}{h} = \lim_{h\to 0} \frac{1}{h}\ln(1+h) = \lim_{h\to 0} \ln(1+h)^{1/h} = 1.$$
Thus, taking $e$ to both sides and using the fact that $e^{\ln A} = A$, we have
$$e^{\lim_{h\to 0}\ln(1+h)^{1/h}} = \lim_{h\to 0} e^{\ln(1+h)^{1/h}} = e^1$$
$$\lim_{h\to 0}(1+h)^{1/h} = e.$$
This limit is sometimes used as the definition of $e$.

(c) Let $n = 1/h$. Then as $h \to 0^+$, we have $n \to \infty$. Since
$$\lim_{h\to 0^+}(1+h)^{1/h} = \lim_{h\to 0}(1+h)^{1/h} = e,$$
we have
$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e.$$
This limit is also sometimes used as the definition of $e$.

## Solutions for Section 3.7

### Exercises

**1.** We differentiate implicitly both sides of the equation with respect to $x$.
$$2x + 2y\frac{dy}{dx} = 0\,,$$
$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

**2.** We differentiate implicitly both sides of the equation with respect to $x$.

$$2x + \left(y + x\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} = y^2 + x(2y)\frac{dy}{dx}\,,$$
$$x\frac{dy}{dx} - 3y^2\frac{dy}{dx} - 2xy\frac{dy}{dx} = y^2 - y - 2x\,,$$
$$\frac{dy}{dx} = \frac{y^2 - y - 2x}{x - 3y^2 - 2xy}.$$

**3.** Implicit differentiation gives

$$1 \cdot y + x \cdot \frac{dy}{dx} + 1 + \frac{dy}{dx} = 0.$$

Solving for $dy/dx$, we have

$$\frac{dy}{dx} = -\frac{1+y}{1+x}.$$

**4.**

$$2xy + x^2\frac{dy}{dx} - 2\frac{dy}{dx} = 0$$
$$(x^2 - 2)\frac{dy}{dx} = -2xy$$
$$\frac{dy}{dx} = \frac{-2xy}{(x^2 - 2)}$$

**5.** We differentiate implicitly both sides of the equation with respect to $x$.

$$x^{1/2} = 5y^{1/2}$$
$$\frac{1}{2}x^{-1/2} = \frac{5}{2}y^{-1/2}\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{\frac{1}{2}x^{-1/2}}{\frac{5}{2}y^{-1/2}} = \frac{1}{5}\sqrt{\frac{y}{x}} = \frac{1}{25}.$$

We can also obtain this answer by realizing that the original equation represents part of the line $x = 25y$ which has slope $1/25$.

**6.** We differentiate implicitly both sides of the equation with respect to $x$.

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 25\,,$$
$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0\,,$$
$$\frac{dy}{dx} = -\frac{\frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{2}y^{-\frac{1}{2}}} = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} = -\frac{\sqrt{y}}{\sqrt{x}} = -\sqrt{\frac{y}{x}}.$$

**7.** We differentiate implicitly with respect to $x$.

$$y + x\frac{dy}{dx} - 1 - \frac{3dy}{dx} = 0$$
$$(x - 3)\frac{dy}{dx} = 1 - y$$
$$\frac{dy}{dx} = \frac{1-y}{x-3}$$

**8.**

$$12x + 8y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-12x}{8y} = \frac{-3x}{2y}$$

**9.**

$$2ax - 2by\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-2ax}{-2by} = \frac{ax}{by}$$

**10.** We differentiate implicitly both sides of the equation with respect to $x$.

$$\begin{aligned} \ln x + \ln(y^2) &= 3 \\ \frac{1}{x} + \frac{1}{y^2}(2y)\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-1/x}{2y/y^2} = -\frac{y}{2x}. \end{aligned}$$

**11.** We differentiate implicitly both sides of the equation with respect to $x$.

$$\begin{aligned} \ln y + x\frac{1}{y}\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= \frac{1}{x} \\ \frac{x}{y}\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= \frac{1}{x} - \ln y \\ \frac{dy}{dx}\left(\frac{x}{y} + 3y^2\right) &= \frac{1 - x\ln y}{x} \\ \frac{dy}{dx}\left(\frac{x + 3y^3}{y}\right) &= \frac{1 - x\ln y}{x} \\ \frac{dy}{dx} &= \frac{(1 - x\ln y)}{x} \cdot \frac{y}{(x + 3y^3)} \end{aligned}$$

**12.** We differentiate implicitly both sides of the equation with respect to $x$.

$$\begin{aligned} \cos(xy)\left(y + x\frac{dy}{dx}\right) &= 2 \\ y\cos(xy) + x\cos(xy)\frac{dy}{dx} &= 2 \\ \frac{dy}{dx} &= \frac{2 - y\cos(xy)}{x\cos(xy)}. \end{aligned}$$

**13.** Using the relation $\cos^2 y + \sin^2 y = 1$, the equation becomes:
$1 = y + 2$ or $y = -1$. Hence, $\frac{dy}{dx} = 0$.

**14.** We differentiate implicitly both sides of the equation with respect to $x$.

$$\begin{aligned} e^{\cos y}(-\sin y)\frac{dy}{dx} &= 3x^2 \arctan y + x^3 \frac{1}{1+y^2}\frac{dy}{dx} \\ \frac{dy}{dx}\left(-e^{\cos y}\sin y - \frac{x^3}{1+y^2}\right) &= 3x^2 \arctan y \\ \frac{dy}{dx} &= \frac{3x^2 \arctan y}{-e^{\cos y}\sin y - x^3(1+y^2)^{-1}}. \end{aligned}$$

**15.** We differentiate implicitly both sides of the equation with respect to $x$.

$$\begin{aligned} \arctan(x^2 y) &= xy^2 \\ \frac{1}{1+x^4y^2}(2xy + x^2\frac{dy}{dx}) &= y^2 + 2xy\frac{dy}{dx} \\ 2xy + x^2\frac{dy}{dx} &= [1 + x^4y^2][y^2 + 2xy\frac{dy}{dx}] \\ \frac{dy}{dx}[x^2 - (1+x^4y^2)(2xy)] &= (1+x^4y^2)y^2 - 2xy \\ \frac{dy}{dx} &= \frac{y^2 + x^4y^4 - 2xy}{x^2 - 2xy - 2x^5y^3}. \end{aligned}$$

**16.** We differentiate implicitly both sides of the equation with respect to $x$.

$$\begin{aligned} e^{x^2} + \ln y &= 0 \\ 2xe^{x^2} + \frac{1}{y}\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -2xye^{x^2}. \end{aligned}$$

**17.** We differentiate implicitly both sides of the equation with respect to $x$.

$$(x-a)^2 + y^2 = a^2$$

$$\begin{aligned} 2(x-a) + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= 2a - 2x \\ \frac{dy}{dx} &= \frac{2a-2x}{2y} = \frac{a-x}{y}. \end{aligned}$$

**18.** $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \cdot \frac{dy}{dx} = 0$, $\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$.

**19.** Differentiating $x^2 + y^2 = 1$ with respect to $x$ gives

$$2x + 2yy' = 0$$

so that

$$y' = -\frac{x}{y}$$

At the point $(0, 1)$ the slope is 0.

**20.** Differentiating $\sin(xy) = x$ with respect to $x$ gives

$$(y + xy')\cos(xy) = 1$$

or

$$xy'\cos(xy) = 1 - y\cos(xy)$$

so that

$$y' = \frac{1 - y\cos(xy)}{x\cos(xy)}.$$

As we move along the curve to the point $(1, \frac{\pi}{2})$, the value of $dy/dx \to \infty$, which tells us the tangent to the curve at $(1, \frac{\pi}{2})$ has infinite slope; the tangent is the vertical line $x = 1$.

**21.** Differentiating with respect to $x$ gives

$$3x^2 + 2xy' + 2y + 2yy' = 0$$

so that

$$y' = -\frac{3x^2 + 2y}{2x + 2y}$$

At the point $(1, 1)$ the slope is $-\frac{5}{4}$.

**22.** The slope is given by $dy/dx$, which we find using implicit differentiation. Notice that the product rule is needed for the second term. We differentiate to obtain:

$$\begin{aligned} 3x^2 + 5x^2\frac{dy}{dx} + 10xy + 4y\frac{dy}{dx} &= 4\frac{dy}{dx} \\ (5x^2 + 4y - 4)\frac{dy}{dx} &= -3x^2 - 10xy \\ \frac{dy}{dx} &= \frac{-3x^2 - 10xy}{5x^2 + 4y - 4}. \end{aligned}$$

At the point $(1, 2)$, we have $dy/dx = (-3-20)/(5+8-4) = -23/9$. The slope of this curve at the point $(1, 2)$ is $-23/9$.

**23.** First, we must find the slope of the tangent, i.e. $\left.\dfrac{dy}{dx}\right|_{(1,-1)}$. Differentiating implicitly, we have:

$$y^2 + x(2y)\frac{dy}{dx} = 0,$$
$$\frac{dy}{dx} = -\frac{y^2}{2xy} = -\frac{y}{2x}.$$

Substitution yields $\left.\dfrac{dy}{dx}\right|_{(1,-1)} = -\dfrac{-1}{2} = \dfrac{1}{2}$. Using the point-slope formula for a line, we have that the equation for the tangent line is $y + 1 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{3}{2}$.

**24.** First we must find the slope of the tangent, $\dfrac{dy}{dx}$, at $(1, e^2)$. Differentiating implicitly, we have:

$$\frac{1}{xy}\left(x\frac{dy}{dx} + y\right) = 2$$
$$\frac{dy}{dx} = \frac{2xy - y}{x}.$$

Evaluating $dy/dx$ at $(1, e^2)$ yields $(2(1)e^2 - e^2)/1 = e^2$. Using the point-slope formula for the equation of the line, we have:

$$y - e^2 = e^2(x - 1),$$

or

$$y = e^2 x.$$

**25.** First, we must find the slope of the tangent, $\left.\dfrac{dy}{dx}\right|_{(4,2)}$. Implicit differentiation yields:

$$2y\frac{dy}{dx} = \frac{2x(xy - 4) - x^2\left(x\frac{dy}{dx} + y\right)}{(xy - 4)^2}.$$

Given the complexity of the above equation, we first want to substitute 4 for $x$ and 2 for $y$ (the coordinates of the point where we are constructing our tangent line), then solve for $\dfrac{dy}{dx}$. Substitution yields:

$$2 \cdot 2\frac{dy}{dx} = \frac{(2 \cdot 4)(4 \cdot 2 - 4) - 4^2\left(4\frac{dy}{dx} + 2\right)}{(4 \cdot 2 - 4)^2} = \frac{8(4) - 16(4\frac{dy}{dx} + 2)}{16} = -4\frac{dy}{dx}.$$
$$4\frac{dy}{dx} = -4\frac{dy}{dx},$$

Solving for $\dfrac{dy}{dx}$, we have:

$$\frac{dy}{dx} = 0.$$

The tangent is a horizontal line through $(4, 2)$, hence its equation is $y = 2$.

**26.** First, we must find the slope of the tangent at the origin, that is $\left.\dfrac{dy}{dx}\right|_{(0,0)}$. Rewriting $y = \dfrac{x}{y + a}$ as $y(y + a) = x$ so that we have

$$y^2 + ay = x$$

and differentiating implicitly gives

$$2y\frac{dy}{dx} + a\frac{dy}{dx} = 1$$
$$\frac{dy}{dx}(2y + a) = 1$$
$$\frac{dy}{dx} = \frac{1}{2y + a}.$$

Substituting $x = 0$, $y = 0$ yields $\left.\dfrac{dy}{dx}\right|_{(0,0)} = \dfrac{1}{a}$. Using the point-slope formula for a line, we have that the equation for the tangent line is

$$y - 0 = \frac{1}{a}(x - 0) \quad \text{or} \quad y = \frac{x}{a}.$$

**27.** First, we must find the slope of the tangent, $\left.\frac{dy}{dx}\right|_{(a,0)}$. We differentiate implicitly, obtaining:

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0\,,$$

$$\frac{dy}{dx} = -\frac{\frac{2}{3}x^{-\frac{1}{3}}}{\frac{2}{3}y^{-\frac{1}{3}}} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}}\,.$$

Substitution yields, $\left.\frac{dy}{dx}\right|_{(a,0)} = \frac{\sqrt[3]{0}}{\sqrt[3]{a}} = 0$. The tangent is a horizontal line through $(a, 0)$, hence its equation is $y = 0$.

## Problems

**28.** (a) By implicit differentiation, we have:

$$\begin{aligned} 2x + 2y\frac{dy}{dx} - 4 + 7\frac{dy}{dx} &= 0 \\ (2y+7)\frac{dy}{dx} &= 4 - 2x \\ \frac{dy}{dx} &= \frac{4-2x}{2y+7}. \end{aligned}$$

(b) The curve has a horizontal tangent line when $dy/dx = 0$, which occurs when $4 - 2x = 0$ or $x = 2$. The curve has a horizontal tangent line at all points where $x = 2$.
The curve has a vertical tangent line when $dy/dx$ is undefined, which occurs when $2y + 7 = 0$ or when $y = -7/2$. The curve has a vertical tangent line at all points where $y = -7/2$.

**29.** (a) Taking derivatives implicitly, we get

$$\begin{aligned} \frac{2}{25}x + \frac{2}{9}y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-9x}{25y}. \end{aligned}$$

(b) The slope is not defined anywhere along the line $y = 0$. This ellipse intersects that line in two places, $(-5, 0)$ and $(5, 0)$. (These are the "ends" of the ellipse where the tangent is vertical.)

**30.** (a) If $x = 4$ then $16 + y^2 = 25$, so $y = \pm 3$. We find $\frac{dy}{dx}$ implicitly:

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

So the slope at $(4, 3)$ is $-\frac{4}{3}$ and at $(4, -3)$ is $\frac{4}{3}$. The tangent lines are:

$$(y-3) = -\frac{4}{3}(x-4) \quad \text{and} \quad (y+3) = \frac{4}{3}(x-4)$$

(b) The normal lines have slopes that are the negative of the reciprocal of the slopes of the tangent lines. Thus,

$$(y-3) = \frac{3}{4}(x-4) \quad \text{so} \quad y = \frac{3}{4}x$$

and

$$(y+3) = -\frac{3}{4}(x-4) \quad \text{so} \quad y = -\frac{3}{4}x$$

are the normal lines.

(c) These lines meet at the origin, which is the center of the circle.

**31.** (a) Solving for $\frac{dy}{dx}$ by implicit differentiation yields

$$3x^2 + 3y^2\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{y^2 - 3x^2}{3y^2 - 2xy}.$$

(b) We can approximate the curve near $x = 1$, $y = 2$ by its tangent line. The tangent line will have slope $\frac{(2)^2-3(1)^2}{3(2)^2-2(1)(2)} = \frac{1}{8} = 0.125$. Thus its equation is

$$y = 0.125x + 1.875$$

Using the $y$-values of the tangent line to approximate the $y$-values of the curve, we get:

| $x$ | 0.96 | 0.98 | 1 | 1.02 | 1.04 |
|---|---|---|---|---|---|
| approximate $y$ | 1.995 | 1.9975 | 2.000 | 2.0025 | 2.005 |

(c) When $x = 0.96$, we get the equation $0.96^3 + y^3 - 0.96y^2 = 5$, whose solution by numerical methods is 1.9945, which is close to the one above.

(d) The tangent line is horizontal when $\frac{dy}{dx}$ is zero and vertical when $\frac{dy}{dx}$ is undefined. These will occur when the numerator is zero and when the denominator is zero, respectively.

Thus, we know that the tangent is horizontal when $y^2 - 3x^2 = 0 \Rightarrow y = \pm\sqrt{3}x$. To find the points that satisfy this condition, we substitute back into the original equation for the curve:

$$x^3 + y^3 - xy^2 = 5$$
$$x^3 \pm 3\sqrt{3}x^3 - 3x^3 = 5$$
$$x^3 = \frac{5}{\pm 3\sqrt{3} - 2}$$

So $x \approx 1.1609$ or $x \approx -0.8857$.

Substituting,

$$y = \pm\sqrt{3}x \text{ so } y \approx 2.0107 \quad \text{or} \quad y \approx 1.5341.$$

Thus, the tangent line is horizontal at $(1.1609, 2.0107)$ and $(-0.8857, 1.5341)$.

Also, we know that the tangent is vertical whenever $3y^2 - 2xy = 0$, that is, when $y = \frac{2}{3}x$ or $y = 0$. Substituting into the original equation for the curve gives us $x^3 + (\frac{2}{3}x)^3 - (\frac{2}{3})^2x^3 = 5$. This means $x^3 \approx 5.8696$, so $x \approx 1.8039$, $y \approx 1.2026$. The other vertical tangent is at $y = 0$, $x = \sqrt[3]{5}$.

**32.** The slope of the tangent to the curve $y = x^2$ at $x = 1$ is 2 so the equation of such a tangent will be of the form $y = 2x + c$. As the tangent must pass through $(1, 1)$, $c = -1$ and so the required tangent is $y = 2x - 1$.

Any circle centered at $(8, 0)$ will be of the form

$$(x - 8)^2 + y^2 = R^2.$$

The slope of this curve at $(x, y)$ is given by implicit differentiation:

$$2(x - 8) + 2yy' = 0$$

or

$$y' = \frac{8 - x}{y}$$

For the tangent to the parabola to be tangential to the circle we need

$$\frac{8 - x}{y} = 2$$

so that at the point of contact of the circle and the line the coordinates are given by $(x, y)$ when $y = 4 - x/2$. Substituting into the equation of the tangent line gives $x = 2$ and $y = 3$. From this we conclude that $R^2 = 45$ so that the equation of the circle is

$$(x - 8)^2 + y^2 = 45.$$

**33.** (a) Differentiating both sides of the equation with respect to $P$ gives

$$\frac{d}{dP}\left(\frac{4f^2P}{1 - f^2}\right) = \frac{dK}{dP} = 0.$$

By the product rule

$$\frac{d}{dP}\left(\frac{4f^2P}{1-f^2}\right) = \frac{d}{dP}\left(\frac{4f^2}{1-f^2}\right)P + \left(\frac{4f^2}{1-f^2}\right)\cdot 1$$
$$= \left(\frac{(1-f^2)(8f) - 4f^2(-2f)}{(1-f^2)^2}\right)\frac{df}{dP}P + \left(\frac{4f^2}{1-f^2}\right)$$
$$= \left(\frac{8f}{(1-f^2)^2}\right)\frac{df}{dP}P + \left(\frac{4f^2}{1-f^2}\right) = 0.$$

So

$$\frac{df}{dP} = \frac{-4f^2/(1-f^2)}{8fP/(1-f^2)^2} = \frac{-1}{2P}f(1-f^2).$$

(b) Since $f$ is a fraction of a gas, $0 \le f \le 1$. Also, in the equation relating $f$ and $P$ we can't have $f = 0$, since that would imply $K = 0$, and we can't have $f = 1$, since the left side is undefined there. So $0 < f < 1$. Thus $1 - f^2 > 0$. Also, pressure can't be negative, and from the equation relating $f$ and $P$, we see that $P$ can't be zero either, so $P > 0$. Therefore $df/dP = -(1/2P)f(1-f^2) < 0$ always. This means that at larger pressures less of the gas decomposes.

