CHAPTER EIGHT

Solutions for Section 8.1

Exercises

1. Each strip is a rectangle of length 3 and width Δx, so

$$\begin{aligned}\text{Area of strip } &= 3\Delta x, \quad \text{so}\\ \text{Area of region } &= \int_0^5 3\,dx = 3x\bigg|_0^5 = 15.\end{aligned}$$

Check: This area can also be computed using Length $\times$ Width $= 5\cdot 3 = 15$.

2. Using similar triangles, the height, y, of the strip is given by

$$\frac{y}{3} = \frac{x}{6} \quad \text{so} \quad y = \frac{x}{2}.$$

Thus,

$$\text{Area of strip } \approx y\Delta x = \frac{x}{2}\Delta x,$$

so

$$\text{Area of region } = \int_0^6 \frac{x}{2}\,dx = \frac{x^2}{4}\bigg|_0^6 = 9.$$

Check: This area can also be computed using the formula $\frac{1}{2}$ Base $\cdot$ Height $= \frac{1}{2}\cdot 6\cdot 3 = 9$.

3. By similar triangles, if w is the length of the strip at height h, we have

$$\frac{w}{3} = \frac{5-h}{5} \quad \text{so} \quad w = 3\left(1-\frac{h}{5}\right).$$

Thus,

$$\begin{aligned}\text{Area of strip } &\approx w\Delta h = 3\left(1-\frac{h}{5}\right)\Delta h.\\ \text{Area of region } &= \int_0^5 3\left(1-\frac{h}{5}\right)dh = \left(3h - \frac{3h^2}{10}\right)\bigg|_0^5 = \frac{15}{2}.\end{aligned}$$

Check: This area can also be computed using the formula $\frac{1}{2}$ Base $\cdot$ Height $= \frac{1}{2}\cdot 3\cdot 5 = \frac{15}{2}$.

4. Suppose the length of the strip shown is w. Then the Pythagorean theorem gives

$$h^2 + \left(\frac{w}{2}\right)^2 = 3^2 \quad \text{so} \quad w = 2\sqrt{3^2-h^2}.$$

Thus

$$\begin{aligned}\text{Area of strip } &\approx w\Delta h = 2\sqrt{3^2-h^2}\,\Delta h,\\ \text{Area of region } &= \int_{-3}^3 2\sqrt{3^2-h^2}\,dh.\end{aligned}$$

Using VI-30 in the Table of Integrals, we have

$$\text{Area } = \left(h\sqrt{3^2-h^2} + 3^2\arcsin\left(\frac{h}{3}\right)\right)\bigg|_{-3}^3 = 9(\arcsin 1 - \arcsin(-1)) = 9\pi.$$

Check: This area can also be computed using the formula $\pi r^2 = 9\pi$.

5. The strip has width Δy, so the variable of integration is y. The length of the strip is x. Since $x^2 + y^2 = 10$ and the region is in the first quadrant, solving for x gives $x = \sqrt{10 - y^2}$. Thus

$$\text{Area of strip } \approx x\Delta y = \sqrt{10 - y^2}\, dy.$$

The region stretches from $y = 0$ to $y = \sqrt{10}$, so

$$\text{Area of region } = \int_0^{\sqrt{10}} \sqrt{10 - y^2}\, dy.$$

Evaluating using VI-30 from the Table of Integrals, we have

$$\text{Area } = \frac{1}{2}\left(y\sqrt{10 - y^2} + 10\arcsin\left(\frac{y}{\sqrt{10}}\right)\right)\Bigg|_0^{\sqrt{10}} = 5(\arcsin 1 - \arcsin 0) = \frac{5}{2}\pi.$$

Check: This area can also be computed using the formula $\frac{1}{4}\pi r^2 = \frac{1}{4}\pi(\sqrt{10})^2 = \frac{5}{2}\pi$.

6. The strip has width Δy, so the variable of integration is y. The length of the strip is $2x$ for $x \geq 0$. For positive x, we have $x = y$. Thus,

$$\text{Area of strip } \approx 2x\Delta y = 2y\Delta y.$$

Since the region extends from $y = 0$ to $y = 4$,

$$\text{Area of region } = \int_0^4 2y\, dy = y^2\Big|_0^4 = 16.$$

Check: The area of the region can be computed by $\frac{1}{2}$ Base $\cdot$ Height $= \frac{1}{2} \cdot 8 \cdot 4 = 16$.

7. The width of the strip is Δy, so the variable of integration is y. Since the graphs are $x = y$ and $x = y^2$, the length of the strip is $y - y^2$, and

$$\text{Area of strip } \approx (y - y^2)\Delta y.$$

The curves cross at the points $(0, 0)$ and $(1, 1)$, so

$$\text{Area of region } = \int_0^1 (y - y^2)\, dy = \frac{y^2}{2} - \frac{y^3}{3}\Bigg|_0^1 = \frac{1}{6}.$$

8. The width of the strip is Δx, so the variable of integration is x. The line has equation $y = 6 - 3x$. The length of the strip is $6 - 3x - (x^2 - 4) = 10 - 3x - x^2$. (Since $x^2 - 4$ is negative where the graph is below the x-axis, subtracting $x^2 - 4$ there adds the length below the x-axis.) Thus

$$\text{Area of strip } \approx (10 - 3x - x^2)\Delta x.$$

Both graphs cross the x-axis where $x = 2$, so

$$\text{Area of region } = \int_0^2 (10 - 3x - x^2)\, dx = 10x - \frac{3}{2}x^2 - \frac{x^3}{3}\Bigg|_0^2 = \frac{34}{3}.$$

9. Each slice is a circular disk with radius $r = 2$ cm.

$$\text{Volume of disk } = \pi r^2\Delta x = 4\pi\Delta x \text{ cm}^3.$$

Summing over all disks, we have

$$\text{Total volume } \approx \sum 4\pi\Delta x \text{ cm}^3.$$

Taking a limit as $\Delta x \to 0$, we get

$$\text{Total volume } = \lim_{\Delta x \to 0} \sum 4\pi\Delta x = \int_0^9 4\pi\, dx \text{ cm}^3.$$

Evaluating gives

$$\text{Total volume } = 4\pi x\Big|_0^9 = 36\pi \text{ cm}^3.$$

Check: The volume of the cylinder can also be calculated using the formula $V = \pi r^2 h = \pi 2^2 \cdot 9 = 36\pi$ cm^3.

10. Each slice is a circular disk. Since the radius of the cone is 2 cm and the length is 6 cm, the radius is one-third of the distance from the vertex. Thus, the radius at x is $r = x/3$ cm. See Figure 8.1.

$$\text{Volume of slice } \approx \pi r^2 \Delta x = \frac{\pi x^2}{9} \Delta x \text{ cm}^3.$$

Summing over all disks, we have

$$\text{Total volume } \approx \sum \pi \frac{x^2}{9} \Delta x \text{ cm}^3.$$

Taking a limit as $\Delta x \to 0$, we get

$$\text{Total volume } = \lim_{\Delta x \to 0} \sum \pi \frac{x^2}{9} \Delta x = \int_0^6 \pi \frac{x^2}{9}\, dx \text{ cm}^3.$$

Evaluating, we get

$$\text{Total volume } = \frac{\pi}{9} \frac{x^3}{3} \bigg|_0^6 = \frac{\pi}{9} \cdot \frac{6^3}{3} = 8\pi \text{ cm}^3.$$

Check: The volume of the cone can also be calculated using the formula $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi 2^2 \cdot 6 = 8\pi$ cm^3.

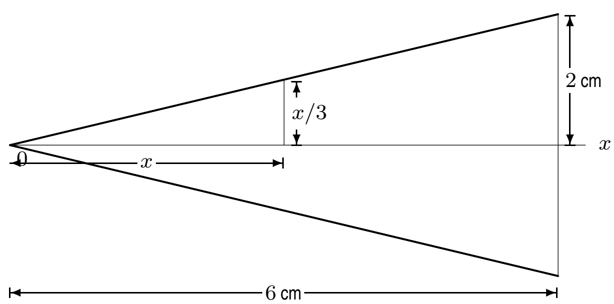


Figure 8.1

11. Each slice is a circular disk. From Figure 8.2, we see that the radius at height y is $r = \frac{2}{5}y$ cm. Thus

$$\text{Volume of disk } \approx \pi r^2 \Delta y = \pi \left(\frac{2}{5}y\right)^2 \Delta y = \frac{4}{25}\pi y^2 \Delta y \text{ cm}^3.$$

Summing over all disks, we have

$$\text{Total volume } \approx \sum \frac{4\pi}{25} y^2 \Delta y \text{ cm}^3.$$

Taking the limit as $\Delta y \to 0$, we get

$$\text{Total volume } = \lim_{\Delta y \to 0} \sum \frac{4\pi}{25} y^2 \Delta y = \int_0^5 \frac{4\pi}{25} y^2 dy \text{ cm}^3.$$

Evaluating gives

$$\text{Total volume } = \frac{4\pi}{25} \frac{y^3}{3} \bigg|_0^5 = \frac{20}{3}\pi \text{ cm}^3.$$

Check: The volume of the cone can also be calculated using the formula $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3} 2^2 \cdot 5 = \frac{20}{3}\pi$ cm^3.

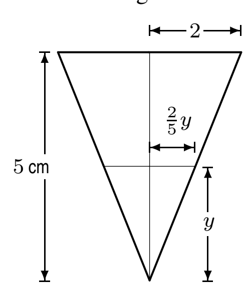


Figure 8.2

12. Each slice is a rectangular slab of length 10 m and width that decreases with height. See Figure 8.3. At height y, the length x is given by the Pythagorean Theorem
$$y^2 + x^2 = 7^2.$$
Solving gives $x = \sqrt{7^2 - y^2}$ m. Thus the width of the slab is $2x = 2\sqrt{7^2 - y^2}$ and
$$\text{Volume of slab} = \text{Length} \cdot \text{Width} \cdot \text{Height} = 10 \cdot 2\sqrt{7^2 - y^2} \cdot \Delta y = 20\sqrt{7^2 - y^2}\Delta y \text{ m}^3.$$
Summing over all slabs, we have
$$\text{Total volume} \approx \sum 20\sqrt{7^2 - y^2}\Delta y \text{ m}^3.$$
Taking a limit as $\Delta y \to 0$, we get
$$\text{Total volume} = \lim_{\Delta y \to 0} \sum 20\sqrt{7^2 - y^2}\Delta y = \int_0^7 20\sqrt{7^2 - y^2}\, dy \text{ m}^3.$$
To evaluate, we use the table of integrals or the fact that $\int_0^7 \sqrt{7^2 - y^2}\, dy$ represents the area of a quarter circle of radius 7, so
$$\text{Total volume} = \int_0^7 20\sqrt{7^2 - y^2}\, dy = 20 \cdot \frac{1}{4}\pi 7^2 = 245\pi \text{ m}^3.$$
Check: the volume of a half cylinder can also be calculated using the formula $V = \frac{1}{2}\pi r^2 h = \frac{1}{2}\pi 7^2 \cdot 10 = 245\pi \text{ m}^3$.

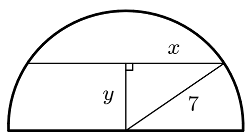


Figure 8.3

13. Each slice is a circular disk. See Figure 8.4. The radius of the sphere is 5 mm, and the radius r at height y is given by the Pythagorean Theorem
$$y^2 + r^2 = 5^2.$$
Solving gives $r = \sqrt{5^2 - y^2}$ mm. Thus,
$$\text{Volume of disk} \approx \pi r^2 \Delta y = \pi(5^2 - y^2)\Delta y \text{ mm}^3.$$
Summing over all disks, we have
$$\text{Total volume} \approx \sum \pi(5^2 - y^2)\Delta y \text{ mm}^3.$$
Taking the limit as $\Delta y \to 0$, we get
$$\text{Total volume} = \lim_{\Delta y \to 0} \sum \pi(5^2 - y^2)\Delta y = \int_0^5 \pi(5^2 - y^2)\, dy \text{ mm}^3.$$
Evaluating gives
$$\text{Total volume} = \pi \left(25y - \frac{y^3}{3}\right)\Bigg|_0^5 = \frac{250}{3}\pi \text{ mm}^3.$$
Check: The volume of a hemisphere can be calculated using the formula $V = \frac{2}{3}\pi r^3 = \frac{2}{3}\pi 5^3 = \frac{250}{3}\pi \text{ mm}^3$.

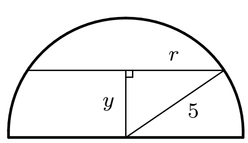


Figure 8.4

14. Each slice is a square; the side length decreases as we go up the pyramid. See Figure 8.5. Since the base of the pyramid is equal to its vertical height, the slice at distance y from the base, or $(2-y)$ from the top, has side $(2-y)$. Thus

$$\text{Volume of slice } \approx (2-y)^2 \Delta y \text{ m}^3.$$

Summing over all slices, we get

$$\text{Total volume } \approx \sum (2-y)^2 \Delta y \text{ m}^3.$$

$$\text{Total volume } = \lim_{\Delta y \to 0} \sum (2-y)^2 \Delta y = \int_0^2 (2-y)^2 dy \text{ m}^3.$$

Evaluating, we find

$$\text{Total volume } = \int_0^2 (4 - 4y + y^2)\, dy = \left(4y - 2y^2 + \frac{y^3}{3} \right) \Bigg|_0^2 = \frac{8}{3} \text{ m}^3.$$

Check: The volume of the pyramid can also be calculated using the formula $V = \frac{1}{3}b^2h = \frac{1}{3}2^2 \cdot 2 = \frac{8}{3}$ m^3.

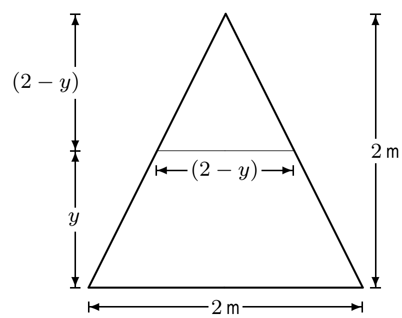


Figure 8.5

Problems

15. Triangle of base and height 1 and 3. See Figure 8.6. (Either 1 or 3 can be the base. A non-right triangle is also possible.)

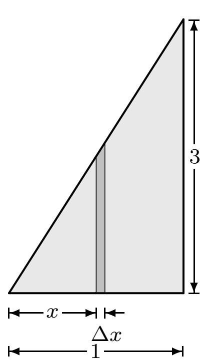


Figure 8.6

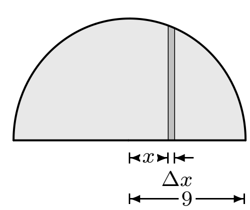


Figure 8.7

16. Semicircle of radius $r = 9$. See Figure 8.7.

17. Quarter circle of radius $r = \sqrt{15}$. See Figure 8.8.

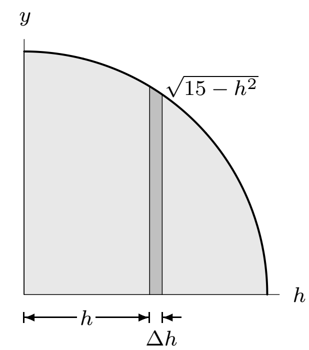


Figure 8.8

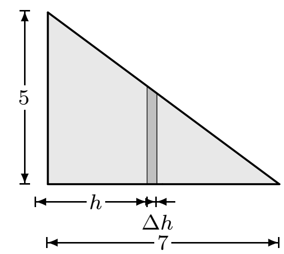


Figure 8.9

18. Triangle of base and height 7 and 5. See Figure 8.9. (Either 7 or 5 can be the base. A non-right triangle is also possible.)

19. Hemisphere with radius 12. See Figure 8.10.

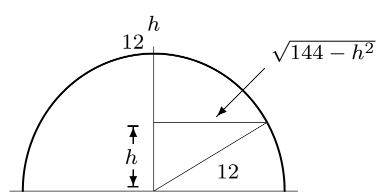


Figure 8.10

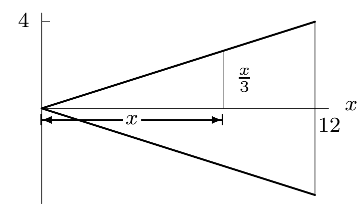


Figure 8.11

20. Cone with height 12 and radius $12/3 = 4$. See Figure 8.11.

21. Cone with height 6 and radius 3. See Figure 8.12.

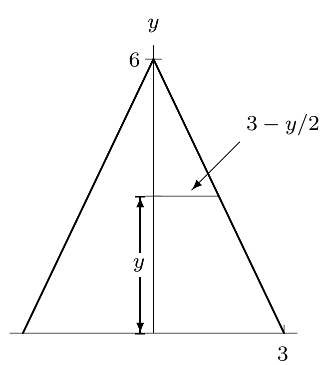


Figure 8.12

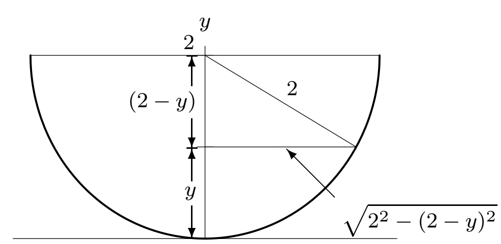


Figure 8.13

22. Hemisphere with radius 2. See Figure 8.13.

23.