**34.** Let the point of intersection of the tangent line with the smaller circle be $(x_1, y_1)$ and the point of intersection with the larger be $(x_2, y_2)$. Let the tangent line be $y = mx + c$. Then at $(x_1, y_1)$ and $(x_2, y_2)$ the slopes of $x^2 + y^2 = 1$ and $y^2 + (x-3)^2 = 4$ are also $m$. The slope of $x^2+y^2 = 1$ is found by implicit differentiation: $2x + 2yy' = 0$ so $y' = -x/y$. Similarly, the slope of $y^2 + (x-3)^2 = 4$ is $y' = -(x-3)/y$. Thus,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_1}{y_1} = -\frac{(x_2-3)}{y_2},$$

where $y_1 = \sqrt{1 - x_1^2}$ and $y_2 = \sqrt{4 - (x_2-3)^2}$. The positive values for $y_1$ and $y_2$ follow from Figure 3.7 and from our choice of $m > 0$. We obtain

$$\frac{x_1}{\sqrt{1-x_1^2}} = \frac{x_2 - 3}{\sqrt{4-(x_2-3)^2}}$$
$$\frac{x_1^2}{1-x_1^2} = \frac{(x_2-3)^2}{4-(x_2-3)^2}$$
$$x_1^2[4-(x_2-3)^2] = (1-x_1^2)(x_2-3)^2$$
$$4x_1^2 - (x_1^2)(x_2-3)^2 = (x_2-3)^2 - x_1^2(x_2-3)^2$$
$$4x_1^2 = (x_2-3)^2$$
$$2|x_1| = |x_2 - 3|.$$

From the picture $x_1 < 0$ and $x_2 < 3$. This gives $x_2 = 2x_1 + 3$ and $y_2 = 2y_1$. From

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_1}{y_1},$$

substituting $y_1 = \sqrt{1 - x_1^2}$, $y_2 = 2y_1$ and $x_2 = 2x_1 + 3$ gives

$$x_1 = -\frac{1}{3}.$$

From $x_2 = 2x_1 + 3$ we get $x_2 = 7/3$. In addition, $y_1 = \sqrt{1 - x_1^2}$ gives $y_1 = 2\sqrt{2}/3$, and finally $y_2 = 2y_1$ gives $y_2 = 4\sqrt{2}/3$.

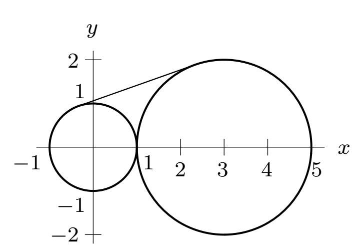



**Figure 3.7**

**35.** $y = x^{\frac{m}{n}}$. Taking $n^{\text{th}}$ powers of both sides of this expression yields $(y)^n = (x^{\frac{m}{n}})^n$, or $y^n = x^m$.

$$\begin{aligned}
\frac{d}{dx}(y^n) &= \frac{d}{dx}(x^m) \\
ny^{n-1}\frac{dy}{dx} &= mx^{m-1} \\
\frac{dy}{dx} &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} \\
&= \frac{m}{n}\frac{x^{m-1}}{(x^{m/n})^{n-1}} \\
&= \frac{m}{n}\frac{x^{m-1}}{x^{m-\frac{m}{n}}} \\
&= \frac{m}{n}x^{(m-1)-(m-\frac{m}{n})} = \frac{m}{n}x^{\frac{m}{n}-1}.
\end{aligned}$$

## Solutions for Section 3.8

### Exercises

**1.** Using the chain rule, $\frac{d}{dx}(\cosh(2x)) = (\sinh(2x)) \cdot 2 = 2\sinh(2x)$.

**2.** Using the chain rule, $\frac{d}{dz}(\sinh(3z+5)) = \cosh(3z+5) \cdot 3 = 3\cosh(3z+5)$.

**3.** Using the chain rule,

$$\frac{d}{dt}(\cosh(\sinh t)) = \sinh(\sinh t) \cdot \cosh t$$

.

**4.** Using the product rule,

$$\frac{d}{dt}\left(t^3 \sinh t\right) = 3t^2 \sinh t + t^3 \cosh t.$$

**5.** Using the chain rule,

$$\frac{d}{dt}\left(\cosh^2 t\right) = 2\cosh t \cdot \sinh t.$$

.

**6.** Using the product and chain rules, $\frac{d}{dt}(\cosh(3t)\sinh(4t)) = 3\sinh(3t)\sinh(4t) + 4\cosh(3t)\cosh(4t)$.

**7.** Using the chain rule twice, $\frac{d}{dt}\left(\cosh(e^{t^2})\right) = \sinh(e^{t^2}) \cdot e^{t^2} \cdot 2t = 2te^{t^2}\sinh(e^{t^2})$.

**8.** Using the chain rule, $\frac{d}{dx}(\tanh(3+\sinh x)) = \frac{1}{\cosh^2(3+\sinh x)} \cdot \cosh x$.

**9.** Using the chain rule twice,

$$\begin{aligned}
\frac{d}{dy}(\sinh(\sinh(3y))) &= \cosh(\sinh(3y)) \cdot \cosh(3y) \cdot 3 \\
&= 3\cosh(3y) \cdot \cosh(\sinh(3y)).
\end{aligned}$$

**10.** Using the chain rule,

$$\frac{d}{d\theta}(\ln(\cosh(1+\theta))) = \frac{1}{\cosh(1+\theta)} \cdot \sinh(1+\theta) = \frac{\sinh(1+\theta)}{\cosh(1+\theta)} = \tanh(1+\theta).$$

**11.** Using the chain rule, $f'(t) = 2\cosh t \sinh t - 2\sinh t \cosh t = 0$. This is to be expected since $\cosh^2 t - \sinh^2 t = 1$.

**12.** Substitute $x = 0$ into the formula for $\sinh x$. This yields

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0.$$

**13.** Substituting $-x$ for $x$ in the formula for $\sinh x$ gives

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2} = -\frac{e^{x} - e^{-x}}{2} = -\sinh x.$$

**14.** Using the formula for $\sinh x$ and the fact that $d(e^{-x})/dx = -e^{-x}$, we see that

$$\frac{d}{dx}\left(\frac{e^{x} - e^{-x}}{2}\right) = \frac{e^{x} + e^{-x}}{2} = \cosh x.$$

**15.** By definition $\sinh x = (e^{x} - e^{-x})/2$ so, since $e^{\ln t} = t$ and $e^{-\ln t} = 1/e^{\ln t} = 1/t$, we have

$$\sinh(\ln t) = \frac{e^{\ln t} - e^{-\ln t}}{2} = \frac{t - 1/t}{2} = \frac{t^2 - 1}{2t}.$$

**16.** By definition $\cosh x = (e^{x} + e^{-x})/2$ so, since $e^{\ln t} = t$ and $e^{-\ln t} = 1/e^{\ln t} = 1/t$, we have

$$\cosh(\ln t) = \frac{e^{\ln t} + e^{-\ln t}}{2} = \frac{t + 1/t}{2} = \frac{t^2 + 1}{2t}.$$

## Problems

**17.** The graph of $\sinh x$ in the text suggests that

$$\text{As } x \to \infty, \quad \sinh x \to \frac{1}{2}e^{x}.$$
$$\text{As } x \to -\infty, \quad \sinh x \to -\frac{1}{2}e^{-x}.$$

Using the facts that

$$\text{As } x \to \infty, \quad e^{-x} \to 0,$$
$$\text{As } x \to -\infty, \quad e^{x} \to 0,$$

we can obtain the same results analytically:

$$\text{As } x \to \infty, \quad \sinh x = \frac{e^{x} - e^{-x}}{2} \to \frac{1}{2}e^{x}.$$
$$\text{As } x \to -\infty, \quad \sinh x = \frac{e^{x} - e^{-x}}{2} \to -\frac{1}{2}e^{-x}.$$

**18.** First we observe that

$$\sinh(2x) = \frac{e^{2x} - e^{-2x}}{2}.$$

Now let's calculate

$$\begin{aligned}(\sinh x)(\cosh x) &= \left(\frac{e^{x} - e^{-x}}{2}\right)\left(\frac{e^{x} + e^{-x}}{2}\right)\\ &= \frac{(e^{x})^2 - (e^{-x})^2}{4}\\ &= \frac{e^{2x} - e^{-2x}}{4}\\ &= \frac{1}{2}\sinh(2x).\end{aligned}$$

Thus, we see that

$$\sinh(2x) = 2\sinh x \cosh x.$$

**19.** First, we observe that

$$\cosh(2x) = \frac{e^{2x} + e^{-2x}}{2}.$$

Now let's use the fact that $e^x \cdot e^{-x} = 1$ to calculate

$$\begin{aligned}
\cosh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 \\
&= \frac{(e^x)^2 + 2e^x \cdot e^{-x} + (e^{-x})^2}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x}}{4}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\sinh^2 x &= \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{(e^x)^2 - 2e^x \cdot e^{-x} + (e^{-x})^2}{4} \\
&= \frac{e^{2x} - 2 + e^{-2x}}{4}.
\end{aligned}$$

Thus, to obtain $\cosh(2x)$, we need to add (rather than subtract) $\cosh^2 x$ and $\sinh^2 x$, giving

$$\begin{aligned}
\cosh^2 x + \sinh^2 x &= \frac{e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}}{4} \\
&= \frac{2e^{2x} + 2e^{-2x}}{4} \\
&= \frac{e^{2x} + e^{-2x}}{2} \\
&= \cosh(2x).
\end{aligned}$$

Thus, we see that the identity relating $\cosh(2x)$ to $\cosh x$ and $\sinh x$ is

$$\cosh(2x) = \cosh^2 x + \sinh^2 x.$$

**20.** Recall that

$$\sinh A = \frac{1}{2}(e^A - e^{-A}) \quad \text{and} \quad \cosh A = \frac{1}{2}(e^A + e^{-A}).$$

Now substitute, expand and collect terms:

$$\begin{aligned}
\sinh A \cosh B + \sinh B \cosh A &= \frac{1}{2}(e^A - e^{-A}) \cdot \frac{1}{2}(e^B + e^{-B}) + \frac{1}{2}(e^B - e^{-B}) \cdot \frac{1}{2}(e^A + e^{-A}) \\
&= \frac{1}{4}\left(e^{A+B} + e^{A-B} - e^{-A+B} - e^{-(A+B)}\right. \\
&\qquad \left. + e^{B+A} + e^{B-A} - e^{-B+A} - e^{-A-B}\right) \\
&= \frac{1}{2}\left(e^{A+B} - e^{-(A+B)}\right) \\
&= \sinh(A+B).
\end{aligned}$$

**21.** Recall that

$$\sinh A = \frac{1}{2}(e^A - e^{-A}) \quad \text{and} \quad \cosh A = \frac{1}{2}(e^A + e^{-A}).$$

Now substitute, expand and collect terms:

$$\begin{aligned}\cosh A\cosh B+\sinh B\sinh A &= \frac{1}{2}(e^{A}+e^{-A})\cdot\frac{1}{2}(e^{B}+e^{-B})+\frac{1}{2}(e^{B}-e^{-B})\cdot\frac{1}{2}(e^{A}-e^{-A})\\ &= \frac{1}{4}\left(e^{A+B}+e^{A-B}+e^{-A+B}+e^{-(A+B)}\right.\\ &\qquad \left.+e^{B+A}-e^{B-A}-e^{-B+A}+e^{-A-B}\right)\\ &= \frac{1}{2}\left(e^{A+B}+e^{-(A+B)}\right)\\ &= \cosh(A+B).\end{aligned}$$

**22.** Using the definition of $\cosh x$ and $\sinh x$, we have $\cosh 2x = \dfrac{e^{2x}+e^{-2x}}{2}$ and $\sinh 3x = \dfrac{e^{3x}-e^{-3x}}{2}$. Therefore

$$\begin{aligned}\lim_{x\to\infty}\frac{\cosh(2x)}{\sinh(3x)} &= \lim_{x\to\infty}\frac{e^{2x}+e^{-2x}}{e^{3x}-e^{-3x}}\\ &= \lim_{x\to\infty}\frac{e^{2x}(1+e^{-4x})}{e^{2x}(e^{x}-e^{-5x})}\\ &= \lim_{x\to\infty}\frac{1+e^{-4x}}{e^{x}-e^{-5x}}\\ &= 0.\end{aligned}$$

**23.** Using the definition of $\sinh x$, we have $\sinh 2x = \dfrac{e^{2x}-e^{-2x}}{2}$. Therefore

$$\begin{aligned}\lim_{x\to\infty}\frac{e^{2x}}{\sinh(2x)} &= \lim_{x\to\infty}\frac{2e^{2x}}{e^{2x}-e^{-2x}}\\ &= \lim_{x\to\infty}\frac{2}{1-e^{-4x}}\\ &= 2.\end{aligned}$$

**24.** Using the definition of $\cosh x$ and $\sinh x$, we have $\cosh x^2 = \dfrac{e^{x^2}+e^{-x^2}}{2}$ and $\sinh x^2 = \dfrac{e^{x^2}-e^{-x^2}}{2}$. Therefore

$$\begin{aligned}\lim_{x\to\infty}\frac{\sinh(x^2)}{\cosh(x^2)} &= \lim_{x\to\infty}\frac{e^{x^2}-e^{-x^2}}{e^{x^2}+e^{-x^2}}\\ &= \lim_{x\to\infty}\frac{e^{x^2}(1-e^{-2x^2})}{e^{x^2}(1+e^{-2x^2})}\\ &= \lim_{x\to\infty}\frac{1-e^{-2x^2}}{1+e^{-2x^2}}\\ &= 1.\end{aligned}$$

**25.** Note that

$$\begin{aligned}\frac{\sinh kx}{\cosh 2x} &= \frac{e^{kx}-e^{-kx}}{e^{2x}+e^{-2x}}\\ &= \frac{e^{2x}(e^{(k-2)x}-e^{-(k+2)x})}{e^{2x}(1+e^{-4x})}\\ &= \frac{e^{(k-2)x}-e^{-(k+2)x}}{1+e^{-4x}}.\end{aligned}$$

If $k=2$, then the limit as $x\to\infty$ is 1.
If $|k|>2$, then the limit as $x\to\infty$ does not exist.
If $|k|<2$, then the limit as $x\to\infty$ is 0.

**26.** Note that

$$\begin{aligned} e^{-3x}\cosh kx &= e^{-3x}\frac{e^{kx}+e^{-kx}}{2} \\ &= \frac{e^{(k-3)x}+e^{-(k+3)x}}{2}. \end{aligned}$$

If $|k| = 3$, then the limit as $x \to \infty$ is 1/2.
If $|k| > 3$, then the limit as $x \to \infty$ does not exist.
If $|k| < 3$, then the limit as $x \to \infty$ is 0.

**27.** (a) The graph in Figure 3.8 looks like the graph of $y = \cosh x$, with the minimum at about $(0.5, 6.3)$.

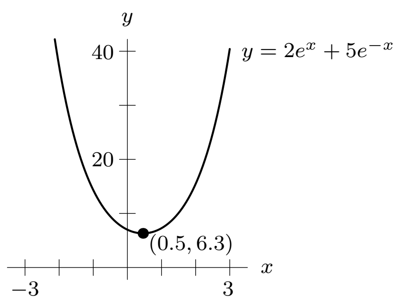



**Figure 3.8**

(b) We want to write

$$\begin{aligned} y = 2e^x + 5e^{-x} = A\cosh(x-c) &= \frac{A}{2}e^{x-c} + \frac{A}{2}e^{-(x-c)} \\ &= \frac{A}{2}e^x e^{-c} + \frac{A}{2}e^{-x}e^c \\ &= \left(\frac{Ae^{-c}}{2}\right)e^x + \left(\frac{Ae^c}{2}\right)e^{-x}. \end{aligned}$$

Thus, we need to choose $A$ and $c$ so that

$$\frac{Ae^{-c}}{2} = 2 \quad \text{and} \quad \frac{Ae^c}{2} = 5.$$

Dividing gives

$$\begin{aligned} \frac{Ae^c}{Ae^{-c}} &= \frac{5}{2} \\ e^{2c} &= 2.5 \\ c &= \frac{1}{2}\ln 2.5 \approx 0.458. \end{aligned}$$

Solving for $A$ gives

$$A = \frac{4}{e^{-c}} = 4e^c \approx 6.325.$$

Thus,

$$y = 6.325\cosh(x - 0.458).$$

Rewriting the function in this way shows that the graph in part (a) is the graph of $\cosh x$ shifted to the right by $0.458$ and stretched vertically by a factor of $6.325$.

**28.** We want to show that for any $A,\ B$ with $A>0,\ B>0$, we can find $K$ and $c$ such that

$$\begin{aligned} y = Ae^x + Be^{-x} &= \frac{Ke^{(x-c)} + Ke^{-(x-c)}}{2} \\ &= \frac{K}{2}e^x e^{-c} + \frac{K}{2}e^{-x}e^c \\ &= \left(\frac{Ke^{-c}}{2}\right)e^x + \left(\frac{Ke^c}{2}\right)e^{-x}. \end{aligned}$$

Thus, we want to find $K$ and $c$ such that

$$\frac{Ke^{-c}}{2} = A \quad \text{and} \quad \frac{Ke^c}{2} = B.$$

Dividing, we have

$$\begin{aligned} \frac{Ke^c}{Ke^{-c}} &= \frac{B}{A} \\ e^{2c} &= \frac{B}{A} \\ c &= \frac{1}{2}\ln\left(\frac{B}{A}\right). \end{aligned}$$

If $A>0,\ B>0$, then there is a solution for $c$. Substituting to find $K$, we have

$$\begin{aligned} \frac{Ke^{-c}}{2} &= A \\ K &= 2Ae^c = 2Ae^{(\ln(B/A))/2} \\ &= 2Ae^{\ln\sqrt{B/A}} = 2A\sqrt{\frac{B}{A}} = 2\sqrt{AB}. \end{aligned}$$

Thus, if $A>0,\ B>0$, there is a solution for $K$ also.

The fact that $y = Ae^x + Be^{-x}$ can be rewritten in this way shows that the graph of $y = Ae^x + Be^{-x}$ is the graph of $\cosh x$, shifted over by $c$ and stretched (or shrunk) vertically by a factor of $K$.