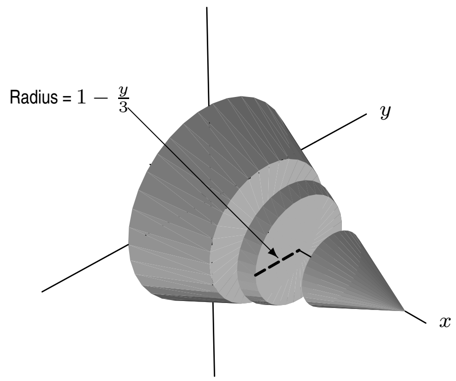
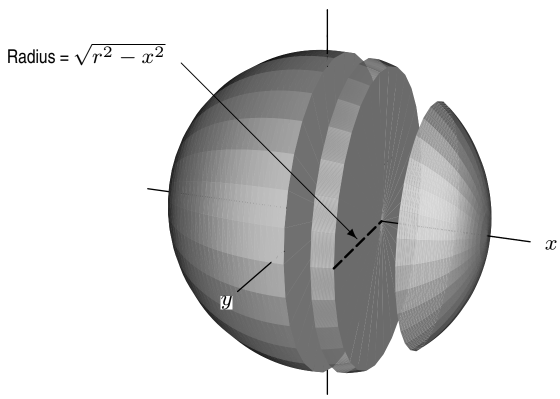


Figure 8.14

Slice parallel to the base of the cone, or, equivalently, rotate the line $x = (3-y)/3$ about the y–axis. (One can also slice the other way.) See Figure 8.14. The volume V is given by

$$\begin{aligned} V &= \int_{y=0}^{y=3} \pi x^2\, dy = \int_0^3 \pi \left(\frac{3-y}{3}\right)^2 dy \\ &= \pi \int_0^3 \left(1 - \frac{2y}{3} + \frac{y^2}{9}\right) dy \\ &= \pi \left(y - \frac{y^2}{3} + \frac{y^3}{27}\right)\Bigg|_0^3 = \pi. \end{aligned}$$

24.



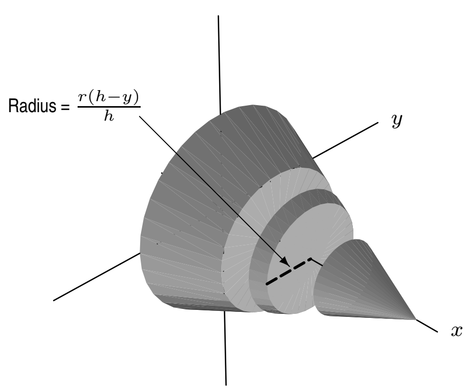
We slice up the sphere in planes perpendicular to the x-axis. Each slice is a circle, with radius $y = \sqrt{r^2 - x^2}$; that's the radius because $x^2 + y^2 = r^2$ when $z = 0$. Then the volume is

$$V \approx \sum \pi(y^2)\,\Delta x = \sum \pi(r^2 - x^2)\,\Delta x.$$

Therefore, as Δx tends to zero, we get

$$\begin{aligned} V &= \int_{x=-r}^{x=r} \pi(r^2 - x^2)\, dx \\ &= 2\int_{x=0}^{x=r} \pi(r^2 - x^2)\, dx \\ &= 2\left(\pi r^2 x - \frac{\pi x^3}{3}\right)\Bigg|_0^r \\ &= \frac{4\pi r^3}{3}. \end{aligned}$$

25.



This cone is what you get when you rotate the line $x = r(h-y)/h$ about the y–axis. So slicing perpendicular to the y–axis yields

$$\begin{aligned} V &= \int_{y=0}^{y=h} \pi x^2\, dy = \pi \int_0^h \left(\frac{(h-y)r}{h}\right)^2 dy \\ &= \pi \frac{r^2}{h^2} \int_0^h (h^2 - 2hy + y^2)\, dy \\ &= \frac{\pi r^2}{h^2}\left[h^2 y - hy^2 + \frac{y^3}{3}\right]\Bigg|_0^h = \frac{\pi r^2 h}{3}. \end{aligned}$$

26. (a) A vertical slice has a triangular shape and thickness Δx. See Figure 8.15.

$$\text{Volume of slice } = \text{ Area of triangle } \cdot \Delta x = \frac{1}{2} \text{ Base } \cdot \text{ Height } \cdot \Delta x = \frac{1}{2} \cdot 2 \cdot 3\Delta x = 3\Delta x \text{ cm}^3.$$

Thus,

$$\text{Total volume } = \lim_{\Delta x \to 0} \sum 3\Delta x = \int_0^4 3\, dx = 3x\Big|_0^4 = 12 \text{ cm}^3.$$

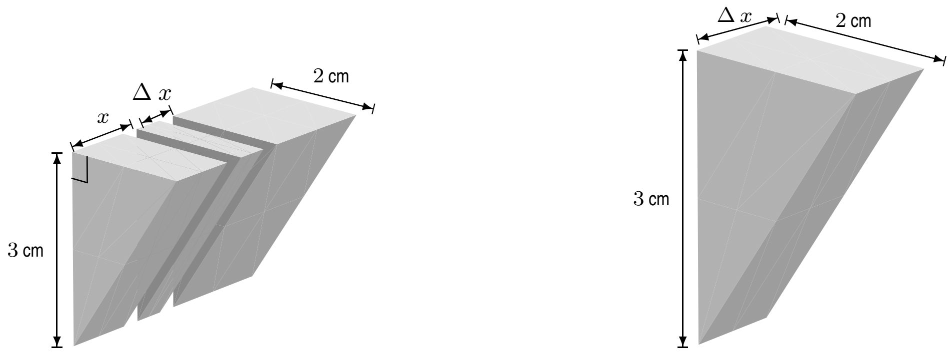


Figure 8.15

(b) A horizontal slice has a rectangular shape and thickness Δh. See Figure 8.16. Using similar triangles, we see that

$$\frac{w}{2} = \frac{3-h}{3},$$

so

$$w = \frac{2}{3}(3-h) = 2 - \frac{2}{3}h.$$

Thus

$$\text{Volume of slice } \approx 4w\Delta h = 4\left(2 - \frac{2}{3}h\right)\Delta h = \left(8 - \frac{8}{3}h\right)\Delta h.$$

So,

$$\text{Total volume } = \lim_{\Delta h \to 0} \sum \left(8 - \frac{8}{3}h\right)\Delta h = \int_0^3 \left(8 - \frac{8}{3}h\right) dh = \left(8h - \frac{4h^2}{3}\right)\Bigg|_0^3 = 12 \text{ cm}^3.$$

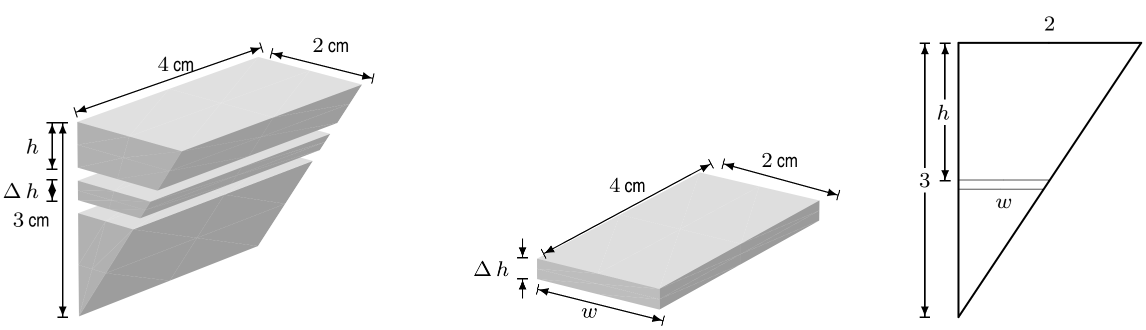


Figure 8.16

27. We slice the water into horizontal slices, each of which is a rectangle. See Figure 8.17.

$$\text{Volume of slice } \approx 150w\Delta h \text{ km}^3.$$

To find w in terms of h, we use the similar triangles in Figure 8.18:

$$\frac{w}{3} = \frac{h}{0.2} \quad \text{so} \quad w = 15h.$$

So

$$\text{Volume of slice } \approx 150 \cdot 15h\Delta h = 2250h\Delta h \text{ km}^3.$$

Summing over all slices and letting $\Delta h \to 0$ gives

$$\text{Total volume } = \lim_{\Delta h \to 0} \sum 2250h\Delta h = \int_0^{0.2} 2250h \, dh \text{ km}^3.$$

Evaluating the integral gives

$$\text{Total volume } = 2250\frac{h^2}{2}\bigg|_0^{0.2} = 45 \text{ km}^3.$$

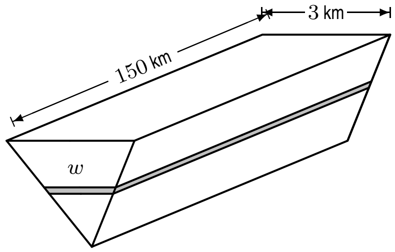


Figure 8.17

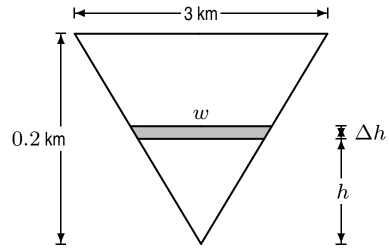


Figure 8.18

28. To calculate the volume of material, we slice the dam horizontally. See Figure 8.19. The slices are rectangular, so

$$\text{Volume of slice } \approx 1400w\Delta h \text{ m}^3.$$

Since w is a linear function of h, and $w = 160$ when $h = 0$, and $w = 10$ when $h = 150$, this function has slope $=$ $(10 - 160)/150 = -1$. Thus

$$w = 160 - h \text{ meters},$$

so

$$\text{Volume of slice } \approx 1400(160 - h)\Delta h \text{ m}^3.$$

Summing over all slices and taking the limit as $\Delta h \to 0$ gives

$$\text{Total volume } = \lim_{\Delta h \to 0} \sum 1400(160 - h)\Delta h = \int_0^{150} 1400(160 - h)\, dh \text{ m}^3.$$

Evaluating the integral gives

$$\text{Total volume } = 1400\left(160h - \frac{h^2}{2}\right)\bigg|_0^{150} = 1.785 \cdot 10^7 \text{ m}^3.$$

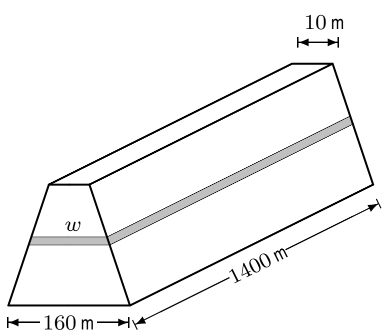


Figure 8.19

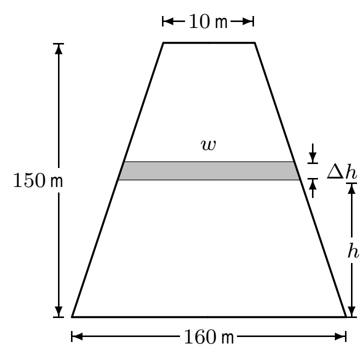


Figure 8.20

Solutions for Section 8.2

Exercises

1. The volume is given by

$$V = \int_0^1 \pi y^2 dx = \int_0^1 \pi x^4 dx = \pi\frac{x^5}{5}\bigg|_0^1 = \frac{\pi}{5}.$$

2. The volume is given by
$$V = \int_1^2 \pi y^2\,dx = \int_1^2 \pi(x+1)^4\,dx = \frac{\pi(x+1)^5}{5}\bigg|_1^2 = \frac{211\pi}{5}.$$

3. The volume is given by
$$V = \int_{-2}^0 \pi(4-x^2)^2\,dx = \pi\int_{-2}^0 (16-8x^2+x^4)\,dx = \pi\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\bigg|_{-2}^0 = \frac{256\pi}{15}.$$

4. The volume is given by
$$V = \int_{-1}^1 \pi(\sqrt{x+1})^2\,dx = \pi\int_{-1}^1 (x+1)\,dx = \pi\left(\frac{x^2}{2}+x\right)\bigg|_{-1}^1 = 2\pi.$$

5. The volume is given by
$$V = \int_{-1}^1 \pi y^2\,dx = \int_{-1}^1 \pi(e^x)^2\,dx = \int_{-1}^1 \pi e^{2x}\,dx = \frac{\pi}{2}e^{2x}\bigg|_{-1}^1 = \frac{\pi}{2}(e^2 - e^{-2}).$$

6. The volume is given by
$$V = \int_0^{\pi/2} \pi y^2\,dx = \int_0^{\pi/2} \pi\cos^2 x\,dx.$$
Integration by parts gives
$$V = \frac{\pi}{2}(\cos x\sin x + x)\bigg|_0^{\pi/2} = \frac{\pi^2}{4}.$$

7. The volume is given by
$$V = \int_0^1 \pi\left(\frac{1}{x+1}\right)^2\,dx = \pi\int_0^1 \frac{dx}{(x+1)^2} = -\pi(x+1)^{-1}\bigg|_0^1 = \pi\left(1-\frac{1}{2}\right) = \frac{\pi}{2}.$$

8. The volume is given by
$$V = \pi\int_0^1 (\sqrt{\cosh 2x})^2\,dx = \pi\int_0^1 \cosh 2x\,dx = \frac{\pi}{2}\sinh 2x\bigg|_0^1 = \frac{\pi}{2}\sinh 2.$$

9. Since the graph of $y = x^2$ is below the graph of $y = x$ for $0 \le x \le 1$, the volume is given by
$$V = \int_0^1 \pi x^2\,dx - \int_0^1 \pi(x^2)^2\,dx = \pi\int_0^1 (x^2 - x^4)\,dx = \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\bigg|_0^1 = \frac{2\pi}{15}.$$

10. Since the graph of $y = e^{3x}$ is above the graph of $y = e^x$ for $0 \le x \le 1$, the volume is given by
$$V = \int_0^1 \pi(e^{3x})^2\,dx - \int_0^1 \pi(e^x)^2\,dx = \int_0^1 \pi(e^{6x} - e^{2x})\,dx = \pi\left(\frac{e^{6x}}{6} - \frac{e^{2x}}{2}\right)\bigg|_0^1 = \pi\left(\frac{e^6}{6} - \frac{e^2}{2} + \frac{1}{3}\right).$$

11. Note that this function is actually $x^{3/2}$ in disguise. So
$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \left[\frac{3}{2}x^{\frac{1}{2}}\right]^2}\,dx = \int_{x=0}^{x=2} \sqrt{1+\frac{9}{4}x}\,dx \\ &= \frac{4}{9}\int_{w=1}^{w=\frac{11}{2}} w^{\frac{1}{2}}\,dw \\ &= \frac{8}{27}w^{\frac{3}{2}}\bigg|_1^{\frac{11}{2}} = \frac{8}{27}\left(\left(\frac{11}{2}\right)^{\frac{3}{2}} - 1\right) \approx 3.526, \end{aligned}$$
where we set $w = 1 + \frac{9}{4}x$, so $dx = \frac{4}{9}dw$.

12. This is a one-quarter of the circumference of a circle of radius 2. That circumference is $2\cdot 2\pi = 4\pi$, so the length is $\frac{4\pi}{4} = \pi$.

13. Since $f'(x) = \sinh x$, the arc length is given by

$$L = \int_0^2 \sqrt{1+\sinh^2 x}\, dx = \int_0^2 \sqrt{\cosh^2 x}\, dx = \int_0^2 \cosh x\, dx = \sinh x \Big|_0^2 = \sinh 2.$$

14. The length is

$$\int_1^2 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt = \int_1^2 \sqrt{5^2 + 4^2 + (-1)^2}\, dt = \sqrt{42}.$$

This is the length of a straight line from the point $(8, 5, 2)$ to $(13, 9, 1)$.

15. We have

$$\begin{aligned} D &= \int_0^1 \sqrt{(-e^t \sin(e^t))^2 + (e^t \cos(e^t))^2}\, dt \\ &= \int_0^1 \sqrt{e^{2t}}\, dt = \int_0^1 e^t\, dt \\ &= e - 1. \end{aligned}$$

This is the length of the arc of a unit circle from the point $(\cos 1, \sin 1)$ to $(\cos e, \sin e)$—in other words between the angles $\theta = 1$ and $\theta = e$. The length of this arc is $(e - 1)$.

16. We have

$$D = \int_0^{2\pi} \sqrt{(-3\sin 3t)^2 + (5\cos 5t)^2}\, dt.$$

We cannot find this integral symbolically, but numerical methods show $D \approx 24.6$.

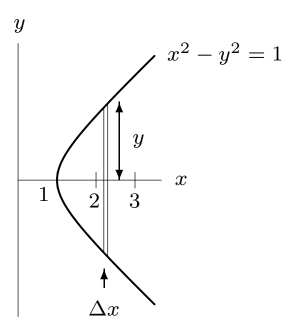
Problems

17. (a) Slicing the region perpendicular to the x-axis gives disks of radius y. See Figure 8.21.

$$\text{Volume of slice} \approx \pi y^2 \Delta x = \pi(x^2 - 1)\Delta x.$$

Thus,

$$\begin{aligned} \text{Total volume} &= \lim_{\Delta x \to 0} \sum \pi(x^2 - 1)\Delta x = \int_2^3 \pi(x^2 - 1)\, dx = \pi\left(\frac{x^3}{3} - x\right)\Bigg|_2^3 \\ &= \pi\left(9 - 3 - \left(\frac{8}{3} - 2\right)\right) = \frac{16\pi}{3}. \end{aligned}$$



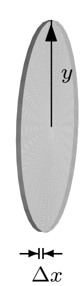


Figure 8.21

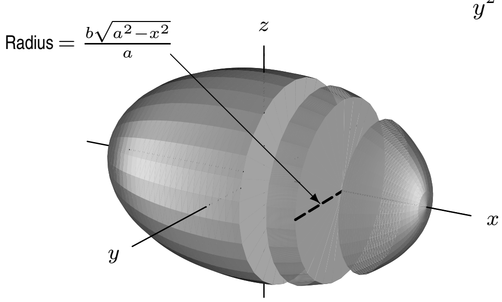
(b) The arc length, L, of the curve $y = f(x)$ is given by $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$. In this problem y is an implicit function of x. Solving for y gives $y = \sqrt{x^2 - 1}$ as the equation of the top half of the hyperbola. Differentiating gives

$$\frac{dy}{dx} = \frac{1}{2}(x^2 - 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 - 1}}.$$