**29.** (a) Since the cosh function is even, the height, $y$, is the same at $x = -T/w$ and $x = T/w$. The height at these endpoints is

$$y = \frac{T}{w}\cosh\left(\frac{w}{T}\cdot\frac{T}{w}\right) = \frac{T}{w}\cosh 1 = \frac{T}{w}\left(\frac{e^1 + e^{-1}}{2}\right).$$

At the lowest point, $x=0$, and the height is

$$y = \frac{T}{w}\cosh 0 = \frac{T}{w}.$$

Thus the "sag" in the cable is given by

$$\text{Sag} = \frac{T}{w}\left(\frac{e+e^{-1}}{2}\right) - \frac{T}{w} = \frac{T}{w}\left(\frac{e+e^{-1}}{2} - 1\right) \approx 0.54\frac{T}{w}.$$

(b) To show that the differential equation is satisfied, take derivatives

$$\begin{aligned} \frac{dy}{dx} &= \frac{T}{w}\cdot\frac{w}{T}\sinh\left(\frac{wx}{T}\right) = \sinh\left(\frac{wx}{T}\right) \\ \frac{d^2y}{dx^2} &= \frac{w}{T}\cosh\left(\frac{wx}{T}\right). \end{aligned}$$

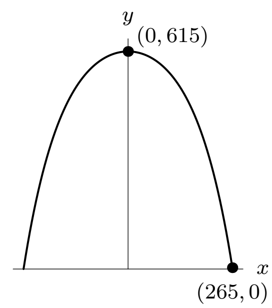
Therefore, using the fact that $1 + \sinh^2 a = \cosh^2 a$ and that cosh is always positive, we have:

$$\begin{aligned} \frac{w}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \frac{w}{T}\sqrt{1 + \sinh^2\left(\frac{wx}{T}\right)} = \frac{w}{T}\sqrt{\cosh^2\left(\frac{wx}{T}\right)} \\ &= \frac{w}{T}\cosh\left(\frac{wx}{T}\right). \end{aligned}$$

So

$$\frac{w}{T}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{d^2y}{dx^2}.$$

**30.**




We know $x = 0$ and $y = 615$ at the top of the arch, so

$$615 = b - a\cosh(0/a) = b - a.$$

This means $b = a + 615$. We also know that $x = 265$ and $y = 0$ where the arch hits the ground, so

$$0 = b - a\cosh(265/a) = a + 615 - a\cosh(265/a).$$

We can solve this equation numerically on a calculator and get $a \approx 100$, which means $b \approx 715$. This results in the equation

$$y \approx 715 - 100\cosh\left(\frac{x}{100}\right).$$

**31.** (a) Substituting $x = 0$ gives

$$\tanh 0 = \frac{e^0 - e^{-0}}{e^0 + e^{-0}} = \frac{1-1}{2} = 0.$$

(b) Since $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ and $e^x + e^{-x}$ is always positive, $\tanh x$ has the same sign as $e^x - e^{-x}$. For $x > 0$, we have $e^x > 1$ and $e^{-x} < 1$, so $e^x - e^{-x} > 0$. For $x < 0$, we have $e^x < 1$ and $e^{-x} > 1$, so $e^x - e^{-x} < 0$. For $x = 0$, we have $e^x = 1$ and $e^{-x} = 1$, so $e^x - e^{-x} = 0$. Thus, $\tanh x$ is positive for $x > 0$, negative for $x < 0$, and zero for $x = 0$.

(c) Taking the derivative, we have

$$\frac{d}{dx}(\tanh x) = \frac{1}{\cosh^2 x}.$$

Thus, for all $x$,

$$\frac{d}{dx}(\tanh x) > 0.$$

Thus, $\tanh x$ is increasing everywhere.

(d) As $x \to \infty$ we have $e^{-x} \to 0$; as $x \to -\infty$, we have $e^x \to 0$. Thus

$$\lim_{x\to\infty} \tanh x = \lim_{x\to\infty}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 1,$$

$$\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = -1.$$

Thus, $y = 1$ and $y = -1$ are horizontal asymptotes to the graph of $\tanh x$. See Figure 3.9.

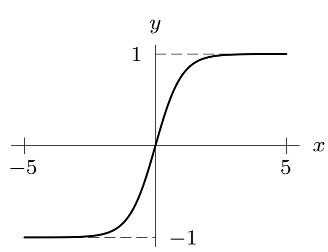



**Figure 3.9**: Graph of $y = \tanh x$

(e) The graph of $\tanh x$ suggests that $\tanh x$ is increasing everywhere; the fact that the derivative of $\tanh x$ is positive for all $x$ confirms this. Since $\tanh x$ is increasing for all $x$, different values of $x$ lead to different values of $y$, and therefore $\tanh x$ does have an inverse.

# Solutions for Section 3.9

## Exercises

**1.** With $f(x) = \sqrt{1+x}$, the chain rule gives $f'(x) = 1/(2\sqrt{1+x})$, so $f(0) = 1$ and $f'(0) = 1/2$. Therefore the tangent line approximation of $f$ near $x = 0$,

$$f(x) \approx f(0) + f'(0)(x-0),$$

becomes

$$\sqrt{1+x} \approx 1 + \frac{x}{2}.$$

This means that, near $x = 0$, the function $\sqrt{1+x}$ can be approximated by its tangent line $y = 1 + x/2$. (See Figure 3.10.)

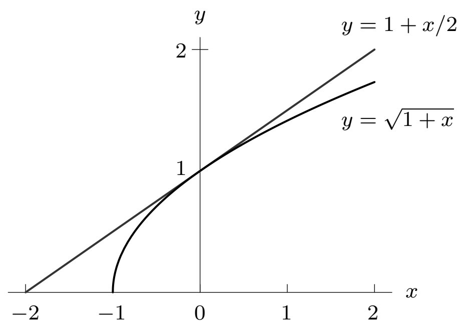



**Figure 3.10**

**2.** With $f(x) = e^x$, the tangent line approximation to $f$ near $x = 0$ is $f(x) \approx f(0) + f'(0)(x-0)$ which becomes $e^x \approx e^0 + e^0 x = 1 + 1x = 1 + x$. Thus, our local linearization of $e^x$ near $x = 0$ is $e^x \approx 1 + x$.

**3.** With $f(x) = 1/x$, we see that the tangent line approximation to $f$ near $x = 1$ is

$$f(x) \approx f(1) + f'(1)(x-1),$$

which becomes

$$\frac{1}{x} \approx 1 + f'(1)(x-1).$$

Since $f'(x) = -1/x^2$, $f'(1) = -1$. Thus our formula reduces to

$$\frac{1}{x} \approx 1 - (x-1) = 2 - x.$$

This is the local linearization of $1/x$ near $x = 1$.

**4.** With $f(x) = 1/(\sqrt{1+x})$, we see that the tangent line approximation to $f$ near $x = 0$ is

$$f(x) \approx f(0) + f'(0)(x-0),$$

which becomes

$$\frac{1}{\sqrt{1+x}} \approx 1 + f'(0)x.$$

Since $f'(x) = (-1/2)(1+x)^{-3/2}$, $f'(0) = -1/2$. Thus our formula reduces to

$$\frac{1}{\sqrt{1+x}} \approx 1 - x/2.$$

This is the local linearization of $\dfrac{1}{\sqrt{1+x}}$ near $x = 0$.

**5.** Let $f(x) = e^{-x}$. Then $f'(x) = -e^{-x}$. So $f(0) = 1$, $f'(0) = -e^0 = -1$. Therefore, $e^{-x} \approx f(0) + f'(0)x = 1 - x$.

**6.** With $f(x) = e^{x^2}$, we get a tangent line approximation of $f(x) \approx f(1) + f'(1)(x-1)$ which becomes $e^{x^2} \approx e + \left(2xe^{x^2}\right)\Big|_{x=1}(x-1) = e + 2e(x-1) = 2ex - e$. Thus, our local linearization of $e^{x^2}$ near $x = 1$ is $e^{x^2} \approx 2ex - e$.

**7.** From Figure 3.11, we see that the error has its maximum magnitude at the end points of the interval, $x = \pm 1$. The magnitude of the error can be read off the graph as less than $0.2$ or estimated as

$$|\text{Error}| \le |1 - \sin 1| = 0.159 < 0.2.$$

The approximation is an overestimate for $x > 0$ and an underestimate for $x < 0$.

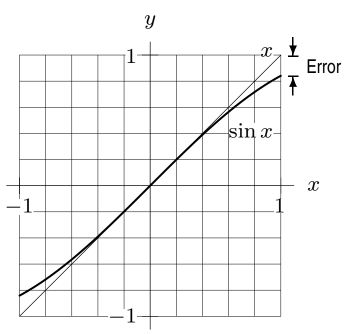



Figure 3.11

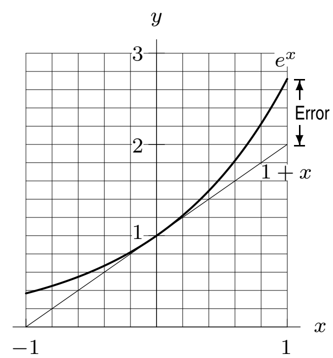



Figure 3.12

**8.** Figure 3.12 shows that $1 + x$ is an underestimate of $e^x$ for $-1 \le x \le 1$. On this interval, the error has the largest magnitude at $x = 1$. Its magnitude can be estimated from the graph as less than $0.8$, or estimated as

$$|\text{Error}| = e - 1 - 1 = 0.718 < 0.8.$$

## Problems

**9.** (a) Since

$$\frac{d}{dx}(\cos x) = -\sin x,$$

the slope of the tangent line is $-\sin(\pi/4) = -1/\sqrt{2}$. Since the tangent line passes through the point $(\pi/4, \cos(\pi/4)) = (\pi/4, 1/\sqrt{2})$, its equation is

$$y - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$
$$y = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}\left(\frac{\pi}{4} + 1\right).$$

Thus, the tangent line approximation to $\cos x$ is

$$\cos x \approx -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}\left(\frac{\pi}{4} + 1\right).$$

(b) From Figure 3.13, we see that the tangent line approximation is an overestimate.

(c) From Figure 3.13, we see that the maximum error for $0 \le x \le \pi/2$ is either at $x = 0$ or at $x = \pi/2$. The error can either be estimated from the graph, or as follows. At $x = 0$,

$$|\text{Error}| = \left|\cos 0 - \frac{1}{\sqrt{2}}\left(\frac{\pi}{4} + 1\right)\right| = 0.262 < 0.3.$$

At $x = \pi/2$,

$$|\text{Error}| = \left|\cos\frac{\pi}{2} + \frac{1}{\sqrt{2}}\frac{\pi}{2} - \frac{1}{\sqrt{2}}\left(\frac{\pi}{4} + 1\right)\right| = 0.152 < 0.2.$$

Thus, for $0 \le x \le \pi/2$, we have

$$|\text{Error}| < 0.3.$$

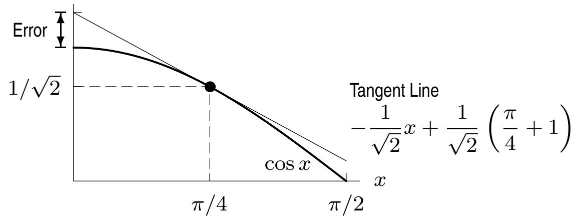



**Figure 3.13**

**10.** (a) Let $f(x) = (1+x)^k$. Then $f'(x) = k(1+x)^{k-1}$. Since

$$f(x) \approx f(0) + f'(0)(x-0)$$

is the tangent line approximation, and $f(0) = 1$, $f'(0) = k$, for small $x$ we get

$$f(x) \approx 1 + kx.$$

(b) Since $\sqrt{1.1} = (1+0.1)^{1/2} \approx 1 + (1/2)0.1 = 1.05$ by the above method, this estimate is about right.

(c) The real answer is less than $1.05$. Since $(1.05)^2 = (1+0.05)^2 = 1 + 2(1)(0.05) + (0.05)^2 = 1.1 + (0.05)^2 > 1.1$, we have $(1.05)^2 > 1.1$ Therefore

$$\sqrt{1.1} < 1.05.$$

Graphically, this because the graph of $\sqrt{1+x}$ is concave down, so it bends below its tangent line. Therefore the true value ($\sqrt{1.1}$) which is on the curve is below the approximate value $(1.05)$ which is on the tangent line.

**11.** Since the line meets the curve at $x = 1$, we have $a = 1$. Since the point with $x = 1$ lies on both the line and the curve, we have

$$f(a) = f(1) = 2 \cdot 1 - 1 = 1.$$

The approximation is an underestimate because the line lies under the curve. Since the linear function approximates $f(x)$, we have

$$f(1.2) \approx 2(1.2) - 1 = 1.4.$$

**12.** (a) From the figure, we see $a = 2$. The point with $x = 2$ lies on both the line and the curve. Since

$$y = -3 \cdot 2 + 7 = 1,$$

we have

$$f(a) = 1.$$

Since the slope of the line is $-3$, we have

$$f'(a) = -3.$$

(b) We use the line to approximate the function, so

$$f(2.1) \approx -3(2.1) + 7 = 0.7.$$

This is an underestimate, because the line is beneath the curve for $x > 2$. Similarly,

$$f(1.98) \approx -3(1.98) + 7 = 1.06.$$

This is an overestimate because the line is above the curve for $x < 2$.

The approximation $f(1.98) \approx 1.06$ is likely to be more accurate because $1.98$ is closer to 2 than $2.1$ is. Since the graph of $f(x)$ appears to bend away from the line at approximately the same rate on either side of $x = 2$, in this example, the error is larger for points farther from $x = 2$.

**13.** We have $f(1) = 1$ and $f'(1) = 4$. Thus

$$E(x) = x^4 - (1 + 4(x-1)).$$

Values of $E(x)/(x-1)$ near $x = 1$ are in Table 3.3.

**Table 3.3**

| $x$ | 1.1 | 1.01 | 1.001 |
|---|---|---|---|
| $E(x)/(x-1)$ | 0.641 | 0.060401 | 0.006004 |

From the table, we can see that

$$\frac{E(x)}{(x-1)} \approx 6(x-1),$$

so $k = 6$ and

$$E(x) \approx 6(x-1)^2.$$

In addition, $f''(1) = 12$, so

$$E(x) \approx 6(x-1)^2 = \frac{f''(1)}{2}(x-1)^2.$$

The same result can be obtained by rewriting the function $x^4$ using $x = 1 + (x-1)$ and expanding:

$$x^4 = (1+(x-1))^4 = 1 + 4(x-1) + 6(x-1)^2 + 4(x-1)^3 + (x-1)^4.$$

Thus,

$$E(x) = x^4 - (1 + 4(x-1)) = 6(x-1)^2 + 4(x-1)^3 + (x-1)^4.$$

For $x$ near 1, the value of $x-1$ is small, so we ignore powers of $x-1$ higher than the first, giving

$$E(x) \approx 6(x-1)^2.$$

**14.** We have $f(0) = 1$ and $f'(0) = 0$. Thus

$$E(x) = \cos x - 1.$$

Values for $E(x)/(x-0)$ near $x = 0$ are in Table 3.4.

**Table 3.4**

| $x$ | 0.1 | 0.01 | 0.001 |
|---|---|---|---|
| $E(x)/(x-0)$ | −0.050 | −0.0050 | −0.00050 |

From the table, we can see that

$$\frac{E(x)}{(x-0)} \approx -0.5(x-0),$$

so $k = -1/2$ and

$$E(x) \approx -\frac{1}{2}(x-0)^2 = -\frac{1}{2}x^2.$$

In addition, $f''(0) = -1$, so

$$E(x) \approx -\frac{1}{2}x^2 = \frac{f''(0)}{2}x^2.$$

**15.** We have $f(0) = 1$ and $f'(0) = 1$. Thus

$$E(x) = e^x - (1+x).$$

Values of $E(x)/(x-0)$ near $x = 0$ are in Table 3.5.

**Table 3.5**

| $x$ | 0.1 | 0.01 | 0.001 |
|---|---|---|---|
| $E(x)/(x-0)$ | 0.052 | 0.0050 | 0.00050 |

From the table, we can see that

$$\frac{E(x)}{(x-0)} \approx 0.5(x-0)$$

so $k = 1/2$ and

$$E(x) \approx \frac{1}{2}(x-0)^2 = \frac{1}{2}x^2.$$

In addition, $f''(0) = 1$, so

$$E(x) \approx \frac{1}{2}x^2 = \frac{f''(0)}{2}x^2$$

**16.** We have $f(1) = 1$ and $f'(1) = 1/2$. Thus

$$E(x) = \sqrt{x} - (1 + \frac{1}{2}(x-1)).$$

Values of $E(x)/(x-1)$ near $x = 1$ are in Table 3.6.

**Table 3.6**

| $x$ | 1.1 | 1.01 | 1.001 |
|---|---|---|---|
| $E(x)/(x-1)$ | $-0.0119$ | $-0.00124$ | $-0.000125$ |

From the table, we can see that

$$\frac{E(x)}{(x-1)} \approx -0.125(x-1)$$

so $k = -1/8$ and

$$E(x) \approx -\frac{1}{8}(x-1)^2.$$

In addition, $f''(1) = -1/4$, so

$$E(x) \approx -\frac{1}{8}(x-1)^2 = \frac{f''(1)}{2}(x-1)^2.$$

**17.** We have $f(1) = 0$ and $f'(1) = 1$. Thus

$$E(x) = \ln x - (x-1).$$

Values of $E(x)/(x-1)$ near $x = 1$ are in Table 3.7.

**Table 3.7**

| $x$ | 1.1 | 1.01 | 1.001 |
|---|---|---|---|
| $E(x)/(x-1)$ | $-0.047$ | $-0.0050$ | $-0.00050$ |

From the table, we see that

$$\frac{E(x)}{(x-1)} \approx -0.5(x-1),$$

so $k = -1/2$ and

$$E(x) \approx -\frac{1}{2}(x-1)^2.$$

In addition, $f''(1) = -1$, so

$$E(x) \approx -\frac{1}{2}(x-1)^2 = \frac{f''(1)}{2}(x-1)^2.$$

**18.** The local linearization of $e^x$ near $x = 0$ is $1 + 1x$ so

$$e^x \approx 1 + x.$$

Squaring this yields, for small $x$,

$$e^{2x} = (e^x)^2 \approx (1+x)^2 = 1 + 2x + x^2.$$

Local linearization of $e^{2x}$ directly yields

$$e^{2x} \approx 1 + 2x$$

for small $x$. The two approximations are consistent because they agree: the tangent line approximation to $1 + 2x + x^2$ is just $1 + 2x$.

The first approximation is more accurate. One can see this numerically or by noting that the approximation for $e^{2x}$ given by $1 + 2x$ is really the same as approximating $e^y$ at $y = 2x$. Since the other approximation approximates $e^y$ at $y = x$, which is twice as close to 0 and therefore a better general estimate, it's more likely to be correct.

**19.** (a) Let $f(x) = 1/(1+x)$. Then $f'(x) = -1/(1+x)^2$ by the chain rule. So $f(0) = 1$, and $f'(0) = -1$. Therefore, for $x$ near 0, $1/(1+x) \approx f(0) + f'(0)x = 1 - x$.

(b) We know that for small $y$, $1/(1+y) \approx 1 - y$. Let $y = x^2$; when $x$ is small, so is $y = x^2$. Hence, for small $x$, $1/(1+x^2) \approx 1 - x^2$.

(c) Since the linearization of $1/(1+x^2)$ is the line $y = 1$, and this line has a slope of 0, the derivative of $1/(1+x^2)$ is zero at $x = 0$.

**20.** The local linearizations of $f(x) = e^x$ and $g(x) = \sin x$ near $x = 0$ are

$$f(x) = e^x \approx 1 + x$$

and

$$g(x) = \sin x \approx x.$$

Thus, the local linearization of $e^x \sin x$ is the local linearization of the product:

$$e^x \sin x \approx (1 + x)x = x + x^2 \approx x.$$

We therefore know that the derivative of $e^x \sin x$ at $x = 0$ must be 1. Similarly, using the local linearization of $1/(1 + x)$ near $x = 0$, $1/(1 + x) \approx 1 - x$, we have

$$\frac{e^x \sin x}{1 + x} = (e^x)(\sin x)\left(\frac{1}{1 + x}\right) \approx (1 + x)(x)(1 - x) = x - x^3$$

so the local linearization of the triple product $\dfrac{e^x \sin x}{1 + x}$ at $x = 0$ is simply $x$. And therefore the derivative of $\dfrac{e^x \sin x}{1 + x}$ at $x = 0$ is 1.

**21.** (a) Suppose

$$g = f(r) = \frac{GM}{r^2}.$$

Then

$$f'(r) = \frac{-2GM}{r^3}.$$

So

$$f(r + \Delta r) \approx f(r) - \frac{2GM}{r^3}(\Delta r).$$

Since $f(r + \Delta r) - f(r) = \Delta g$, and $g = GM/r^2$, we have

$$\Delta g \approx -2\frac{GM}{r^3}(\Delta r) = -2g\frac{\Delta r}{r}.$$

(b) The negative sign tells us that the acceleration due to gravity decreases as the distance from the center of the earth increases.

(c) The fractional change in $g$ is given by

$$\frac{\Delta g}{g} \approx -2\frac{\Delta r}{r}.$$

So, since $\Delta r = 4.315$ km and $r = 6400$ km, we have

$$\frac{\Delta g}{g} \approx -2\left(\frac{4.315}{6400}\right) = -0.00135 = -0.135\%.$$

**22.** (a) Suppose $g$ is a constant and

$$T = f(l) = 2\pi\sqrt{\frac{l}{g}}.$$

Then

$$f'(l) = \frac{2\pi}{\sqrt{g}}\frac{1}{2}l^{-1/2} = \frac{\pi}{\sqrt{gl}}.$$

Thus, local linearity tells us that

$$f(l + \Delta l) \approx f(l) + \frac{\pi}{\sqrt{gl}}\Delta l.$$

Now $T = f(l)$ and $\Delta T = f(l + \Delta l) - f(l)$, so

$$\Delta T \approx \frac{\pi}{\sqrt{gl}}\Delta l = 2\pi\sqrt{\frac{l}{g}} \cdot \frac{1}{2}\frac{\Delta l}{l} = \frac{T}{2}\frac{\Delta l}{l}.$$

(b) Knowing that the length of the pendulum increases by 2% tells us that

$$\frac{\Delta l}{l} = 0.02.$$

Thus,

$$\Delta T \approx \frac{T}{2}(0.02) = 0.01T.$$

So

$$\frac{\Delta T}{T} \approx 0.01.$$

Thus, $T$ increases by 1%.

**23.** (a) Considering $l$ as a constant, we have

$$T = f(g) = 2\pi\sqrt{\frac{l}{g}}.$$

Then,

$$f'(g) = 2\pi\sqrt{l}\left(-\frac{1}{2}g^{-3/2}\right) = -\pi\sqrt{\frac{l}{g^3}}.$$

Thus, local linearity gives

$$f(g+\Delta g) \approx f(g) - \pi\sqrt{\frac{l}{g^3}}(\Delta g).$$

Since $T = f(g)$ and $\Delta T = f(g+\Delta g) - f(g)$, we have

$$\Delta T \approx -\pi\sqrt{\frac{l}{g^3}}\Delta g = -2\pi\sqrt{\frac{l}{g}}\frac{\Delta g}{2g} = \frac{-T}{2}\frac{\Delta g}{g}.$$

$$\Delta T \approx \frac{-T}{2}\frac{\Delta g}{g}.$$

(b) If $g$ increases by 1%, we know

$$\frac{\Delta g}{g} = 0.01.$$

Thus,

$$\frac{\Delta T}{T} \approx -\frac{1}{2}\frac{\Delta g}{g} = -\frac{1}{2}(0.01) = -0.005,$$

So, $T$ decreases by 0.5%.

**24.** Since $f$ has a positive second derivative, its graph is concave up, as in Figure 3.14 or 3.15. This means that the graph of $f(x)$ is above its tangent line. We see that in both cases

$$f(1+\Delta x) \geq f(1) + f'(1)\Delta x.$$

(The diagrams show $\Delta x$ positive, but the result is also true if $\Delta x$ is negative.)

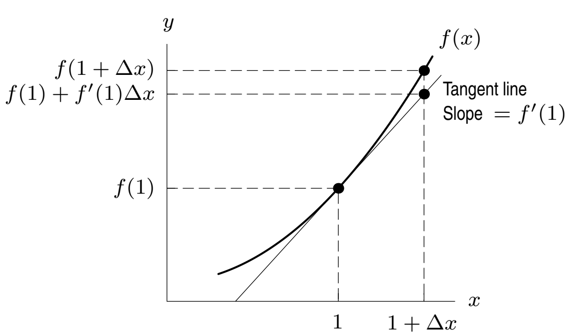



Figure 3.14

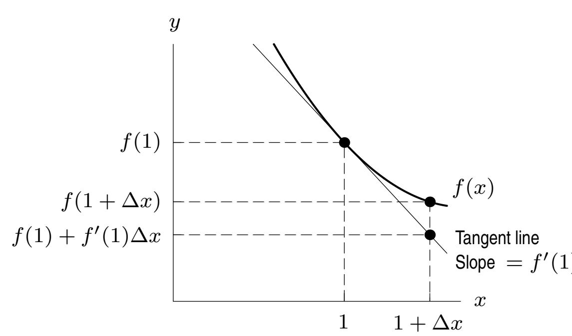



Figure 3.15

**25.** (a) Since $f'$ is decreasing, $f'(5)$ is larger.
(b) Since $f'$ is decreasing, its derivative, $f''$, is negative. Thus, $f''(5)$ is negative, so 0 is larger.
(c) Since $f''(x)$ is negative for all $x$, the graph of $f$ is concave down. Thus the graph of $f(x)$ is below its tangent line. From Figure 3.16, we see that $f(5+\Delta x)$ is below $f(5)+f'(5)\Delta x$. Thus, $f(5)+f'(5)\Delta x$ is larger.

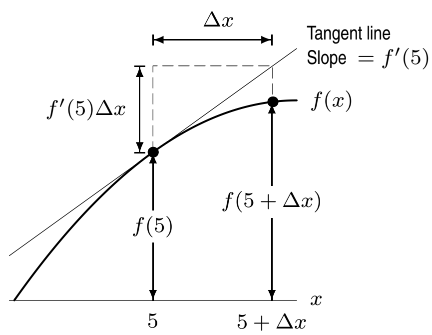



**Figure 3.16**

**26.** Note that

$$[f(x)g(x)]' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

We use the hint: For small $h$, $f(x+h) \approx f(x) + f'(x)h$, and $g(x+h) \approx g(x) + g'(x)h$. Therefore

$$\begin{aligned} f(x+h)g(x+h) - f(x)g(x) &\approx [f(x) + hf'(x)][g(x) + hg'(x)] - f(x)g(x) \\ &= f(x)g(x) + hf'(x)g(x) + hf(x)g'(x) \\ &\quad + h^2 f'(x)g'(x) - f(x)g(x) \\ &= hf'(x)g(x) + hf(x)g'(x) + h^2 f'(x)g'(x). \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \lim_{h\to 0} \frac{hf'(x)g(x) + hf(x)g'(x) + h^2 f'(x)g'(x)}{h} \\ &= \lim_{h\to 0} \frac{h\left(f'(x)g(x) + f(x)g'(x) + hf'(x)g'(x)\right)}{h} \\ &= \lim_{h\to 0} \left(f'(x)g(x) + f(x)g'(x) + hf'(x)g'(x)\right) \\ &= f'(x)g(x) + f(x)g'(x). \end{aligned}$$

A more complete derivation can be given using the error term discussed in the section on Differentiability and Linear Approximation in Chapter 2. Adapting the notation of that section to this problem, we write

$$f(x+h) = f(x) + f'(x)h + E_f(h) \quad \text{and} \quad g(x+h) = g(x) + g'(x)h + E_g(h),$$

where $\lim_{h\to 0} \frac{E_f(h)}{h} = \lim_{h\to 0} \frac{E_g(h)}{h} = 0$. (This implies that $\lim_{h\to 0} E_f(h) = \lim_{h\to 0} E_g(h) = 0$.)

We have

$$\begin{aligned} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \frac{f(x)g(x)}{h} + f(x)g'(x) + f'(x)g(x) + f(x)\frac{E_g(h)}{h} + g(x)\frac{E_f(h)}{h} \\ &\quad + f'(x)g'(x)h + f'(x)E_g(h) + g'(x)E_f(h) + \frac{E_f(h)E_g(h)}{h} - \frac{f(x)g(x)}{h} \end{aligned}$$

The terms $f(x)g(x)/h$ and $-f(x)g(x)/h$ cancel out. All the remaining terms on the right, with the exception of the second and third terms, go to zero as $h \to 0$. Thus, we have

$$[f(x)g(x)]' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = f(x)g'(x) + f'(x)g(x).$$

**27.** Note that

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}.$$

Using the local linearizations of $f$ and $g$, we get that

$$\begin{aligned} f(g(x+h)) - f(g(x)) &\approx f\left(g(x) + g'(x)h\right) - f(g(x)) \\ &\approx f(g(x)) + f'(g(x))g'(x)h - f(g(x)) \\ &= f'(g(x))g'(x)h. \end{aligned}$$

Therefore,

$$\begin{aligned} [f(g(x))]' &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h\to 0} \frac{f'(g(x))g'(x)h}{h} \\ &= \lim_{h\to 0} f'(g(x))g'(x) = f'(g(x))g'(x). \end{aligned}$$

A more complete derivation can be given using the error term discussed in the section on Differentiability and Linear Approximation in Chapter 2. Adapting the notation of that section to this problem, we write

$$f(z+k) = f(z) + f'(z)k + E_f(k) \quad \text{and} \quad g(x+h) = g(x) + g'(x)h + E_g(h),$$

where $\lim_{h\to 0} \frac{E_g(h)}{h} = \lim_{k\to 0} \frac{E_f(k)}{k} = 0.$

Now we let $z = g(x)$ and $k = g(x+h) - g(x)$. Then we have $k = g'(x)h + E_g(h)$. Thus,

$$\begin{aligned} \frac{f(g(x+h)) - f(g(x))}{h} &= \frac{f(z+k) - f(z)}{h} \\ &= \frac{f(z) + f'(z)k + E_f(k) - f(z)}{h} = \frac{f'(z)k + E_f(k)}{h} \\ &= \frac{f'(z)g'(x)h + f'(z)E_g(h)}{h} + \frac{E_f(k)}{k} \cdot \left(\frac{k}{h}\right) \\ &= f'(z)g'(x) + \frac{f'(z)E_g(h)}{h} + \frac{E_f(k)}{k}\left[\frac{g'(x)h + E_g(h)}{h}\right] \\ &= f'(z)g'(x) + \frac{f'(z)E_g(h)}{h} + \frac{g'(x)E_f(k)}{k} + \frac{E_g(h)\cdot E_f(k)}{h\cdot k} \end{aligned}$$

Now, if $h \to 0$ then $k \to 0$ as well, and all the terms on the right except the first go to zero, leaving us with the term $f'(z)g'(x)$. Substituting $g(x)$ for $z$, we obtain

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = f'(g(x))g'(x).$$

**28.** We want to show that

$$\lim_{x\to a} \frac{f(x) - f(a)}{x-a} = L.$$

Substituting for $f(x)$ we have

$$\begin{aligned} \lim_{x\to a} \frac{f(x) - f(a)}{x-a} &= \lim_{x\to a} \frac{f(a) + L(x-a) + E_L(x) - f(a)}{x-a} \\ &= \lim_{x\to a}\left(L + \frac{E_L(x)}{x-a}\right) = L + \lim_{x\to 0} \frac{E_L(x)}{x-a} = L. \end{aligned}$$

Thus, we have shown that $f$ is differentiable at $x = a$ and that its derivative is $L$, that is, $f'(a) = L$.

## Solutions for Section 3.10

### Exercises

1. False. The derivative, $f'(x)$, is not equal to zero everywhere, because the function is not continuous at integral values of $x$, so $f'(x)$ does not exist there. Thus, the Constant Function Theorem does not apply.

2. False. The horse that wins the race may have been moving faster for some, but not all, of the race. The Racetrack Principle guarantees the converse—that if the horses start at the same time and one moves faster throughout the race, then that horse wins.

3. True. If $g(x)$ is the position of the slower horse at time $x$ and $h(x)$ is the position of the faster, then $g'(x) \leq h'(x)$ for $a < x < b$. Since the horses start at the same time, $g(a) = h(a)$, so, by the Racetrack Principle, $g(x) \leq h(x)$ for $a \leq x \leq b$. Therefore, $g(b) \leq h(b)$, so the slower horse loses the race.

4. True. If $f'$ is positive on $[a, b]$, then $f$ is continuous and the Increasing Function Theorem applies. Thus, $f$ is increasing on $[a, b]$, so $f(a) < f(b)$.

5. False. Let $f(x) = x^3$ on $[-1, 1]$. Then $f(x)$ is increasing but $f'(x) = 0$ for $x = 0$.

6. No, it does not satisfy the hypotheses. The function does not appear to be differentiable. There appears to be no tangent line, and hence no derivative, at the "corner."
   No, it does not satisfy the conclusion as there is no horizontal tangent.

7. Yes, it satisfies the hypotheses and the conclusion. This function has two points, $c$, at which the tangent to the curve is parallel to the secant joining $(a, f(a))$ to $(b, f(b))$, but this does not contradict the Mean Value Theorem. The function is continuous and differentiable on the interval $[a, b]$.

8. No, it does not satisfy the hypotheses. This function does not appear to be continuous.
   No, it does not satisfy the conclusion as there is no horizontal tangent.

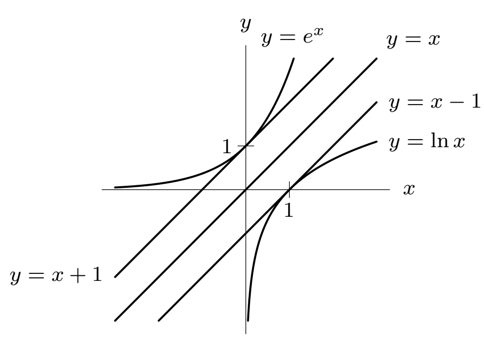
9. No. This function does not satisfy the hypotheses of the Mean Value Theorem, as it is not continuous.
   However, the function has a point $c$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$
   Thus, this satisfies the conclusion of the theorem.

### Problems

10. Let $f(x) = \sin x$ and $g(x) = x$. Then $f(0) = 0$ and $g(0) = 0$. Also $f'(x) = \cos x$ and $g'(x) = 1$, so for all $x \geq 0$ we have $f'(x) \leq g'(x)$. So the graphs of $f$ and $g$ both go through the origin and the graph of $f$ climbs slower than the graph of $g$. Thus the graph of $f$ is below the graph of $g$ for $x \geq 0$ by the Racetrack Principle. In other words, $\sin x \leq x$ for $x \geq 0$.

11. Let $g(x) = \ln x$ and $h(x) = x - 1$. For $x \geq 1$, we have $g'(x) = 1/x \leq 1 = h'(x)$. Since $g(1) = h(1)$, the Racetrack Principle with $a = 1$ says that $g(x) \leq h(x)$ for $x \geq 1$, that is, $\ln x \leq x - 1$ for $x \geq 1$. For $0 < x \leq 1$, we have $h'(x) = 1 \leq 1/x = g'(x)$. Since $g(1) = h(1)$, the Racetrack Principle with $b = 1$ says that $g(x) \leq h(x)$ for $0 < x \leq 1$, that is, $\ln x \leq x - 1$ for $0 < x \leq 1$.

12.




Graphical solution: If $f$ and $g$ are inverse functions then the graph of $g$ is just the graph of $f$ reflected through the line $y = x$. But $e^x$ and $\ln x$ are inverse functions, and so are the functions $x + 1$ and $x - 1$. Thus the equivalence is clear from the figure.

Algebraic solution: If $x > 0$ and

$$x + 1 \leq e^x,$$

then, replacing $x$ by $x - 1$, we have

$$x \leq e^{x-1}.$$

Taking logarithms, and using the fact that ln is an increasing function, gives

$$\ln x \leq x - 1.$$

We can also go in the opposite direction, which establishes the equivalence.

**13.** The Decreasing Function Theorem is: Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. If $f'(x) < 0$ on $(a, b)$, then $f$ is decreasing on $[a, b]$. If $f'(x) \leq 0$ on $(a, b)$, then $f$ is nonincreasing on $[a, b]$.

To prove the theorem, we note that if $f$ is decreasing then $-f$ is increasing and vice-versa. Similarly, if $f$ is nonincreasing, then $-f$ is nondecreasing. Thus if $f'(x) < 0$, then $-f'(x) > 0$, so $-f$ is increasing, which means $f$ is decreasing. And if $f'(x) \leq 0$, then $-f'(x) \geq 0$, so $-f$ is nondecreasing, which means $f$ is nonincreasing.

**14.** Use the Racetrack Principle, Theorem 3.10, with $g(x) = x$. Since $f'(x) \leq g'(x)$ for all $x$ and $f(0) = g(0)$, then $f(x) \leq g(x) = x$ for all $x \geq 0$.

**15.** First apply the Racetrack Principle, Theorem 3.10, to $f'(t)$ and $g(t) = 3t$. Since $f''(t) \leq g'(t)$ for all $t$ and $f'(0) = 0 = g(0)$, then $f'(t) \leq 3t$ for all $t \geq 0$. Next apply the Racetrack Principle again to $f(t)$ and $h(t) = \frac{3}{2}t^2$. Since $f'(t) \leq h'(t)$ for all $t \geq 0$ and $f(0) = 0 = h(0)$, then $f(t) \leq h(t) = \frac{3}{2}t^2$ for all $t \geq 0$.

**16.** Apply the Constant Function Theorem, Theorem 3.9, to $h(x) = f(x) - g(x)$. Then $h'(x) = 0$ for all $x$, so $h(x)$ is constant for all $x$. Since $h(5) = f(5) - g(5) = 0$, we have $h(x) = 0$ for all $x$. Therefore $f(x) - g(x) = 0$ for all $x$, so $f(x) = g(x)$ for all $x$.