Thus

$$\text{Arc length} = \int_2^3 \sqrt{1+\left(\frac{x}{\sqrt{x^2-1}}\right)^2}\,dx = \int_2^3 \sqrt{1+\frac{x^2}{x^2-1}}\,dx = \int_2^3 \sqrt{\frac{2x^2-1}{x^2-1}}\,dx = 1.48.$$

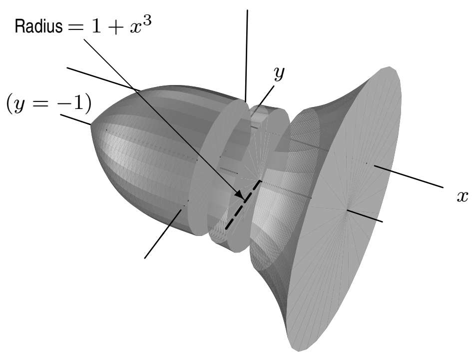
18.



$$y^2 = b^2\left(1-\frac{x^2}{a^2}\right).$$

$$\begin{aligned}
V &= \int_{-a}^{a} \pi y^2\,dx = \pi\int_{-a}^{a} b^2\left(1-\frac{x^2}{a^2}\right)dx\\
&= 2\pi b^2\int_0^a \left(1-\frac{x^2}{a^2}\right)dx = 2\pi b^2\left[x-\frac{x^3}{3a^2}\right]_0^a\\
&= 2\pi b^2\left(a-\frac{a^3}{3a^2}\right) = 2\pi b^2\left(a-\frac{1}{3}a\right)\\
&= \frac{4}{3}\pi ab^2.
\end{aligned}$$

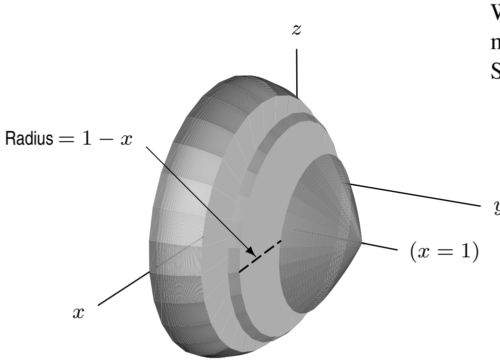
19.



We slice the region perpendicular to the x–axis. The Riemann sum we get is $\sum \pi(x^3+1)^2\Delta x$. So the volume V is the integral

$$\begin{aligned}
V &= \int_{-1}^{1} \pi(x^3+1)^2\,dx\\
&= \pi\int_{-1}^{1}(x^6+2x^3+1)\,dx\\
&= \pi\left(\frac{x^7}{7}+\frac{x^4}{2}+x\right)\Bigg|_{-1}^{1}\\
&= (16/7)\pi \approx 7.18.
\end{aligned}$$

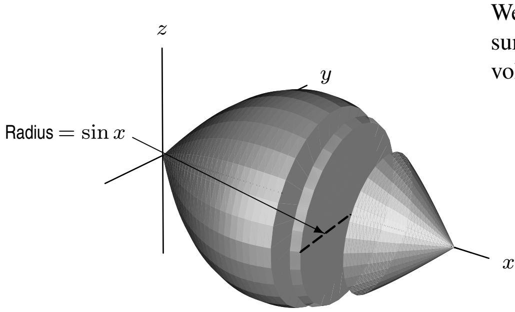
20.



We slice the region perpendicular to the y–axis. The Riemann sum we get is $\sum \pi(1-x)^2\Delta y = \sum \pi(1-y^2)^2\Delta y$. So the volume V is the integral

$$\begin{aligned}
V &= \int_0^1 \pi(1-y^2)^2\,dy\\
&= \pi\int_0^1(1-2y^2+y^4)\,dy\\
&= \pi\left(y-\frac{2y^3}{3}+\frac{y^5}{5}\right)\Bigg|_0^1\\
&= (8/15)\pi \approx 1.68.
\end{aligned}$$

21.



We take slices perpendicular to the x–axis. The Riemann sum for approximating the volume is $\sum \pi \sin^2 x\Delta x$. The volume is the integral corresponding to that sum, namely

$$\begin{aligned}
V &= \int_0^\pi \pi\sin^2 x\,dx\\
&= \pi\left[-\frac{1}{2}\sin x\cos x+\frac{1}{2}x\right]\Bigg|_0^\pi = \frac{\pi^2}{2} \approx 4.935.
\end{aligned}$$

22. Slice the object into disks horizontally, as in Figure 8.22. A typical disk has thickness Δy and radius $x = \sqrt{y}$. Thus

$$\text{Volume of slice } \approx \pi x^2 \Delta y = \pi y\, \Delta y.$$

$$\text{Volume of solid } = \lim_{\Delta y \to 0} \sum \pi y\, \Delta y = \int_0^1 \pi y\, dy = \pi \frac{y^2}{2}\bigg|_0^1 = \frac{\pi}{2}.$$

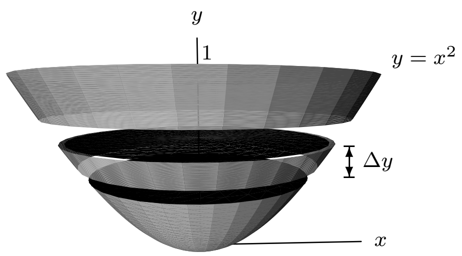


Figure 8.22

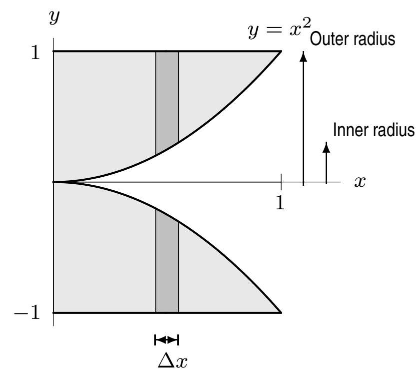


Figure 8.23: Cross-section of solid

23. Slice the object into rings vertically, as is Figure 8.23. A typical ring has thickness Δx and outer radius $y = 1$ and inner radius $y = x^2$.

$$\text{Volume of slice } \approx \pi 1^2 \Delta x - \pi y^2 \Delta x = \pi(1 - x^4)\, \Delta x.$$

$$\text{Volume of solid } = \lim_{\Delta x \to 0} \sum \pi(1 - x^4)\, \Delta x = \int_0^1 \pi(1 - x^4)\, dx = \pi \left(x - \frac{x^5}{5}\right)\bigg|_0^1 = \frac{4}{5}\pi.$$

24. The region is cylindrical with a hole around the axis of rotation, $y = -2$. Slice it into rings vertically, as in Figure 8.24. A typical ring has thickness Δx and outer radius $1 + 2 = 3$ and inner radius $y + 2 = x^2 + 2$. Thus

$$\text{Volume of slice } \approx \pi 3^2 \Delta x - \pi(x^2 + 2)^2 \Delta x = \pi(5 - x^4 - 4x^2)\, \Delta x.$$

$$\text{Volume of solid } = \int_0^1 \pi(5 - x^4 - 4x^2)\, \Delta x = \pi \left(5x - \frac{x^5}{5} - \frac{4}{3}x^3\right)\bigg|_0^1 = \frac{52\pi}{15}.$$

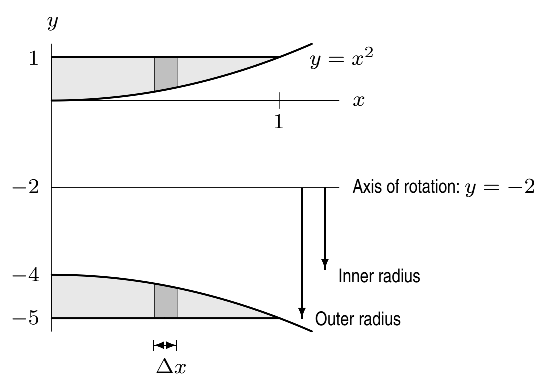


Figure 8.24: Cross-section of solid

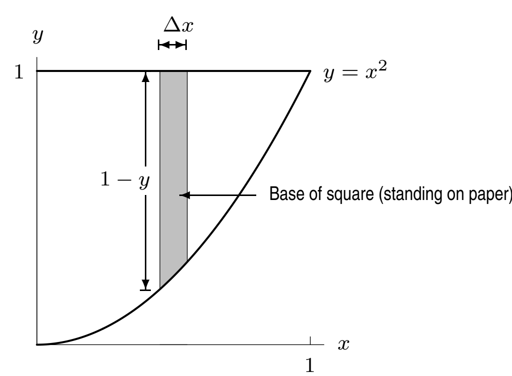


Figure 8.25: Base of solid

25. Slicing perpendicularly to the x-axis gives squares whose thickness is Δx and whose side is $1 - y = 1 - x^2$. See Figure 8.25. Thus

$$\text{Volume of square slice } \approx (1-x^2)^2\Delta x = (1 - 2x^2 + x^4)\,\Delta x.$$

$$\text{Volume of solid } = \int_0^1 (1-2x^2+x^4)\,dx = x - \frac{2}{3}x^3 + \frac{x^5}{5}\bigg|_0^1 = \frac{8}{15}.$$