**17.** By the Mean Value Theorem, Theorem 3.7, there is a number $c$, with $0 < c < 1$, such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}.$$

Since $f(1) - f(0) > 0$, we have $f'(c) > 0$.

Alternatively if $f'(c) \leq 0$ for all $c$ in $(0, 1)$, then by the Increasing Function Theorem, $f(0) \geq f(1)$.

**18.** Since $f''(t) \leq 7$ for $0 \leq t \leq 2$, if we apply the Racetrack Principle with $a = 0$ to the functions $f'(t) - f'(0)$ and $7t$, both of which go through the origin, we get

$$f'(t) - f'(0) \leq 7t \quad \text{for } 0 \leq t \leq 2.$$

The left side of this inequality is the derivative of $f(t) - f'(0)t$, so if we apply the Racetrack Principle with $a = 0$ again, this time to the functions $f(t) - f'(0)t$ and $(7/2)t^2 + 3$, both of which have the value 3 at $t = 0$, we get

$$f(t) - f'(0)t \leq \frac{7}{2}t^2 + 3 \quad \text{for } 0 \leq t \leq 2.$$

That is,

$$f(t) \leq 3 + 4t + \frac{7}{2}t^2 \quad \text{for } 0 \leq t \leq 2.$$

In the same way, we can show that the lower bound on the acceleration, $5 \leq f''(t)$ leads to:

$$f(t) \geq 3 + 4t + \frac{5}{2}t^2 \quad \text{for } 0 \leq t \leq 2.$$

If we substitute $t = 2$ into these two inequalities, we get bounds on the position at time 2:

$$21 \leq f(2) \leq 25.$$

**19.** Consider the function $f(x) = h(x) - g(x)$. Since $f'(x) = h'(x) - g'(x) \geq 0$, we know that $f$ is nondecreasing by the Increasing Function Theorem. This means $f(x) \leq f(b)$ for $a \leq x \leq b$. However, $f(b) = h(b) - g(b) = 0$, so $f(x) \leq 0$, which means $h(x) \leq g(x)$.

**20.** If $f'(x) = 0$, then both $f'(x) \geq 0$ and $f'(x) \leq 0$. By the Increasing and Decreasing Function Theorems, $f$ is both nondecreasing and nonincreasing, so $f$ is constant.

**21.** Let $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) = 0$ for all $x$ in $(a, b)$. Hence, by the Constant Function Theorem, there is a constant $C$ such that $h(x) = C$ on $(a, b)$. Thus $f(x) = g(x) + C$.

**22.** We will show $f(x) = Ce^x$ by deducing that $f(x)/e^x$ is a constant. By the Constant Function Theorem, we need only show the derivative of $g(x) = f(x)/e^x$ is zero. By the quotient rule (since $e^x \neq 0$), we have

$$g'(x) = \frac{f'(x)e^x - e^x f(x)}{(e^x)^2}.$$

Since $f'(x) = f(x)$, we simplify and obtain

$$g'(x) = \frac{f(x)e^x - e^x f(x)}{(e^x)^2} = \frac{0}{e^{2x}} = 0,$$

which is what we needed to show.

**23.** Apply the Racetrack Principle to the functions $f(x) - f(a)$ and $M(x-a)$; we can do this since $f(a) - f(a) = M(a-a)$ and $f'(x) \le M$. We conclude that $f(x) - f(a) \le M(x - a)$. Similarly, apply the Racetrack Principle to the functions $m(x - a)$ and $f(x) - f(a)$ to obtain $m(x - a) \le f(x) - f(a)$. If we substitute $x = b$ into these inequalities we get

$$m(b - a) \le f(b) - f(a) \le M(b - a).$$

Now, divide by $b - a$.

**24.** (a) Since $f''(x) \ge 0$, $f'(x)$ is nondecreasing on $(a, b)$. Thus $f'(c) \le f'(x)$ for $c \le x < b$ and $f'(x) \le f'(c)$ for $a < x \le c$.

(b) Let $g(x) = f(c) + f'(c)(x-c)$ and $h(x) = f(x)$. Then $g(c) = f(c) = h(c)$, and $g'(x) = f'(c)$ and $h'(x) = f'(x)$. If $c \le x < b$, then $g'(x) \le h'(x)$, and if $a < x \le c$, then $g'(x) \ge h'(x)$, by (a). By the Racetrack Principle, $g(x) \le h'(x)$ for $c \le x < b$ and for $a < x \le c$, as we wanted.

## Solutions for Chapter 3 Review

### Exercises

**1.** $f'(t) = \dfrac{d}{dt}\left(2te^t - \dfrac{1}{\sqrt{t}}\right) = 2e^t + 2te^t + \dfrac{1}{2t^{3/2}}.$

**2.**

$$\begin{aligned}\frac{dw}{dz} &= \frac{(-3)(5+3z) - (5-3z)(3)}{(5+3z)^2} \\ &= \frac{-15 - 9z - 15 + 9z}{(5+3z)^2} = \frac{-30}{(5+3z)^2}\end{aligned}$$

**3.** $f'(x) = \dfrac{3x^2}{9}(3\ln x - 1) + \dfrac{x^3}{9}\left(\dfrac{3}{x}\right) = x^2 \ln x - \dfrac{x^2}{3} + \dfrac{x^2}{3} = x^2 \ln x$

**4.** $f'(\theta) = -1(1 + e^{-\theta})^{-2}(e^{-\theta})(-1) = \dfrac{e^{-\theta}}{(1+e^{-\theta})^2}.$

**5.** Since $h(\theta) = \theta(\theta^{-1/2} - \theta^{-2}) = \theta\theta^{-1/2} - \theta\theta^{-2} = \theta^{1/2} - \theta^{-1}$, we have $h'(\theta) = \dfrac{1}{2}\theta^{-1/2} + \theta^{-2}$.

**6.** $f'(\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta.$

**7.** $\dfrac{d}{dy}\ln\ln(2y^3) = \dfrac{1}{\ln(2y^3)}\dfrac{1}{2y^3}6y^2 = \dfrac{3}{y\ln(2y^3)}.$

**8.** $g'(x) = \dfrac{d}{dx}\left(x^k + k^x\right) = kx^{k-1} + k^x \ln k.$

**9.** $y' = 0$

**10.** $\dfrac{dz}{d\theta} = 3\sin^2\theta\cos\theta$

**11.**

$$\begin{aligned} f'(t) &= 2\cos(3t+5)\cdot(-\sin(3t+5))3 \\ &= -6\cos(3t+5)\cdot\sin(3t+5) \end{aligned}$$

**12.**

$$\begin{aligned} M'(\alpha) &= 2\tan(2+3\alpha)\cdot\frac{1}{\cos^2(2+3\alpha)}\cdot 3 \\ &= 6\cdot\frac{\tan(2+3\alpha)}{\cos^2(2+3\alpha)} \end{aligned}$$

**13.** $s'(\theta) = \dfrac{d}{d\theta}\sin^2(3\theta-\pi) = 6\cos(3\theta-\pi)\sin(3\theta-\pi)$.

**14.** $h'(t) = \dfrac{1}{e^{-t}-t}\left(-e^{-t}-1\right)$.

**15.**

$$\begin{aligned} \frac{d}{d\theta}\left(\frac{\sin(5-\theta)}{\theta^2}\right) &= \frac{\cos(5-\theta)(-1)\theta^2-\sin(5-\theta)(2\theta)}{\theta^4} \\ &= -\frac{\theta\cos(5-\theta)+2\sin(5-\theta)}{\theta^3}. \end{aligned}$$

**16.** $w'(\theta) = \dfrac{1}{\sin^2\theta} - \dfrac{2\theta\cos\theta}{\sin^3\theta}$

**17.** $g'(x) = \dfrac{d}{dx}\left(x^{\frac{1}{2}}+x^{-1}+x^{-\frac{3}{2}}\right) = \dfrac{1}{2}x^{-\frac{1}{2}} - x^{-2} - \dfrac{3}{2}x^{-\frac{5}{2}}$.

**18.** $g'(w) = \dfrac{d}{dw}\left(\dfrac{1}{2^w+e^w}\right) = -\dfrac{2^w\ln 2+e^w}{(2^w+e^w)^2}$.

**19.** $s'(x) = \dfrac{d}{dx}\left(\arctan(2-x)\right) = \dfrac{-1}{1+(2-x)^2}$.

**20.** $r'(\theta) = \dfrac{d}{d\theta}\left(e^{\left(e^\theta+e^{-\theta}\right)}\right) = e^{\left(e^\theta+e^{-\theta}\right)}\left(e^\theta-e^{-\theta}\right)$.

**21.** Using the chain rule, we get:

$$m'(n) = \cos(e^n)\cdot(e^n)$$

**22.** Using the chain rule we get:

$$k'(\alpha) = e^{\tan(\sin\alpha)}(\tan(\sin\alpha))' = e^{\tan(\sin\alpha)}\cdot\frac{1}{\cos^2(\sin\alpha)}\cdot\cos\alpha.$$

**23.** Here we use the product rule, and then the chain rule, and then the product rule.

$$\begin{aligned} g'(t) = \cos(\sqrt{t}e^t) + t(\cos\sqrt{t}e^t)' &= \cos(\sqrt{t}e^t) + t(-\sin(\sqrt{t}e^t)\cdot(\sqrt{t}e^t)') \\ &= \cos(\sqrt{t}e^t) - t\sin(\sqrt{t}e^t)\cdot\left(\sqrt{t}e^t + \frac{1}{2\sqrt{t}}e^t\right) \end{aligned}$$

**24.** $f'(r) = e(\tan 2+\tan r)^{e-1}(\tan 2+\tan r)' = e(\tan 2+\tan r)^{e-1}\left(\dfrac{1}{\cos^2 r}\right)$

**25.** $\dfrac{d}{dx}xe^{\tan x} = e^{\tan x} + xe^{\tan x}\dfrac{1}{\cos^2 x}$.

**26.** $\dfrac{dy}{dx} = 2e^{2x}\sin^2(3x) + e^{2x}(2\sin(3x)\cos(3x)3) = 2e^{2x}\sin(3x)(\sin(3x)+3\cos(3x))$

**27.** $g'(x) = \dfrac{6x}{1+(3x^2+1)^2} = \dfrac{6x}{9x^4+6x^2+2}$

**28.** $\dfrac{dy}{dx} = (\ln 2)2^{\sin x}\cos x\cdot\cos x + 2^{\sin x}(-\sin x) = 2^{\sin x}\left((\ln 2)\cos^2 x - \sin x\right)$

**29.** $h(x) = ax \cdot \ln e = ax$, so $h'(x) = a$.

**30.** $k'(x) = a$

**31.** $f'(\theta) = ke^{k\theta}$

**32.** Using the product rule and factoring gives $f'(t) = e^{-4kt}(\cos t - 4k \sin t)$.

**33.** Using the product rule gives

$$\begin{aligned} H'(t) &= 2ate^{-ct} - c(at^2 + b)e^{-ct} \\ &= (-cat^2 + 2at - bc)e^{-ct}. \end{aligned}$$

**34.** $\dfrac{d}{d\theta}\sqrt{a^2 - \sin^2\theta} = \dfrac{1}{2\sqrt{a^2 - \sin^2\theta}}(-2\sin\theta\cos\theta) = -\dfrac{\sin\theta\cos\theta}{\sqrt{a^2 - \sin^2\theta}}.$

**35.** Using the chain rule gives $f'(x) = 5\ln(a)a^{5x}$.

**36.** Using the quotient rule gives

$$f'(x) = \frac{(-2x)(a^2 + x^2) - (2x)(a^2 - x^2)}{(a^2 + x^2)^2} = \frac{-4a^2x}{(a^2 + x^2)^2}.$$

**37.** Using the quotient rule gives

$$\begin{aligned} w'(r) &= \frac{2ar(b + r^3) - 3r^2(ar^2)}{(b + r^3)^2} \\ &= \frac{2abr - ar^4}{(b + r^3)^2}. \end{aligned}$$

**38.** Using the quotient rule gives

$$\begin{aligned} f'(s) &= \frac{-2s\sqrt{a^2 + s^2} - \frac{s}{\sqrt{a^2+s^2}}(a^2 - s^2)}{(a^2 + s^2)} \\ &= \frac{-2s(a^2 + s^2) - s(a^2 - s^2)}{(a^2 + s^2)^{3/2}} \\ &= \frac{-2a^2s - 2s^3 - a^2s + s^3}{(a^2 + s^2)^{3/2}} \\ &= \frac{-3a^2s - s^3}{(a^2 + s^2)^{3/2}}. \end{aligned}$$

**39.** $\dfrac{dy}{dx} = \dfrac{1}{1 + \left(\frac{2}{x}\right)^2}\left(\dfrac{-2}{x^2}\right) = \dfrac{-2}{x^2 + 4}$

**40.** Using the chain rule gives $r'(t) = \dfrac{\cos(\frac{t}{k})}{\sin(\frac{t}{k})}\left(\dfrac{1}{k}\right)$.

**41.** Since $g(w) = 5(a^2 - w^2)^{-2}$, $g'(w) = -10(a^2 - w^2)^{-3}(-2w) = \dfrac{20w}{(a^2 - w^2)^3}$

**42.**

$$\begin{aligned} \frac{dy}{dx} &= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} \\ &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} \\ &= \frac{4}{(e^x + e^{-x})^2} \end{aligned}$$

**43.** $g'(u) = \dfrac{ae^{au}}{a^2 + b^2}$

**44.** Using the quotient and chain rules, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{(ae^{ax} + ae^{-ax})(e^{ax} + e^{-ax}) - (e^{ax} - e^{-ax})(ae^{ax} - ae^{-ax})}{(e^{ax} + e^{-ax})^2} \\ &= \frac{a(e^{ax} + e^{-ax})^2 - a(e^{ax} - e^{-ax})^2}{(e^{ax} + e^{-ax})^2} \\ &= \frac{a[(e^{2ax} + 2 + e^{-2ax}) - (e^{2ax} - 2 + e^{-2ax})]}{(e^{ax} + e^{-ax})^2} \\ &= \frac{4a}{(e^{ax} + e^{-ax})^2}\end{aligned}$$

**45.** Using the quotient rule gives

$$\begin{aligned}g'(t) &= \frac{\left(\frac{k}{kt} + 1\right)(\ln(kt) - t) - (\ln(kt) + t)\left(\frac{k}{kt} - 1\right)}{(\ln(kt) - t)^2} \\ g'(t) &= \frac{\left(\frac{1}{t} + 1\right)(\ln(kt) - t) - (\ln(kt) + t)\left(\frac{1}{t} - 1\right)}{(\ln(kt) - t)^2} \\ g'(t) &= \frac{\ln(kt)/t - 1 + \ln(kt) - t - \ln(kt)/t - 1 + \ln(kt) + t}{(\ln(kt) - t)^2} \\ g'(t) &= \frac{2\ln(kt) - 2}{(\ln(kt) - t)^2}.\end{aligned}$$

**46.** Using the quotient and chain rules

$$\begin{aligned}\frac{dz}{dt} &= \frac{\frac{d}{dt}(e^{t^2} + t)\cdot\sin(2t) - (e^{t^2} + t)\frac{d}{dt}(\sin(2t))}{(\sin(2t))^2} \\ &= \frac{\left(e^{t^2}\cdot\frac{d}{dt}(t^2) + 1\right)\sin(2t) - (e^{t^2} + t)\cos(2t)\frac{d}{dt}(2t)}{\sin^2(2t)} \\ &= \frac{(2te^{t^2} + 1)\sin(2t) - (e^{t^2} + t)2\cos(2t)}{\sin^2(2t)}.\end{aligned}$$

**47.** Using the chain rule twice:

$$f'(t) = \cos\sqrt{e^t + 1}\frac{d}{dt}\sqrt{e^t + 1} = \cos\sqrt{e^t + 1}\frac{1}{2\sqrt{e^t + 1}}\cdot\frac{d}{dt}(e^t + 1) = \cos\sqrt{e^t + 1}\frac{1}{2\sqrt{e^t + 1}}e^t = e^t\frac{\cos\sqrt{e^t + 1}}{2\sqrt{e^t + 1}}.$$

**48.** Using the chain rule twice:

$$g'(y) = e^{2e^{(y^3)}}\frac{d}{dy}\left(2e^{(y^3)}\right) = 2e^{2e^{(y^3)}}e^{(y^3)}\frac{d}{dy}(y^3) = 6y^2e^{(y^3)}e^{2e^{(y^3)}}.$$

**49.** $g'(x) = -\dfrac{1}{2}(5x^4 + 2)$.

**50.** $y' = -12x^3 - 12x^2 - 6$.

**51.** $g(z) = z^5 + 5z^4 - z$
$g'(z) = 5z^4 + 20z^3 - 1$.

**52.** $f'(z) = (2\ln 3)z + (\ln 4)e^z$.

**53.** $g'(x) = \dfrac{d}{dx}(2x - x^{-1/3} + 3^x - e) = 2 + \dfrac{1}{3x^{\frac{4}{3}}} + 3^x\ln 3$.

**54.** $f'(x) = 6x(e^x - 4) + (3x^2 + \pi)e^x = 6xe^x - 24x + 3x^2e^x + \pi e^x$.

**55.** $f'(\theta) = 2\theta\sin\theta + \theta^2\cos\theta + 2\cos\theta - 2\theta\sin\theta - 2\cos\theta = \theta^2\cos\theta$.

**56.**

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{1}{2}(\cos(5\theta))^{-\frac{1}{2}}(-\sin(5\theta)\cdot 5) + 2\sin(6\theta)\cos(6\theta)\cdot 6\\ &= -\frac{5}{2}\frac{\sin(5\theta)}{\sqrt{\cos(5\theta)}} + 12\sin(6\theta)\cos(6\theta)\end{aligned}$$

**57.** $r'(\theta) = \dfrac{d}{d\theta}\sin[(3\theta-\pi)^2] = \cos[(3\theta-\pi)^2]\cdot 2(3\theta-\pi)\cdot 3 = 6(3\theta-\pi)\cos[(3\theta-\pi)^2]$.

**58.** Using the product and chain rules, we have

$$\begin{aligned}\frac{dy}{dx} &= 3(x^2+5)^2(2x)(3x^3-2)^2 + (x^2+5)^3[2(3x^3-2)(9x^2)]\\ &= 3(2x)(x^2+5)^2(3x^3-2)[(3x^3-2)+(x^2+5)(3x)]\\ &= 6x(x^2+5)^2(3x^3-2)[6x^3+15x-2].\end{aligned}$$

**59.** Since $\tan(\arctan(k\theta)) = k\theta$, because tangent and arctangent are inverse functions, we have $N'(\theta) = k$.

**60.** Using the product rule gives $h'(t) = ke^{kt}(\sin at + \cos bt) + e^{kt}(a\cos at - b\sin bt)$.

**61.** $f'(x) = \dfrac{d}{dx}(2-4x-3x^2)(6x^e-3\pi) = (-4-6x)(6x^e-3\pi) + (2-4x-3x^2)(6ex^{e-1})$.

**62.** $f'(t) = 4(\sin(2t)-\cos(3t))^3[2\cos(2t)+3\sin(3t)]$

**63.** Since $\cos^2 y + \sin^2 y = 1$, we have $s(y) = \sqrt[3]{1+3} = \sqrt[3]{4}$. Thus $s'(y) = 0$.

**64.**

$$\begin{aligned}f'(x) &= (-2x+6x^2)(6-4x+x^7) + (4-x^2+2x^3)(-4+7x^6)\\ &= (-12x+44x^2-24x^3-2x^8+6x^9) + (-16+4x^2-8x^3+28x^6-7x^8+14x^9)\\ &= -16-12x+48x^2-32x^3+28x^6-9x^8+20x^9\end{aligned}$$

**65.**

$$\begin{aligned}h'(x) &= \left(-\frac{1}{x^2}+\frac{2}{x^3}\right)\left(2x^3+4\right) + \left(\frac{1}{x}-\frac{1}{x^2}\right)\left(6x^2\right)\\ &= -2x+4-\frac{4}{x^2}+\frac{8}{x^3}+6x-6\\ &= 4x-2-4x^{-2}+8x^{-3}\end{aligned}$$

**66.** Note: $f(z) = (5z)^{1/2} + 5z^{1/2} + 5z^{-1/2} - \sqrt{5}z^{-1/2} + \sqrt{5}$, so $f'(z) = \dfrac{5}{2}(5z)^{-1/2} + \dfrac{5}{2}z^{-1/2} - \dfrac{5}{2}z^{-3/2} + \dfrac{\sqrt{5}}{2}z^{-3/2}$.

**67.**

$$\begin{aligned}3x^2 + 3y^2\frac{dy}{dx} - 8xy - 4x^2\frac{dy}{dx} &= 0\\ (3y^2-4x^2)\frac{dy}{dx} &= 8xy - 3x^2\\ \frac{dy}{dx} &= \frac{8xy-3x^2}{3y^2-4x^2}\end{aligned}$$

**68.** Differentiating implicitly on both sides with respect to $x$,

$$\begin{aligned}a\cos(ay)\frac{dy}{dx} - b\sin(bx) &= y + x\frac{dy}{dx}\\ (a\cos(ay)-x)\frac{dy}{dx} &= y + b\sin(bx)\\ \frac{dy}{dx} &= \frac{y+b\sin(bx)}{a\cos(ay)-x}.\end{aligned}$$

**69.** We wish to find the slope $m = dy/dx$. To do this, we can implicitly differentiate the given formula in terms of $x$:

$$\begin{aligned} x^2 + 3y^2 &= 7 \\ 2x + 6y\frac{dy}{dx} &= \frac{d}{dx}(7) = 0 \\ \frac{dy}{dx} &= \frac{-2x}{6y} = \frac{-x}{3y}. \end{aligned}$$

Thus, at $(2, -1)$, $m = -(2)/3(-1) = 2/3$.

**70.** Taking derivatives implicitly, we find

$$\begin{aligned} \frac{dy}{dx} + \cos y\frac{dy}{dx} + 2x &= 0 \\ \frac{dy}{dx} &= \frac{-2x}{1+\cos y} \end{aligned}$$

So, at the point $x = 3, y = 0$,

$$\frac{dy}{dx} = \frac{(-2)(3)}{1+\cos 0} = \frac{-6}{2} = -3.$$

**71.** First, we differentiate with respect to $x$:

$$\begin{aligned} x \cdot \frac{dy}{dx} + y \cdot 1 + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}(x + 2y) &= -y \\ \frac{dy}{dx} &= \frac{-y}{x+2y}. \end{aligned}$$

At $x = 3$, we have

$$\begin{aligned} 3y + y^2 &= 4 \\ y^2 + 3y - 4 &= 0 \\ (y-1)(y+4) &= 0. \end{aligned}$$

Our two points, then, are $(3, 1)$ and $(3, -4)$.

$$\text{At } (3,1), \quad \frac{dy}{dx} = \frac{-1}{3+2(1)} = -\frac{1}{5}; \quad \text{Tangent line: } (y-1) = -\frac{1}{5}(x-3).$$

$$\text{At } (3,-4), \quad \frac{dy}{dx} = \frac{-(-4)}{3+2(-4)} = -\frac{4}{5}; \quad \text{Tangent line: } (y+4) = -\frac{4}{5}(x-3).$$

## Problems

**72.** (a) Applying the product rule to $h(x)$ we get $h'(1) = t'(1)s(1) + t(1)s'(1) \approx (-2) \cdot 3 + 0 \cdot 0 = -6$.
(b) Applying the product rule to $h(x)$ we get $h'(0) = t'(0)s(0) + t(0)s'(0) \approx (-2) \cdot 2 + 2 \cdot 2 = 0$.
(c) Applying the quotient rule to $p(x)$ we get $p'(0) = \dfrac{t'(0)s(0) - t(0)s'(0)}{(s(0))^2} \approx \dfrac{(-2) \cdot 2 - 2 \cdot 2}{2^2} = -2$.

Note that since $t(x)$ is a linear function whose slope looks like $-2$ from the graph, $t'(x) \approx -2$ everywhere. To find $s'(1)$, draw a line tangent to the curve at the point $(1, s(1))$, and estimate the slope.

**73.** Since $r(x) = s(t(x))$, the chain rule gives $r'(x) = s'(t(x)) \cdot t'(x)$. Thus,

$$r'(0) = s'(t(0)) \cdot t'(0) \approx s'(2) \cdot (-2) \approx (-2)(-2) = 4.$$

Note that since $t(x)$ is a linear function whose slope looks like $-2$ from the graph, $t'(x) \approx -2$ everywhere. To find $s'(2)$, draw a line tangent to the curve at the point $(2, s(2))$, and estimate the slope.

**74.** (a) Applying the chain rule we get $h'(1) = s'(s(1)) \cdot s'(1) \approx s'(3) \cdot 0 = 0$.
(b) Applying the chain rule we get $h'(2) = s'(s(2)) \cdot s'(2) \approx s'(2) \cdot s'(2) = (-2)^2 = 4$.
To find $s'(2)$, draw a line tangent to the curve at the point $(2, s(2))$, and estimate the slope.

**75.** We need to find all values for $x$ such that

$$\frac{dy}{dx} = s'(s(x)) \cdot s'(x) = 0.$$

This is the case when either $s'(s(x)) = 0$ or $s'(x) = 0$. From the graph we see that $s'(x) = 0$ when $x \approx 1$. Also, $s'(s(x)) = 0$ when $s(x) \approx 1$, which happens when $x \approx -0.4$ or $x \approx 2.4$.
To find $s'(a)$, for any $a$, draw a line tangent to the curve at the point $(a, s(a))$, and estimate the slope.

**76.** (a) Applying the product rule we get $h'(-1) = 2 \cdot (-1) \cdot t(-1) + (-1)^2 \cdot t'(-1) \approx (-2) \cdot 4 + 1 \cdot (-2) = -10$.
(b) Applying the chain rule we get $p'(-1) = t'((-1)^2) \cdot 2 \cdot (-1) = -2 \cdot t'(1) \approx (-2) \cdot (-2) = 4$.
Note that since $t(x)$ is a linear function whose slope looks like $-2$ from the graph, $t'(x) \approx -2$ everywhere.

**77.** We have $r(1) = s(t(1)) \approx s(0) \approx 2$. By the chain rule, $r'(x) = s'(t(x)) \cdot t'(x)$, so

$$r'(1) = s'(t(1)) \cdot t'(1) \approx s'(0) \cdot (-2) \approx 2(-2) = -4.$$

Thus the equation of the tangent line is

$$y - 2 = -4(x - 1)$$
$$y = -4x + 6.$$

Note that since $t(x)$ is a linear function whose slope looks like $-2$ from the graph, $t'(x) \approx -2$ everywhere. To find $s'(0)$, draw a line tangent to the curve at the point $(0, s(0))$, and estimate the slope.

**78.** We have

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))}.$$

From the graph of $f(x)$ we see that $f^{-1}(5) = 13$. From the graph of $f'(x)$ we see that $f'(13) = 0.36$. Thus $(f^{-1})'(5) = 1/0.36 = 2.8$.

**79.** We have

$$(f^{-1})'(10) = \frac{1}{f'(f^{-1}(10))}.$$

From the graph of $f(x)$ we see that $f^{-1}(10) = 23$. From the graph of $f'(x)$ we see that $f'(23) = 0.62$. Thus $(f^{-1})'(10) = 1/0.62 = 1.6$.

**80.** We have

$$(f^{-1})'(15) = \frac{1}{f'(f^{-1}(15))}.$$

From the graph of $f(x)$ we see that $f^{-1}(15) = 30$. From the graph of $f'(x)$ we see that $f'(30) = 0.73$. Thus $(f^{-1})'(15) = 1/0.73 = 1.4$.

**81.** Since $W$ is proportional to $r^3$, we have $W = kr^3$ for some constant $k$. Thus, $dW/dr = k(3r^2) = 3kr^2$. Thus, $dW/dr$ is proportional to $r^2$.

**82.** Taking the values of $f$, $f'$, $g$, and $g'$ from the table we get:
(a) $h(4) = f(g(4)) = f(3) = 1$.
(b) $h'(4) = f'(g(4))g'(4) = f'(3) \cdot 1 = 2$.
(c) $h(4) = g(f(4)) = g(4) = 3$.
(d) $h'(4) = g'(f(4))f'(4) = g'(4) \cdot 3 = 3$.
(e) $h'(4) = (f(4)g'(4) - g(4)f'(4))/f^2(4) = -5/16$.
(f) $h'(4) = f(4)g'(4) + g(4)f'(4) = 13$.

**83.** (a) $H'(2) = r'(2)s(2) + r(2)s'(2) = -1 \cdot 1 + 4 \cdot 3 = 11$.
(b) $H'(2) = \dfrac{r'(2)}{2\sqrt{r(2)}} = \dfrac{-1}{2\sqrt{4}} = -\dfrac{1}{4}$.
(c) $H'(2) = r'(s(2))s'(2) = r'(1) \cdot 3$, but we don't know $r'(1)$.
(d) $H'(2) = s'(r(2))r'(2) = s'(4)r'(2) = -3$.

**84.** (a) $f(x) = x^2 - 4g(x)$
$f'(x) = 2x - 4g'(x)$
$f'(2) = 2(2) - 4(-4) = 4 + 16 = 20$

(b) $f(x) = \frac{x}{g(x)}$
$f'(x) = \frac{g(x)-xg'(x)}{(g(x))^2}$
$f'(2) = \frac{g(2)-2g'(2)}{(g(2))^2} = \frac{3-2(-4)}{(3)^2} = \frac{11}{9}$

(c) $f(x) = x^2 g(x)$
$f'(x) = 2xg(x) + x^2 g'(x)$
$f'(2) = 2(2)(3) + (2)^2(-4) = 12 - 16 = -4$

(d) $f(x) = (g(x))^2$
$f'(x) = 2g(x) \cdot g'(x)$
$f'(2) = 2(3)(-4) = -24$

(e) $f(x) = x\sin(g(x))$
$f'(x) = \sin(g(x)) + x\cos(g(x)) \cdot g'(x)$

$$\begin{aligned} f'(2) &= \sin(g(2)) + 2\cos(g(2)) \cdot g'(2) \\ &= \sin 3 + 2\cos(3) \cdot (-4) \\ &= \sin 3 - 8\cos 3 \end{aligned}$$

(f) $f(x) = x^2 \ln(g(x))$
$f'(x) = 2x\ln(g(x)) + x^2(\frac{g'(x)}{g(x)})$

$$\begin{aligned} f'(2) &= 2(2)\ln 3 + (2)^2(\tfrac{-4}{3}) \\ &= 4\ln 3 - \tfrac{16}{3} \end{aligned}$$

**85.** (a) $f(x) = x^2 - 4g(x)$
$f(2) = 4 - 4(3) = -8$
$f'(2) = 20$
Thus, we have a point $(2, -8)$ and slope $m = 20$. This gives

$$\begin{aligned} -8 &= 2(20) + b \\ b &= -48, \quad \text{so} \\ y &= 20x - 48. \end{aligned}$$

(b) $f(x) = \dfrac{x}{g(x)}$
$f(2) = \dfrac{2}{3}$
$f'(2) = \dfrac{11}{9}$
Thus, we have point $(2, \frac{2}{3})$ and slope $m = \frac{11}{9}$. This gives

$$\begin{aligned} \frac{2}{3} &= (\frac{11}{9})(2) + b \\ b &= \frac{2}{3} - \frac{22}{9} = \frac{-16}{9}, \quad \text{so} \\ y &= \frac{11}{9}x - \frac{16}{9}. \end{aligned}$$

(c) $f(x) = x^2 g(x)$
$f(2) = 4 \cdot g(2) = 4(3) = 12$
$f'(2) = -4$
Thus, we have point $(2, 12)$ and slope $m = -4$. This gives

$$\begin{aligned} 12 &= 2(-4) + b \\ b &= 20, \quad \text{so} \\ y &= -4x + 20. \end{aligned}$$

(d) $f(x) = (g(x))^2$
$f(2) = (g(2))^2 = (3)^2 = 9$
$f'(2) = -24$
Thus, we have point $(2, 9)$ and slope $m = -24$. This gives

$$\begin{aligned} 9 &= 2(-24) + b \\ b &= 57, \quad \text{so} \\ y &= -24x + 57. \end{aligned}$$

(e) $f(x) = x\sin(g(x))$
$f(2) = 2\sin(g(2)) = 2\sin 3$
$f'(2) = \sin 3 - 8\cos 3$
We will use a decimal approximation for $f(2)$ and $f'(2)$, so the point $(2, 2\sin 3) \approx (2, 0.28)$ and $m \approx 8.06$. Thus,

$$\begin{aligned} 0.28 &= 2(8.06) + b \\ b &= -15.84, \quad \text{so} \\ y &= 8.06x - 15.84. \end{aligned}$$

(f) $f(x) = x^2 \ln g(x)$
$f(2) = 4\ln g(2) = 4\ln 3 \approx 4.39$
$f'(2) = 4\ln 3 - \dfrac{16}{3} \approx -0.94.$
Thus, we have point $(2, 4.39)$ and slope $m = -0.94$. This gives

$$\begin{aligned} 4.39 &= 2(-0.94) + b \\ b &= 6.27, \quad \text{so} \\ y &= -0.94x + 6.27. \end{aligned}$$

**86.** When we zoom in on the origin, we find that two functions are not defined there. The other functions all look like straight lines through the origin. The only way we can tell them apart is their slope.

The following functions all have slope 0 and are therefore indistinguishable:
$\sin x - \tan x$, $\frac{x^2}{x^2+1}$, $x - \sin x$, and $\frac{1-\cos x}{\cos x}$.

These functions all have slope 1 at the origin, and are thus indistinguishable:
$\arcsin x$, $\frac{\sin x}{1+\sin x}$, $\arctan x$, $e^x - 1$, $\frac{x}{x+1}$, and $\frac{x}{x^2+1}$.

Now, $\frac{\sin x}{x} - 1$ and $-x\ln x$ both are undefined at the origin, so they are distinguishable from the other functions. In addition, while $\frac{\sin x}{x} - 1$ has a slope that approaches zero near the origin, $-x\ln x$ becomes vertical near the origin, so they are distinguishable from each other.

Finally, $x^{10} + \sqrt[10]{x}$ is the only function defined at the origin and with a vertical tangent there, so it is distinguishable from the others.

**87.** It makes sense to define the angle between two curves to be the angle between their tangent lines. (The tangent lines are the best linear approximations to the curves). See Figure 3.17. The functions $\sin x$ and $\cos x$ are equal at $x = \frac{\pi}{4}$.

$$\text{For } f_1(x) = \sin x, \quad f_1'(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$$
$$\text{For } f_2(x) = \cos x, \quad f_2'(\frac{\pi}{4}) = -\sin(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}.$$

Using the point $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$ for each tangent line we get $y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}(1 - \frac{\pi}{4})$ and $y = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}(1 + \frac{\pi}{4})$, respectively.

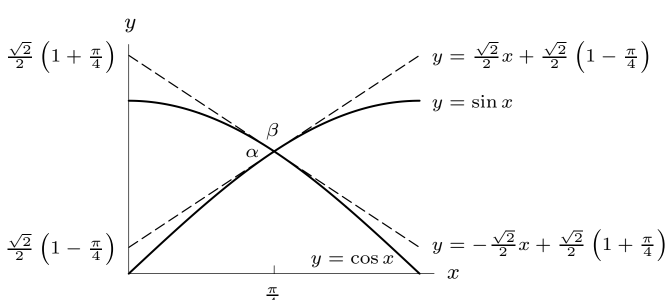



Figure 3.17

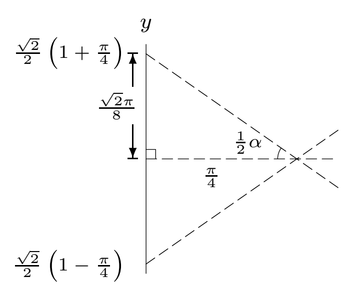



Figure 3.18

There are two possibilities of how to define the angle between the tangent lines, indicated by $\alpha$ and $\beta$ above. The choice is arbitrary, so we will solve for both. To find the angle, $\alpha$, we consider the triangle formed by these two lines and the $y$-axis. See Figure 3.18.

$$\tan\left(\frac{1}{2}\alpha\right) = \frac{\sqrt{2}\pi/8}{\pi/4} = \frac{\sqrt{2}}{2}$$
$$\frac{1}{2}\alpha = 0.61548 \text{ radians}$$
$$\alpha = 1.231 \text{ radians, or } 70.5^\circ.$$

Now let us solve for $\beta$, the other possible measure of the angle between the two tangent lines. Since $\alpha$ and $\beta$ are supplementary, $\beta = \pi - 1.231 = 1.909$ radians, or $109.4^\circ$.

**88.** The curves meet when $1 + x - x^2 = 1 - x + x^2$, that is when $2x(1-x) = 0$ so that $x = 1$ or $x = 0$. Let

$$y_1(x) = 1 + x - x^2 \quad \text{and} \quad y_2(x) = 1 - x + x^2.$$

Then

$$y_1{}' = 1 - 2x \quad \text{and} \quad y_2{}' = -1 + 2x.$$

At $x = 0$, $y_1{}' = 1$, $y_2{}' = -1$ so that $y_1{}' \cdot y_2{}' = -1$ and the curves are perpendicular. At $x = 1$, $y_1{}' = -1$, $y_2{}' = 1$ so that $y_1{}' \cdot y_2{}' = -1$ and the curves are perpendicular.

**89.** The curves meet when $1 - x^3/3 = x - 1$, that is when $x^3 + 3x - 6 = 0$. So the roots of this equation give us the $x$-coordinates of the intersection point. By numerical methods, we see there is one solution near $x = 1.3$. See Figure 3.19. Let

$$y_1(x) = 1 - \frac{x^3}{3} \quad \text{and} \quad y_2(x) = x - 1.$$

So we have

$$y_1{}' = -x^2 \quad \text{and} \quad y_2{}' = 1.$$

However, $y_2{}'(x) = +1$, so if the curves are to be perpendicular when they cross, then $y_1{}'$ must be $-1$. Since $y_1{}' = -x^2$, $y_1{}' = -1$ only at $x = \pm 1$ which is not the point of intersection. The curves are therefore not perpendicular when they cross.