26. Slicing perpendicularly to the x-axis gives semicircles whose thickness is Δx and whose diameter is $1 - y = 1 - x^2$. See Figure 8.26. Thus

$$\text{Volume of semicircular slice } \approx \pi\left(\frac{1-x^2}{2}\right)^2\Delta x = \frac{\pi}{4}(1-2x^2+x^4)\,\Delta x.$$

$$\text{Volume of solid } = \int_0^1 \frac{\pi}{4}(1-2x^2+x^4)\,dx = \frac{\pi}{4}\left(x - \frac{2}{3}x^3 + \frac{x^5}{5}\right)\bigg|_0^1 = \frac{\pi}{4}\cdot\frac{8}{15} = \frac{2\pi}{15}.$$

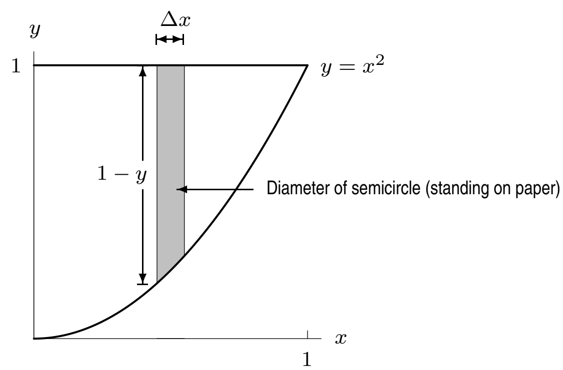


Figure 8.26: Base of solid

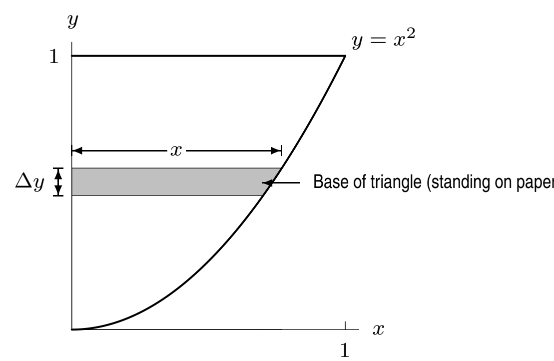


Figure 8.27: Base of solid

27. An equilateral triangle of side s has height $\sqrt{3}s/2$ and

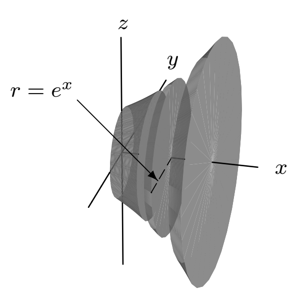
$$\text{Area } = \frac{1}{2}\cdot s\cdot\frac{\sqrt{3}s}{2} = \frac{\sqrt{3}}{4}s^2.$$

Slicing perpendicularly to the y-axis gives equilateral triangles whose thickness is Δy and whose side is $x = \sqrt{y}$. See Figure 8.27. Thus

$$\text{Volume of triangular slice } \approx \frac{\sqrt{3}}{4}(\sqrt{y})^2\Delta y = \frac{\sqrt{3}}{4}y\,\Delta y.$$

$$\text{Volume of solid } = \int_0^1 \frac{\sqrt{3}}{4}y\,dy = \frac{\sqrt{3}}{4}\frac{y^2}{2}\bigg|_0^1 = \frac{\sqrt{3}}{8}.$$

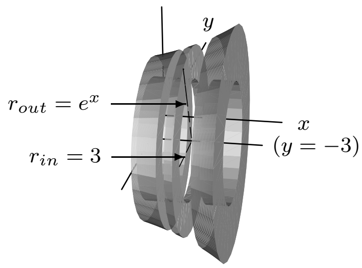
28.



This is the volume of revolution gotten from the rotating the curve $y = e^x$. Take slices perpendicular to the x-axis. They will be circles with radius e^x, so

$$V = \int_{x=0}^{x=1} \pi y^2\,dx = \pi\int_0^1 e^{2x}\,dx$$
$$= \frac{\pi e^{2x}}{2}\bigg|_0^1 = \frac{\pi(e^2-1)}{2} \approx 10.036.$$

29.



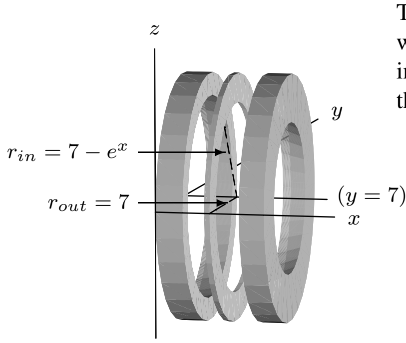
We slice the volume with planes perpendicular to the line $y = -3$. This divides the curve into thin washers, as in Example 3 on page 376 of the text, whose volumes are

$$\pi r_{\text{out}}^2 dx - \pi r_{\text{in}}^2 dx = \pi(3+y)^2 dx - \pi 3^2 dx.$$

So the integral we get from adding all these washers up is

$$\begin{aligned} V &= \int_{x=0}^{x=1} [\pi(3+y)^2 - \pi 3^2]\, dx \\ &= \pi \int_0^1 [(3+e^x)^2 - 9]\, dx \\ &= \pi \int_0^1 [e^{2x} + 6e^x]\, dx = \pi[\frac{e^{2x}}{2} + 6e^x]\Big|_0^1 \\ &= \pi[(e^2/2 + 6e) - (1/2 + 6)] \approx 42.42. \end{aligned}$$

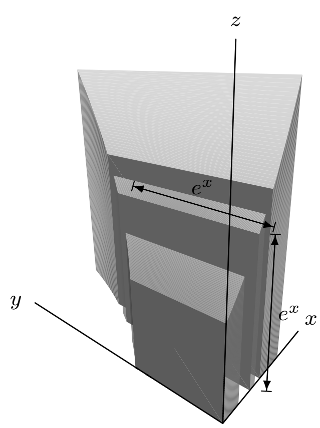
30.



This problem can be done by slicing the volume into washers with planes perpendicular to the axis of rotation, $y = 7$, just like in Example 3. This time the outside radius of a washer is 7, and the inside radius is $7 - e^x$. Therefore, the volume V is

$$\begin{aligned} V &= \int_{x=0}^{x=1} [\pi 7^2 - \pi(7-e^x)^2]\, dx = \pi \int_0^1 (14e^x - e^{2x})\, dx \\ &= \pi \left[14e^x - \frac{1}{2}e^{2x}\right]\Bigg|_0^1 = \pi\left[14e - \frac{1}{2}e^2 - \left(14 - \frac{1}{2}\right)\right] \\ &\approx 65.54. \end{aligned}$$

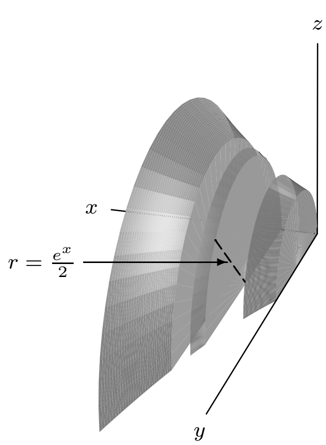
31.



We now slice perpendicular to the x-axis. As stated in the problem, the cross-sections obtained thereby will be squares, with base length e^x. The volume of one square slice is $(e^x)^2\, dx$. (Look at the picture.) Adding up the volumes of the slices yields

$$\begin{aligned} \text{Volume } &= \int_{x=0}^{x=1} y^2\, dx = \int_0^1 e^{2x}\, dx \\ &= \frac{e^{2x}}{2}\Bigg|_0^1 = \frac{e^2 - 1}{2} \approx 3.195. \end{aligned}$$

32.



We slice perpendicular to the x-axis. As stated in the problem, the cross-sections obtained thereby will be semicircles, with radius $\frac{e^x}{2}$. The volume of one semicircular slice is $\frac{1}{2}\pi\left(\frac{e^x}{2}\right)^2 dx$. (Look at the picture.) Adding up the volumes of the slices yields

$$\begin{aligned} \text{Volume } &= \int_{x=0}^{x=1} \pi\frac{y^2}{2}\, dx = \frac{\pi}{8}\int_0^1 e^{2x}\, dx \\ &= \frac{\pi e^{2x}}{16}\Bigg|_0^1 = \frac{\pi(e^2-1)}{16} \approx 1.25. \end{aligned}$$

33. (a) We can begin by slicing the pie into horizontal slabs of thickness Δh located at height h. To find the radius of each slice, we note that radius increases linearly with height. Since $r = 4.5$ when $h = 3$ and $r = 3.5$ when $h = 0$, we should have $r = 3.5 + h/3$. Then the volume of each slab will be $\pi r^2 \, \Delta h = \pi(3.5 + h/3)^2 \, \Delta h$. To find the total volume of the pie, we integrate this from $h = 0$ to $h = 3$:

$$\begin{aligned} V &= \pi \int_0^3 \left(3.5 + \frac{h}{3}\right)^2 \, dh \\ &= \pi \left[\frac{h^3}{27} + \frac{7h^2}{6} + \frac{49h}{4}\right]\Bigg|_0^3 \\ &= \pi \left[\frac{3^3}{27} + \frac{7(3^2)}{6} + \frac{49(3)}{4}\right] \approx 152 \text{ in}^3. \end{aligned}$$

(b) We use 1.5 in as a rough estimate of the radius of an apple. This gives us a volume of $(4/3)\pi(1.5)^3 \approx 10 \text{ in}^3$. Since $152/10 \approx 15$, we would need about 15 apples to make a pie.