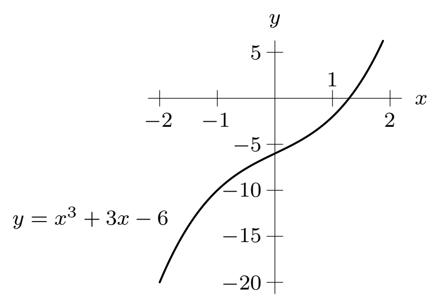



**Figure 3.19**

**90.** Differentiating gives $\dfrac{dy}{dx} = \ln x + 1 - b$.

To find the point at which the graph crosses the $x$-axis, set $y = 0$ and solve for $x$:

$$0 = x\ln x - bx$$
$$0 = x(\ln x - b).$$

Since $x > 0$, we have

$$\ln x - b = 0$$
$$x = e^b.$$

At the point $(e^b, 0)$, the slope is

$$\frac{dy}{dx} = \ln\left(e^b\right) + 1 - b = b + 1 - b = 1.$$

Thus the equation of the tangent line is

$$y - 0 = 1(x - e^b)$$
$$y = x - e^b.$$

**91.** Using the definition of $\cosh x$ and $\sinh x$, we have $\cosh 2x = \dfrac{e^{2x}+e^{-2x}}{2}$ and $\sinh 3x = \dfrac{e^{3x}-e^{-3x}}{2}$. Therefore

$$\begin{aligned}\lim_{x\to-\infty}\frac{\cosh(2x)}{\sinh(2x)} &= \lim_{x\to-\infty}\frac{e^{2x}+e^{-2x}}{e^{3x}-e^{-3x}}\\ &= \lim_{x\to-\infty}\frac{e^{-2x}(e^{4x}+1)}{e^{-2x}(e^{5x}-e^{-x})}\\ &= \lim_{x\to-\infty}\frac{e^{4x}+1}{e^{5x}-e^{-x}}\\ &= 0.\end{aligned}$$

**92.** Using the definition of $\sinh x$ we have $\sinh 2x = \dfrac{e^{2x}-e^{-2x}}{2}$. Therefore

$$\begin{aligned}\lim_{x\to-\infty}\frac{e^{-2x}}{\sinh(2x)} &= \lim_{x\to-\infty}\frac{2e^{-2x}}{e^{2x}-e^{-2x}}\\ &= \lim_{x\to-\infty}\frac{2}{e^{4x}-1}\\ &= -2.\end{aligned}$$

**93.** Using the definition of $\cosh x$ and $\sinh x$, we have $\cosh x^2 = \dfrac{e^{x^2}+e^{-x^2}}{2}$ and $\sinh x^2 = \dfrac{e^{x^2}-e^{-x^2}}{2}$. Therefore

$$\begin{aligned}\lim_{x\to-\infty}\frac{\sinh(x^2)}{\cosh(x^2)} &= \lim_{x\to-\infty}\frac{e^{x^2}-e^{-x^2}}{e^{x^2}+e^{-x^2}}\\ &= \lim_{x\to-\infty}\frac{e^{x^2}(1-e^{-2x^2})}{e^{x^2}(1+e^{-2x^2})}\\ &= \lim_{x\to-\infty}\frac{1-e^{-2x^2}}{1+e^{-2x^2}}\\ &= 1.\end{aligned}$$

**94.** (a) $\dfrac{dg}{dr} = GM\dfrac{d}{dr}\left(\dfrac{1}{r^2}\right) = GM\dfrac{d}{dr}\left(r^{-2}\right) = GM(-2)r^{-3} = -\dfrac{2GM}{r^3}$.

(b) $\dfrac{dg}{dr}$ is the rate of change of acceleration due to the pull of gravity. The further away from the center of the earth, the weaker the pull of gravity is. So $g$ is decreasing and therefore its derivative, $\dfrac{dg}{dr}$, is negative.

(c) By part (a),

$$\left.\frac{dg}{dr}\right|_{r=6400} = -\left.\frac{2GM}{r^3}\right|_{r=6400} = -\frac{2(6.67\times10^{-20})(6\times10^{24})}{(6400)^3} \approx -3.05\times10^{-6}.$$

(d) It is reasonable to assume that $g$ is a constant near the surface of the earth.

**95.** The population of Mexico is given by the formula

$$M = 84(1+0.026)^t = 84(1.026)^t \text{ million}$$

and that of the US by

$$U = 250(1+0.007)^t = 250(1.007)^t \text{ million},$$

where $t$ is measured in years ($t = 0$ corresponds to the year 1990). So,

$$\left.\frac{dM}{dt}\right|_{t=0} = 84\frac{d}{dt}(1.026)^t\Big|_{t=0} = 84(1.026)^t\ln(1.026)\Big|_{t=0} \approx 2.156$$

and

$$\left.\frac{dU}{dt}\right|_{t=0} = 250\frac{d}{dt}(1.007)^t\Big|_{t=0} = 250(1.007)^t\ln(1.007)\Big|_{t=0} \approx 1.744$$

Since $\left.\frac{dM}{dt}\right|_{t=0} > \left.\frac{dU}{dt}\right|_{t=0}$, the population of Mexico was growing faster in 1990.

**96.** (a) If the distance $s(t) = 20e^{\frac{t}{2}}$, then the velocity, $v(t)$, is given by

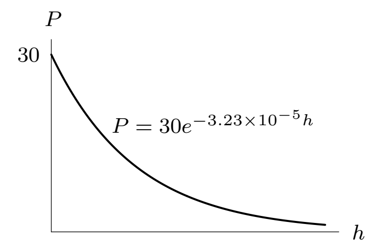
$$v(t) = s'(t) = \left(20e^{\frac{t}{2}}\right)' = \left(\frac{1}{2}\right)\left(20e^{\frac{t}{2}}\right) = 10e^{\frac{t}{2}}.$$

(b) Observing the differentiation in (a), we note that

$$s'(t) = v(t) = \frac{1}{2}\left(20e^{\frac{t}{2}}\right) = \frac{1}{2}s(t).$$

Substituting $s(t)$ for $20e^{\frac{t}{2}}$, we obtain $s'(t) = \frac{1}{2}s(t)$.

**97.** (a)




(b)

$$\frac{dP}{dh} = 30e^{-3.23\times10^{-5}h}(-3.23\times10^{-5})$$

so

$$\left.\frac{dP}{dh}\right|_{h=0} = -30(3.23\times10^{-5}) = -9.69\times10^{-4}$$

Hence, at $h = 0$, the slope of the tangent line is $-9.69\times10^{-4}$, so the equation of the tangent line is

$$y - 30 = (-9.69\times10^{-4})(h-0)$$
$$y = (-9.69\times10^{-4})h + 30.$$

(c) The rule of thumb says

$$\text{Drop in pressure from sea level to height } h = \frac{h}{1000}$$

But since the pressure at sea level is 30 inches of mercury, this drop in pressure is also $(30 - P)$, so

$$30 - P = \frac{h}{1000}$$

giving

$$P = 30 - 0.001h.$$

(d) The equations in (b) and (c) are almost the same: both have $P$ intercepts of 30, and the slopes are almost the same $(9.69\times10^{-4} \approx 0.001)$. The rule of thumb calculates values of $P$ which are very close to the tangent lines, and therefore yields values very close to the curve.

(e) The tangent line is slightly below the curve, and the rule of thumb line, having a slightly more negative slope, is slightly below the tangent line (for $h > 0$). Thus, the rule of thumb values are slightly smaller.

**98.**

$$\frac{dy}{dt} = -7.5(0.507)\sin(0.507t) = -3.80\sin(0.507t)$$

(a) When $t = 6$, we have $\frac{dy}{dt} = -3.80\sin(0.507\cdot6) = -0.38$ meters/hour. So the tide is falling at $0.38$ meters/hour.

(b) When $t = 9$, we have $\frac{dy}{dt} = -3.80\sin(0.507\cdot9) = 3.76$ meters/hour. So the tide is rising at $3.76$ meters/hour.

(c) When $t = 12$, we have $\frac{dy}{dt} = -3.80\sin(0.507\cdot12) = 0.75$ meters/hour. So the tide is rising at $0.75$ meters/hour.

(d) When $t = 18$, we have $\frac{dy}{dt} = -3.80\sin(0.507\cdot18) = -1.12$ meters/hour. So the tide is falling at $1.12$ meters/hour.

**99.** Since we're given that the instantaneous rate of change of $T$ at $t = 30$ is 2, we want to choose $a$ and $b$ so that the derivative of $T$ agrees with this value. Differentiating, $T'(t) = ab \cdot e^{-bt}$. Then we have

$$2 = T'(30) = abe^{-30b} \text{ or } e^{-30b} = \frac{2}{ab}$$

We also know that at $t = 30$, $T = 120$, so

$$120 = T(30) = 200 - ae^{-30b} \text{ or } e^{-30b} = \frac{80}{a}$$

Thus $\frac{80}{a} = e^{-30b} = \frac{2}{ab}$, so $b = \frac{1}{40} = 0.025$ and $a = 169.36$.

**100.** (a) Differentiating, we see

$$v = \frac{dy}{dt} = -2\pi\omega y_0 \sin(2\pi\omega t)$$
$$a = \frac{dv}{dt} = -4\pi^2\omega^2 y_0 \cos(2\pi\omega t).$$

(b) We have

$$y = y_0 \cos(2\pi\omega t)$$
$$v = -2\pi\omega y_0 \sin(2\pi\omega t)$$
$$a = -4\pi^2\omega^2 y_0 \cos(2\pi\omega t).$$

So

Amplitude of $y$ is $|y_0|$,
Amplitude of $v$ is $|2\pi\omega y_0| = 2\pi\omega|y_0|$,
Amplitude of $a$ is $|4\pi^2\omega^2 y_0| = 4\pi^2\omega^2|y_0|$.

The amplitudes are different (provided $2\pi\omega \neq 1$). The periods of the three functions are all the same, namely $1/\omega$.

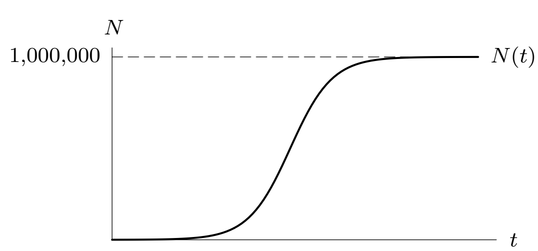
(c) Looking at the answer to part (a), we see

$$\frac{d^2y}{dt^2} = a = -4\pi^2\omega^2 (y_0 \cos(2\pi\omega t)) = -4\pi^2\omega^2 y.$$

So we see that

$$\frac{d^2y}{dt^2} + 4\pi^2\omega^2 y = 0.$$

**101.** (a) Since $\lim_{t\to\infty} e^{-0.1t} = 0$, we see that $\lim_{t\to\infty} \frac{1000000}{1 + 5000e^{-0.1t}} = 1000000$. Thus, in the long run, close to 1,000,000 people will have had the disease. This can be seen in the figure below.




(b) The rate at which people fall sick is given by the first derivative $N'(t)$.
$N'(t) \approx \frac{\Delta N}{\Delta t}$, where $\Delta t = 1$ day.

$$N'(t) = \frac{500{,}000{,}000}{e^{0.1t}(1 + 5000e^{-0.1t})^2} = \frac{500{,}000{,}000}{e^{0.1t} + 25{,}000{,}000e^{-0.1t} + 10^4}$$

In Figure 3.20, we see that the maximum value of $N'(t)$ is approximately $25{,}000$. Therefore the maximum number of people to fall sick on any given day is $25{,}000$. Thus there are no days on which a quarter million or more get sick.

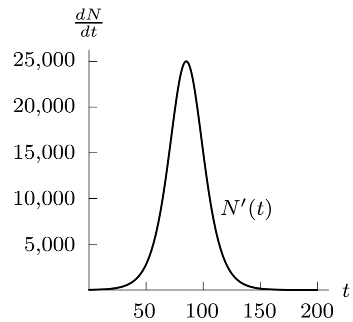



**Figure 3.20**

**102.** (a) The statement $f(2) = 4023$ tells us that when the price is \$2 per gallon, 4023 gallons of gas are sold.
(b) Since $f(2) = 4023$, we have $f^{-1}(4023) = 2$. Thus, 4023 gallons are sold when the price is \$2 per gallon.
(c) The statement $f'(2) = -1250$ tells us that if the price increases from \$2 per gallon, the sales decrease at a rate of 1250 gallons per \$1 increase in price.
(d) The units of $(f^{-1})'(4023)$ are dollars per gallon. We have

$$(f^{-1})'(4023) = \frac{1}{f'(f^{-1}(4023))} = \frac{1}{f'(2)} = -\frac{1}{1250} = -0.0008.$$

Thus, when 4023 gallons are already sold, sales decrease at the rate of one gallon per price increase of 0.0008 dollars. In others words, an additional gallon is sold if the price drops by 0.0008 dollars.

**103.** Since $f(20) = 10$, we have $f^{-1}(10) = 20$, so $(f^{-1})'(10) = \dfrac{1}{f'(f^{-1}(10))} = \dfrac{1}{f'(20)}$. Therefore $(f^{-1})'(10)f'(20) = 1$.

Option (b) is wrong.

**104.** Since $f(x)$ is decreasing, its inverse function $f^{-1}(x)$ is also decreasing. Thus $(f^{-1})'(x) \leq 0$ for all $x$. Option (b) is incorrect.

**105.** (a) If $y = \ln x$, then

$$\begin{aligned} y' &= \frac{1}{x} \\ y'' &= -\frac{1}{x^2} \\ y''' &= \frac{2}{x^3} \\ y'''' &= -\frac{3\cdot 2}{x^4} \end{aligned}$$

and so

$$y^{(n)} = (-1)^{n+1}(n-1)!x^{-n}.$$

(b) If $y = xe^x$, then

$$\begin{aligned} y' &= xe^x + e^x \\ y'' &= xe^x + 2e^x \\ y''' &= xe^x + 3e^x \end{aligned}$$

so that

$$y^{(n)} = xe^x + ne^x.$$

(c) If $y = e^x \cos x$, then

$$\begin{aligned} y' &= e^x(\cos x - \sin x) \\ y'' &= -2e^x \sin x \\ y''' &= e^x(-2\cos x - 2\sin x) \\ y^{(4)} &= -4e^x \cos x \\ y^{(5)} &= e^x(-4\cos x + 4\sin x) \\ y^{(6)} &= 8e^x \sin x. \end{aligned}$$

Combining these results we get

$$\begin{aligned} y^{(n)} &= (-4)^{(n-1)/4} e^x(\cos x - \sin x), && n = 4m+1, \quad m = 0,1,2,3,\ldots \\ y^{(n)} &= -2(-4)^{(n-2)/4} e^x \sin x, && n = 4m+2, \quad m = 0,1,2,3,\ldots \\ y^{(n)} &= -2(-4)^{(n-3)/4} e^x(\cos x + \sin x), && n = 4m+3, \quad m = 0,1,2,3,\ldots \\ y^{(n)} &= (-4)^{(n/4)} e^x \cos x, && n = 4m, \quad m = 1,2,3,\ldots. \end{aligned}$$

**106.** (a) We multiply through by $h = f \cdot g$ and cancel as follows:

$$\begin{aligned} \frac{f'}{f} + \frac{g'}{g} &= \frac{h'}{h} \\ \left(\frac{f'}{f} + \frac{g'}{g}\right) \cdot fg &= \frac{h'}{h} \cdot fg \\ \frac{f'}{f} \cdot fg + \frac{g'}{g} \cdot fg &= \frac{h'}{h} \cdot h \\ f' \cdot g + g' \cdot f &= h', \end{aligned}$$

which is the product rule.

(b) We start with the product rule, multiply through by $1/(fg)$ and cancel as follows:

$$\begin{aligned} f' \cdot g + g' \cdot f &= h' \\ (f' \cdot g + g' \cdot f) \cdot \frac{1}{fg} &= h' \cdot \frac{1}{fg} \\ (f' \cdot g) \cdot \frac{1}{fg} + (g' \cdot f) \cdot \frac{1}{fg} &= h' \cdot \frac{1}{fg} \\ \frac{f'}{f} + \frac{g'}{g} &= \frac{h'}{h}, \end{aligned}$$

which is the additive rule shown in part (a).

**107.** This problem can be solved by using either the quotient rule or the fact that

$$\frac{f'}{f} = \frac{d}{dx}(\ln f) \quad \text{and} \quad \frac{g'}{g} = \frac{d}{dx}(\ln g).$$

We use the second method. The relative rate of change of $f/g$ is $(f/g)'/(f/g)$, so

$$\frac{(f/g)'}{f/g} = \frac{d}{dx} \ln\left(\frac{f}{g}\right) = \frac{d}{dx}(\ln f - \ln g) = \frac{d}{dx}(\ln f) - \frac{d}{dx}(\ln g) = \frac{f'}{f} - \frac{g'}{g}.$$

Thus, the relative rate of change of $f/g$ is the difference between the relative rates of change of $f$ and of $g$.

## CAS Challenge Problems

**108.** (a) Answers from different computer algebra systems may be in different forms. One form is:

$$\begin{aligned} \frac{d}{dx}(x+1)^x &= x(x+1)^{x-1} + (x+1)^x \ln(x+1) \\ \frac{d}{dx}(\sin x)^x &= x\cos x(\sin x)^{x-1} + (\sin x)^x \ln(\sin x) \end{aligned}$$

(b) Both the answers in part (a) follow the general rule:

$$\frac{d}{dx}f(x)^x = xf'(x)\,(f(x))^{x-1} + (f(x))^x \ln(f(x)).$$

(c) Applying this rule to $g(x)$, we get

$$\frac{d}{dx}(\ln x)^x = x(1/x)(\ln x)^{x-1} + (\ln x)^x \ln(\ln x) = (\ln x)^{x-1} + (\ln x)^x \ln(\ln x).$$

This agrees with the answer given by the computer algebra system.

(d) We can write $f(x) = e^{\ln(f(x))}$. So

$$(f(x))^x = (e^{\ln(f(x))})^x = e^{x\ln(f(x))}.$$

Therefore, using the chain rule and the product rule,

$$\begin{aligned}\frac{d}{dx}\,(f(x))^x &= \frac{d}{dx}\,(x\ln(f(x)))\cdot e^{x\ln(f(x))} = \left(\ln(f(x)) + x\frac{d}{dx}\ln(f(x))\right)e^{x\ln(f(x))}\\ &= \left(\ln(f(x)) + x\frac{f'(x)}{f(x)}\right)(f(x))^x = \ln(f(x))\,(f(x))^x + xf'(x)\,(f(x))^{x-1}\\ &= xf'(x)\,(f(x))^{x-1} + (f(x))^x\ln(f(x)).\end{aligned}$$

**109.** (a) A CAS gives $f'(x) = 1$.

(b) By the chain rule,

$$f'(x) = \cos(\arcsin x)\cdot\frac{1}{\sqrt{1-x^2}}.$$

Now $\cos t = \pm\sqrt{1-\sin^2 t}$. Furthermore, if $-\pi/2 \le t \le \pi/2$ then $\cos t \ge 0$, so we take the positive square root and get $\cos t = \sqrt{1-\sin^2 t}$. Since $-\pi/2 \le \arcsin x \le \pi/2$ for all $x$ in the domain of arcsin, we have

$$\cos(\arcsin x) = \sqrt{1-(\sin(\arcsin x))^2} = \sqrt{1-x^2},$$

so

$$\frac{d}{dx}\sin(\arcsin(x)) = \sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}} = 1.$$

(c) Since $\sin(\arcsin(x)) = x$, its derivative is 1.

**110.** (a) A CAS gives $g'(r) = 0$.

(b) Using the product rule,

$$\begin{aligned}g'(r) &= \frac{d}{dr}(2^{-2r})\cdot 4^r + 2^{-2r}\frac{d}{dr}(4^r) = -2\ln 2\cdot 2^{-2r}4^r + 2^{-2r}\ln 4\cdot 4^r\\ &= -\ln 4\cdot 2^{-2r}4^r + \ln 4\cdot 2^{-2r}4^r = (-\ln 4 + \ln 4)2^{-2r}4^r = 0\cdot 2^{-2r}4^r = 0.\end{aligned}$$

(c) By the laws of exponents, $4^r = (2^2)^r = 2^{2r}$, so $2^{-2r}4^r = 2^{-2r}2^{2r} = 2^0 = 1$. Therefore, its derivative is zero.

**111.** (a) A CAS gives $h'(t) = 0$

(b) By the chain rule

$$\begin{aligned}h'(t) &= \frac{\frac{d}{dt}\left(1-\frac{1}{t}\right)}{1-\frac{1}{t}} + \frac{\frac{d}{dt}\left(\frac{t}{t-1}\right)}{\frac{t}{t-1}} = \frac{\frac{1}{t^2}}{\frac{t-1}{t}} + \frac{\frac{1}{t-1} - \frac{t}{(t-1)^2}}{\frac{t}{t-1}}\\ &= \frac{1}{t^2-t} + \frac{(t-1)-t}{t^2-t} = \frac{1}{t^2-t} + \frac{-1}{t^2-t} = 0.\end{aligned}$$

(c) The expression inside the first logarithm is $1-(1/t) = (t-1)/t$. Using the property $\log A + \log B = \log(AB)$, we get

$$\begin{aligned}\ln\left(1-\frac{1}{t}\right) + \ln\left(\frac{t}{t-1}\right) &= \ln\left(\frac{t-1}{t}\right) + \ln\left(\frac{t}{t-1}\right)\\ &= \ln\left(\frac{t-1}{t}\cdot\frac{t}{1-t}\right) = \ln 1 = 0.\end{aligned}$$

Thus $h(t) = 0$, so $h'(t) = 0$ also.

## CHECK YOUR UNDERSTANDING

**1.** True. Since $d(x^n)/dx = nx^{n-1}$, the derivative of a power function is a power function, so the derivative of a polynomial is a polynomial.

**2.** False, since
$$\frac{d}{dx}\left(\frac{\pi}{x^2}\right) = \frac{d}{dx}\left(\pi x^{-2}\right) = -2\pi x^{-3} = \frac{-2\pi}{x^3}.$$

**3.** True, since $\cos\theta$ and therefore $\cos^2\theta$ are periodic, and
$$\frac{d}{d\theta}(\tan\theta) = \frac{1}{\cos^2\theta}.$$

**4.** False. Since
$$\frac{d}{dx}\ln(x^2) = \frac{1}{x^2}\cdot 2x = \frac{2}{x} \quad \text{and} \quad \frac{d^2}{dx^2}\ln(x^2) = \frac{d}{dx}\left(\frac{2}{x}\right) = -\frac{2}{x^2},$$
we see that the second derivative of $\ln(x^2)$ is negative for $x > 0$. Thus, the graph is concave down.

**5.** True. Since $f'(x)$ is the limit
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$
the function $f$ must be defined for all $x$.

**6.** True. The slope of $f(x) + g(x)$ at $x = 2$ is the sum of the derivatives, $f'(2) + g'(2) = 3.1 + 7.3 = 10.4$.

**7.** False. The product rule gives
$$(fg)' = fg' + f'g.$$
Differentiating this and using the product rule again, we get
$$(fg)'' = f'g' + fg'' + f'g' + f''g = fg'' + 2f'g' + f''g.$$
Thus, the right hand side is not equal to $fg'' + f''g$ in general.

**8.** True. If $f(x)$ is periodic with period $c$, then $f(x+c) = f(x)$ for all $x$. By the definition of the derivative, we have
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
and
$$f'(x+c) = \lim_{h\to 0}\frac{f(x+c+h)-f(x+c)}{h}.$$
Since $f$ is periodic, for any $h \neq 0$, we have
$$\frac{f(x+h)-f(x)}{h} = \frac{f(x+c+h)-f(x+c)}{h}.$$
Taking the limit as $h \to 0$, we get that $f'(x) = f'(x+c)$, so $f'$ is periodic with the same period as $f(x)$.

**9.** True; differentiating the equation with respect to $x$, we get
$$2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0.$$
Solving for $dy/dx$, we get that
$$\frac{dy}{dx} = \frac{-y}{2y+x}.$$
Thus $dy/dx$ exists where $2y + x \neq 0$. Now if $2y + x = 0$, then $x = -2y$. Substituting for $x$ in the original equation, $y^2 + xy - 1 = 0$, we get
$$y^2 - 2y^2 - 1 = 0.$$
This simplifies to $y^2 + 1 = 0$, which has no solutions. Thus $dy/dx$ exists everywhere.

**10.** True. We have $\tanh x = (\sinh x)/\cosh x = (e^x - e^{-x})/(e^x + e^{-x})$. Replacing $x$ by $-x$ in this expression gives $(e^{-x} - e^x)/(e^{-x} + e^x) = -\tanh x$.

**11.** False. The second, fourth and all even derivatives of $\sinh x$ are all $\sinh x$.

**12.** True. The definitions of $\sinh x$ and $\cosh x$ give

$$\sinh x + \cosh x = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = \frac{2e^x}{2} = e^x.$$

**13.** False. Since $(\sinh x)' = \cosh x > 0$, the function $\sinh x$ is increasing everywhere so can never repeat any of its values.

**14.** False. Since $(\sinh^2 x)' = 2\sinh x \cosh x$ and $(2\sinh x \cosh x)' = 2\sinh^2 x + 2\cosh^2 x > 0$, the function $\sinh^2 x$ is concave up everywhere.

**15.** False. If $f(x) = |x|$, then $f(x)$ is not differentiable at $x = 0$ and $f'(x)$ does not exist at $x = 0$.

**16.** False. If $f(x) = \ln x$, then $f'(x) = 1/x$, which is decreasing for $x > 0$.

**17.** False; the fourth derivative of $\cos t + C$, where $C$ is any constant, is indeed $\cos t$. But any function of the form $\cos t + p(t)$, where $p(t)$ is a polynomial of degree less than or equal to 3, also has its fourth derivative equal to $\cos t$. So $\cos t + t^2$ will work.

**18.** False; For example, the inverse function of $f(x) = x^3$ is $x^{1/3}$, and the derivative of $x^{1/3}$ is $(1/3)x^{-2/3}$, which is not $1/f'(x) = 1/(3x^2)$.

**19.** False; for example, if both $f(x)$ and $g(x)$ are constant functions, such as $f(x) = 6, g(x) = 10$, then $(fg)'(x) = 0$, and $f'(x) = 0$ and $g'(x) = 0$.

**20.** True; looking at the statement from the other direction, if both $f(x)$ and $g(x)$ are differentiable at $x = 1$, then so is their quotient, $f(x)/g(x)$, as long as it is defined there, which requires that $g(1) \neq 0$. So the only way in which $f(x)/g(x)$ can be defined but not differentiable at $x = 1$ is if either $f(x)$ or $g(x)$, or both, is not differentiable there.

**21.** False; for example, if both $f$ and $g$ are constant functions, then the derivative of $f(g(x))$ is zero, as is the derivative of $f(x)$. Another example is $f(x) = 5x + 7$ and $g(x) = x + 2$.

**22.** True. Since $f''(x) > 0$ and $g''(x) > 0$ for all $x$, we have $f''(x) + g''(x) > 0$ for all $x$, which means that $f(x) + g(x)$ is concave up.

**23.** False. Let $f(x) = x^2$ and $g(x) = x^2 - 1$. Let $h(x) = f(x)g(x)$. Then $h''(x) = 12x^2 - 2$. Since $h''(0) < 0$, clearly $h$ is not concave up for all $x$.

**24.** False. Let $f(x) = 2x^2$ and $g(x) = x^2$. Then $f(x) - g(x) = x^2$, which is concave up for all $x$.

**25.** False. Let $f(x) = e^{-x}$ and $g(x) = x^2$. Let $h(x) = f(g(x)) = e^{-x^2}$. Then $h'(x) = -2xe^{-x^2}$ and $h''(x) = (-2 + 4x^2)e^{-x^2}$. Since $h''(0) < 0$, clearly $h$ is not concave up for all $x$.

**26.** (a) False. Only if $k = f'(a)$ is $L$ the local linearization of $f$.
(b) False. Since $f(a) = L(a)$ for any $k$, we have $\lim_{x\to a}(f(x) - L(x)) = f(a) - L(a) = 0$, but only if $k = f'(a)$ is $L$ the local linearization of $f$.

**27.** (a) This is not a counterexample. Although the product rule says that $(fg)' = f'g + fg'$, that does not rule out the possibility that also $(fg)' = f'g'$. In fact, if $f$ and $g$ are both constant functions, then both $f'g + fg'$ and $f'g'$ are zero, so they are equal to each other.
(b) This is not a counterexample. In fact, it agrees with the product rule:

$$\frac{d}{dx}(xf(x)) = \left(\frac{d}{dx}(x)\right) f(x) + x\frac{d}{dx}f(x) = f(x) + xf'(x) = xf'(x) + f(x).$$

(c) This is not a counterexample. Although the product rule says that

$$\frac{d}{dx}(f(x)^2) = \frac{d}{dx}f(x) \cdot f(x) = f'(x)f(x) + f(x)f'(x) = 2f(x)f'(x),$$

it could be true that $f'(x) = 1$, so that the derivative is also just $2f(x)$. In fact, $f(x) = x$ is an example where this happens.
(d) This would be a counterexample. If $f'(a) = g'(a) = 0$, then

$$\frac{d}{dx}(f(x)g(x))\Big|_{x=a} = f'(a)g(a) + f(a)g'(a) = 0.$$

So $fg$ cannot have positive slope at $x = a$. Of course such a counterexample could not exist, since the product rule is true.

**28.** True, by the Increasing Function Theorem, Theorem 3.8.

**29.** False. For example, let $f(x) = x + 5$, and $g(x) = 2x - 3$. Then $f'(x) \leq g'(x)$ for all $x$, but $f(0) > g(0)$.

**30.** False. For example, let $f(x) = 3x + 1$ and $g(x) = 3x + 7$.

**31.** False. For example, if $f(x) = -x$, then $f'(x) \leq 1$ for all $x$, but $f(-2) = 2$, so $f(-2) > -2$.

**32.** The function $f(x) = |x|$ is continuous on $[-1, 1]$, but there is no number $c$, with $-1 < c < 1$, such that

$$f'(c) = \frac{|1| - |-1|}{1 - (-1)} = 0;$$

that is, the slope of $f(x) = |x|$ is never 0.

**33.** Let $f$ be defined by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 2 \\ 19 & \text{if } x = 2 \end{cases}$$

Then $f$ is differentiable on $(0, 2)$ and $f'(x) = 1$ for all $x$ in $(0, 2)$. Thus there is no $c$ in $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{19}{2}.$$

The reason that this function does not satisfy the conclusion of the Mean Value Theorem is that it is not continuous at $x = 2$.

**34.** Let $f$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x < 1 \\ 1/2 & \text{if } x = 1. \end{cases}$$

Then $f$ is not continuous at $x = 1$, but $f$ is differentiable on $(0, 1)$ and $f'(x) = 2x$ for $0 < x < 1$. Thus, $c = 1/4$ satisfies

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{1}{2}, \quad \text{since} \quad f'\left(\frac{1}{4}\right) = 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

## PROJECTS FOR CHAPTER THREE

**1.** Let $r = i/100$. (For example if $i = 5\%$, $r = 0.05$.) Then the balance, $\$B$, after $t$ years is given by

$$B = P(1 + r)^t,$$

where $\$P$ is the original deposit. If we are doubling our money, then $B = 2P$, so we wish to solve for $t$ in the equation $2P = P(1 + r)^t$. This is equivalent to

$$2 = (1 + r)^t.$$

Taking natural logarithms of both sides and solving for $t$ yields

$$\ln 2 = t \ln(1 + r),$$
$$t = \frac{\ln 2}{\ln(1 + r)}.$$

We now approximate $\ln(1 + r)$ near $r = 0$. Let $f(r) = \ln(1 + r)$. Then $f'(r) = 1/(1 + r)$. Thus, $f(0) = 0$ and $f'(0) = 1$, so

$$f(r) \approx f(0) + f'(0)r$$

becomes

$$\ln(1 + r) \approx r.$$

Therefore,

$$t = \frac{\ln 2}{\ln(1 + r)} \approx \frac{\ln 2}{r} = \frac{100 \ln 2}{i} \approx \frac{70}{i},$$

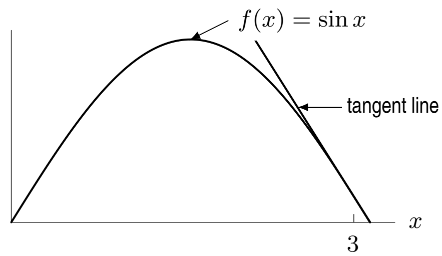
as claimed. We expect this approximation to hold for small values of $i$; it turns out that values of $i$ up to 10 give good enough answers for most everyday purposes.

**2. (a)** (i) Set $f(x) = \sin x$, so $f'(x) = \cos x$. Guess $x_0 = 3$. Then

$$x_1 = 3 - \frac{\sin 3}{\cos 3} \approx 3.1425$$
$$x_2 = x_1 - \frac{\sin x_1}{\cos x_1} \approx 3.1415926533,$$

which is correct to one billionth!

(ii) Newton's method uses the tangent line at $x = 3$, i.e. $y - \sin 3 = \cos(3)(x - 3)$. Around $x = 3$, however, $\sin x$ is almost linear, since the second derivative $\sin''(\pi) = 0$. Thus using the tangent line to get an approximate value for the root gives us a very good approximation.




(iii) For $f(x) = \sin x$, we have

$$f(3) = 0.14112$$
$$f(4) = -0.7568,$$

so there is a root in $[3, 4]$. We now continue bisecting:

$$[3, 3.5] : f(3.5) = -0.35078 \text{ (bisection 1)}$$
$$[3, 3.25] : f(3.25) = -0.10819 \text{ (bisection 2)}$$
$$[3.125, 3.25] : f(3.125) = \quad 0.01659 \text{ (bisection 3)}$$
$$[3.125, 3.1875] : f(3.1875) = -0.04584 \text{ (bisection 4)}$$

We continue this process; after 11 bisections, we know the root lies between 3.1411 and 3.1416, which still is not as good an approximation as what we get from Newton's method in just two steps.

**(b)** (i) We have $f(x) = \sin x - \frac{2}{3}x$ and $f'(x) = \cos x - \frac{2}{3}$.
Using $x_0 = 0.904$,

$$x_1 = 0.904 - \frac{\sin(0.904) - \frac{2}{3}(0.904)}{\cos(0.904) - \frac{2}{3}} \approx 4.704,$$

$$x_2 = 4.704 - \frac{\sin(4.704) - \frac{2}{3}(4.704)}{\cos(4.704) - \frac{2}{3}} \approx -1.423,$$

$$x_3 = -1.433 - \frac{\sin(-1.423) - \frac{2}{3}(-1.423)}{\cos(-1.423) - \frac{2}{3}} \approx -1.501,$$

$$x_4 = -1.499 - \frac{\sin(-1.501) - \frac{2}{3}(-1.501)}{\cos(-1.501) - \frac{2}{3}} \approx -1.496,$$

$$x_5 = -1.496 - \frac{\sin(-1.496) - \frac{2}{3}(-1.496)}{\cos(-1.496) - \frac{2}{3}} \approx -1.496.$$

Using $x_0 = 0.905$,

$$x_1 = 0.905 - \frac{\sin(0.905) - \frac{2}{3}(0.905)}{\cos(0.905) - \frac{2}{3}} \approx 4.643,$$

$$x_2 = 4.643 - \frac{\sin(4.643) - \frac{2}{3}(4.643)}{\cos(4.643) - \frac{2}{3}} \approx -0.918,$$

$$x_3 = -0.918 - \frac{\sin(-0.918) - \frac{2}{3}(-0.918)}{\cos(-0.918) - \frac{2}{3}} \approx -3.996,$$

$$x_4 = -3.996 - \frac{\sin(-3.996) - \frac{2}{3}(-3.996)}{\cos(-3.996) - \frac{2}{3}} \approx -1.413,$$

$$x_5 = -1.413 - \frac{\sin(-1.413) - \frac{2}{3}(-1.413)}{\cos(-1.413) - \frac{2}{3}} \approx -1.502,$$

$$x_6 = -1.502 - \frac{\sin(-1.502) - \frac{2}{3}(-1.502)}{\cos(-1.502) - \frac{2}{3}} \approx -1.496.$$

Now using $x_0 = 0.906$,

$$x_1 = 0.906 - \frac{\sin(0.906) - \frac{2}{3}(0.906)}{\cos(0.906) - \frac{2}{3}} \approx 4.584,$$

$$x_2 = 4.584 - \frac{\sin(4.584) - \frac{2}{3}(4.584)}{\cos(4.584) - \frac{2}{3}} \approx -0.509,$$

$$x_3 = -0.510 - \frac{\sin(-0.509) - \frac{2}{3}(-0.509)}{\cos(-0.509) - \frac{2}{3}} \approx .207,$$

$$x_4 = -1.300 - \frac{\sin(.207) - \frac{2}{3}(.207)}{\cos(.207) - \frac{2}{3}} \approx -0.009,$$

$$x_5 = -1.543 - \frac{\sin(-0.009) - \frac{2}{3}(-0.009)}{\cos(-0.009) - \frac{2}{3}} \approx 0,$$

(ii) Starting with 0.904 and 0.905 yields the same value, but the two paths to get to the root are very different. Starting with 0.906 leads to a different root. Our starting points were near the maximum value of $f$. Consequently, a small change in $x_0$ makes a large change in $x_1$.