34. (a) The volume can be computed by several methods, not all of them requiring integration. We will slice horizontally, forming rectangular slabs of length 100 cm, height Δy, width w and integrate. See Figure 8.28.

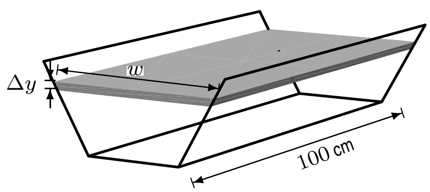


Figure 8.28

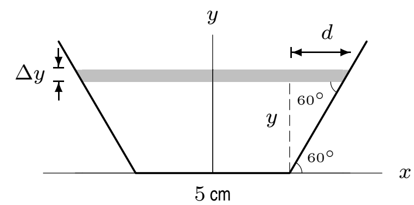


Figure 8.29

From the right triangle, we see

$$\frac{y}{d} = \tan 60° = \sqrt{3}$$

so

$$d = \frac{y}{\sqrt{3}}.$$

Thus

$$w = 5 + 2d = 5 + \frac{2y}{\sqrt{3}}.$$

The volume of the slab is

$$\Delta V \approx 100 w \Delta y = 100 \left(5 + \frac{2y}{\sqrt{3}}\right) \Delta y,$$

so the total volume is given by

$$\begin{aligned} \text{Volume} &= \lim_{\Delta y \to 0} \sum \Delta V = \lim_{\Delta y \to 0} \sum 100 \left(5 + \frac{2y}{\sqrt{3}}\right) \Delta y \\ &= \int_0^h 100 \left(5 + \frac{2y}{\sqrt{3}}\right) dy = 100 \left(5y + \frac{y^2}{\sqrt{3}}\right)\Bigg|_0^h = 100 \left(5h + \frac{h^2}{\sqrt{3}}\right) \text{ cm}^3. \end{aligned}$$

(b) The maximum value of h is $h = 5 \sin 60° = 5\sqrt{3}/2$ cm ≈ 4.33 cm.
(c) The maximum volume of water that the gutter can hold is given by substituting $h = 5\sqrt{3}/2$ into the volume:

$$\text{Maximum volume} = 100 \left(5 \cdot \frac{5\sqrt{3}}{2} + \left(\frac{5\sqrt{3}}{2}\right)^2 \Big/ \sqrt{3}\right) = \frac{2500}{4}(2\sqrt{3} + \sqrt{3}) = 1875\sqrt{3} \approx 3247.6 \text{ cm}^3.$$

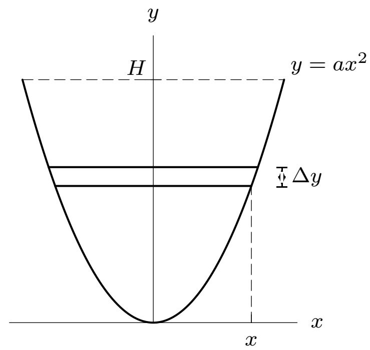
(d) Because the gutter is narrower at the bottom than the top, if it is filled with half the maximum possible volume of water, the gutter will be filled to a depth of more than half of 4.33 cm.

(e) We want to solve for the value of h such that

$$\text{Volume} = 100\left(5h + \frac{h^2}{\sqrt{3}}\right) = \frac{1}{2}\cdot 1875\sqrt{3} = \frac{1}{2}V_{\max}$$
$$5h + \frac{h^2}{\sqrt{3}} = 16.238.$$

Solving gives $h = 2.52$ and $h = -11.18$. Since only positive values of h are meaningful, $h = 2.52$ cm.

35.



We divide the interior of the boat into flat slabs of thickness Δy and width $2x = 2\sqrt{y/a}$. (See above.) We have

$$\text{Volume of slab } \approx 2xL\Delta y = 2L\sqrt{\frac{y}{a}}\Delta y.$$

We are interested in the total volume of the region $0 \le y \le H$, so

$$\begin{aligned}\text{Total volume } &= \lim_{\Delta y\to 0}\sum 2L\left(\frac{y}{a}\right)^{(1/2)}\Delta y = \int_0^H 2L\left(\frac{y}{a}\right)^{(1/2)} dy\\ &= \frac{2L}{\sqrt{a}}\int_0^H y^{(1/2)} dy = \frac{4LH^{(3/2)}}{3\sqrt{a}}.\end{aligned}$$

If L and H are in meters,

$$\text{Buoyancy force } = \frac{40,000LH^{(3/2)}}{3\sqrt{a}} \text{ newtons.}$$

36. We can find the volume of the tree by slicing it into a series of thin horizontal cylinders of height dh and circumference C. The volume of each cylindrical disk will then be

$$V = \pi r^2\, dh = \pi\left(\frac{C}{2\pi}\right)^2 dh = \frac{C^2\, dh}{4\pi}.$$

Summing all such cylinders, we have the total volume of the tree as

$$\text{Total volume} = \frac{1}{4\pi}\int_0^{120} C^2\, dh.$$

We can estimate this volume using a trapezoidal approximation to the integral with $\Delta h = 20$:

$$\begin{aligned}\text{LEFT estimate } &= \frac{1}{4\pi}[20(31^2 + 28^2 + 21^2 + 17^2 + 12^2 + 8^2)] = \frac{1}{4\pi}(53660).\\ \text{RIGHT estimate } &= \frac{1}{4\pi}[20(28^2 + 21^2 + 17^2 + 12^2 + 8^2 + 2^2)] = \frac{1}{4\pi}(34520).\\ \text{TRAP } &= \frac{1}{4\pi}(44090) \approx 3509 \text{ cubic inches.}\end{aligned}$$

37. (a) The volume, V, contained in the bowl when the surface has height h is

$$V = \int_0^h \pi x^2 \, dy.$$

However, since $y = x^4$, we have $x^2 = \sqrt{y}$ so that

$$V = \int_0^h \pi \sqrt{y} \, dy = \frac{2}{3}\pi h^{3/2}.$$

Differentiating gives $dV/dh = \pi h^{1/2} = \pi\sqrt{h}$. We are given that $dV/dt = -6\sqrt{h}$, where the negative sign reflects the fact that V is decreasing. Using the chain rule we have

$$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{1}{dV/dh} \cdot \frac{dV}{dt} = \frac{1}{\pi\sqrt{h}} \cdot (-6\sqrt{h}) = -\frac{6}{\pi}.$$

Thus, $dh/dt = -6/\pi$, a constant.

(b) Since $dh/dt = -6/\pi$ we know that $h = -6t/\pi + C$. However, when $t = 0$, $h = 1$, therefore $h = 1 - 6t/\pi$. The bowl is empty when $h = 0$, that is when $t = \pi/6$ units.

38. The problem appears complicated, because we are now working in three dimensions. However, if we take one dimension at a time, we will see that the solution is not too difficult. For example, let's just work at a constant depth, say 0. We apply the trapezoid rule to find the approximate area along the length of the boat. For example, by the trapezoid rule the approximate area at depth 0 from the front of the boat to 10 feet toward the back is $\frac{(2+8)\cdot 10}{2} = 50$. Overall, at depth 0 we have that the area for each length span is as follows:

Table 8.1

length span:		0–10	10–20	20–30	30–40	40–50	50–60
depth	0	50	105	145	165	165	130

We can fill in the whole chart the same way:

Table 8.2

length span:		0–10	10–20	20–30	30–40	40–50	50–60
	0	50	105	145	165	165	130
	2	25	60	90	105	105	90
depth	4	15	35	50	65	65	50
	6	5	15	25	35	35	25
	8	0	5	10	10	10	10

Now, to find the volume, we just apply the trapezoid rule to the depths and areas. For example, according to the trapezoid rule the approximate volume as the depth goes from 0 to 2 and the length goes from 0 to 10 is $\frac{(50+25)\cdot 2}{2} = 75$. Again, we fill in a chart:

Table 8.3

length span:		0–10	10–20	20–30	30–40	40–50	50–60
	0–2	75	165	235	270	270	220
depth	2–4	40	95	140	170	170	140
span	4–6	20	50	75	100	100	75
	6–8	5	20	35	45	45	35

Adding all this up, we find the volume is approximately 2595 cubic feet.

You might wonder what would have happened if we had done our trapezoids along the depth axis first instead of along the length axis. If you try this, you'd find that you come up with the same answers in the volume chart! For the trapezoid rule, it does not matter which axis you choose first.

39. (a) The equation of a circle of radius r around the origin is $x^2+y^2=r^2$. This means that $y^2=r^2-x^2$, so $2y(dy/dx)=-2x$, and $dy/dx=-x/y$. Since the circle is symmetric about both axes, its arc length is 4 times the arc length in the first quadrant, namely

$$4\int_0^r \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 4\int_0^r \sqrt{1+\left(-\frac{x}{y}\right)^2}\,dx.$$

(b) Evaluating this integral yields

$$\begin{aligned}4\int_0^r \sqrt{1+\left(-\frac{x}{y}\right)^2}\,dx &= 4\int_0^r \sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = 4\int_0^r \sqrt{\frac{r^2}{r^2-x^2}}\,dx \\ &= 4r\int_0^r \sqrt{\frac{1}{r^2-x^2}}\,dx = 4r(\arcsin(x/r))\Big|_0^r = 2\pi r.\end{aligned}$$

This is the expected answer.

40. As can be seen in Figure 8.30, the region has three straight sides and one curved one. The lengths of the straight sides are 1, 1, and e. The curved side is given by the equation $y=f(x)=e^x$. We can find its length by the formula

$$\int_0^1 \sqrt{1+f'(x)^2}\,dx = \int_0^1 \sqrt{1+(e^x)^2}\,dx = \int_0^1 \sqrt{1+e^{2x}}\,dx.$$

Evaluating the integral numerically gives 2.0035. The total length, therefore, is about $1+1+e+2.0035\approx 6.722$.

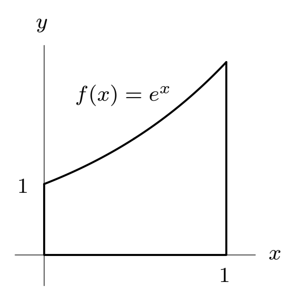


Figure 8.30

41. Since $y=(e^x+e^{-x})/2$, $y'=(e^x-e^{-x})/2$. The length of the catenary is

$$\begin{aligned}\int_{-1}^1 \sqrt{1+(y')^2}\,dx &= \int_{-1}^1 \sqrt{1+\left[\frac{e^x-e^{-x}}{2}\right]^2}\,dx = \int_{-1}^1 \sqrt{1+\frac{e^{2x}}{4}-\frac{1}{2}+\frac{e^{-2x}}{4}}\,dx \\ &= \int_{-1}^1 \sqrt{\left[\frac{e^x+e^{-x}}{2}\right]^2}\,dx = \int_{-1}^1 \frac{e^x+e^{-x}}{2}\,dx \\ &= \left[\frac{e^x-e^{-x}}{2}\right]\Bigg|_{-1}^1 = e-e^{-1}.\end{aligned}$$

42. Since the ellipse is symmetric about its axes, we can just find its arc length in the first quadrant and multiply that by 4. To determine the arc length of this section, we first solve for y in terms of x: since $x^2/4+y^2=1$ is the equation for the ellipse, we have $y^2=1-x^2/4$, so $y=\sqrt{1-x^2/4}$. We also need to find dy/dx; we can do this by differentiating $y^2=1-x^2/4$ implicitly, obtaining $2y\,dy/dx=-x/2$, whence $dy/dx=-x/(4y)$. We now set up the integral:

$$\begin{aligned}\text{Circumference of ellipse in first quadrant} &= \int_0^2 \sqrt{1+\left(-\frac{x}{4y}\right)^2}\,dx = \int_0^2 \sqrt{1+\frac{x^2}{16y^2}}\,dx \\ &= \int_0^2 \sqrt{1+\frac{x^2}{16-4x^2}}\,dx = \int_0^2 \sqrt{\frac{16-3x^2}{16-4x^2}}\,dx.\end{aligned}$$

This is an improper integral, since $16 - 4x^2 = 0$ for $x = 2$. Hence, integrating it numerically is somewhat tricky. However, we can integrate numerically from 0 to 1.999, and then use a vertical line to approximate the last section. The upper point of the line is $(1.999, 0.016)$; the lower point is $(2, 0)$. The length of the line connecting these two points is $\sqrt{(2-1.999)^2 + (0-0.016)^2} \approx 0.016$. Approximating the integral from 0 to 1.999 gives 2.391; hence the total arc length of the first quadrant is approximately $2.391 + 0.016 = 2.407$. So the arc length of the whole ellipse is about $4 \cdot 2.407 \approx 9.63$.

43. Here are many functions which "work."

- Any linear function $y = mx + b$ "works." This follows because $\frac{dy}{dx} = m$ is constant for such functions. So

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1+m^2}\, dx = (b-a)\sqrt{1+m^2}.$$

- The function $y = \frac{x^4}{8} + \frac{1}{4x^2}$ "works": $\frac{dy}{dx} = \frac{1}{2}(x^3 - 1/x^3)$, and

$$\begin{aligned}\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx &= \int \sqrt{1+\frac{\left(x^3-\frac{1}{x^3}\right)^2}{4}}\, dx = \int \sqrt{1+\frac{x^6}{4}-\frac{1}{2}+\frac{1}{4x^6}}\, dx \\ &= \int \sqrt{\frac{1}{4}\left(x^3+\frac{1}{x^3}\right)^2}\, dx = \int \frac{1}{2}\left(x^3+\frac{1}{x^3}\right)\, dx \\ &= \left[\frac{x^4}{8} - \frac{1}{4x^2}\right] + C.\end{aligned}$$

- One more function that "works" is $y = \ln(\cos x)$; we have $\frac{dy}{dx} = -\sin x/\cos x$. Hence

$$\begin{aligned}\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx. &= \int \sqrt{1+\left(\frac{-\sin x}{\cos x}\right)^2}\, dx = \int \sqrt{1+\frac{\sin^2 x}{\cos^2 x}}\, dx \\ &= \int \sqrt{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}}\, dx = \int \sqrt{\frac{1}{\cos^2 x}}\, dx \\ &= \int \frac{1}{\cos x}\, dx = \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + C,\end{aligned}$$

where the last integral comes from IV-22 of the integral tables.

44. (a) If $f(x) = \int_0^x \sqrt{g'(t)^2 - 1}\, dt$, then, by the Fundamental Theorem of Calculus, $f'(x) = \sqrt{g'(x)^2 - 1}$. So the arc length of f from 0 to x is

$$\begin{aligned}\int_0^x \sqrt{1+(f'(t))^2}\, dt &= \int_0^x \sqrt{1+(\sqrt{g'(t)^2-1})^2}\, dt \\ &= \int_0^x \sqrt{1+g'(t)^2-1}\, dt \\ &= \int_0^x g'(t)\, dt = g(x) - g(0) = g(x).\end{aligned}$$

(b) If g is the arc length of any function f, then by the Fundamental Theorem of Calculus, $g'(x) = \sqrt{1+f'(x)^2} \geq 1$. So if $g'(x) < 1$, g cannot be the arc length of a function.

(c) We find a function f whose arc length from 0 to x is $g(x) = 2x$. Using part (a), we see that

$$f(x) = \int_0^x \sqrt{(g'(t))^2 - 1}\, dt = \int_0^x \sqrt{2^2-1}\, dt = \sqrt{3}x.$$

This is the equation of a line. Does it make sense to you that the arc length of a line segment depends linearly on its right endpoint?

Solutions for Section 8.3

Exercises

1. With $r = 1$ and $\theta = 2\pi/3$, we find $x = r\cos\theta = 1\cdot\cos(2\pi/3) = -1/2$ and $y = r\sin\theta = 1\cdot\sin(2\pi/3) = \sqrt{3}/2$.
The rectangular coordinates are $(-1/2, \sqrt{3}/2)$.

2. With $r = \sqrt{3}$ and $\theta = -3\pi/4$, we find $x = r\cos\theta = \sqrt{3}\cos(-3\pi/4) = \sqrt{3}(-\sqrt{2}/2) = -\sqrt{6}/2$ and $y = r\sin\theta = \sqrt{3}\sin(-3\pi/4) = \sqrt{3}(-\sqrt{2}/2) = -\sqrt{6}/2$.
The rectangular coordinates are $(-\sqrt{6}/2, -\sqrt{6}/2)$.

3. With $r = 2\sqrt{3}$ and $\theta = -\pi/6$, we find $x = r\cos\theta = 2\sqrt{3}\cos(-\pi/6) = 2\sqrt{3}\cdot\sqrt{3}/2 = 3$ and $y = r\sin\theta = 2\sqrt{3}\sin(-\pi/6) = 2\sqrt{3}(-1/2) = -\sqrt{3}$.
The rectangular coordinates are $(3, -\sqrt{3})$.

4. With $r = 2$ and $\theta = 5\pi/6$, we find $x = r\cos\theta = 2\cos(5\pi/6) = 2(-\sqrt{3}/2) = -\sqrt{3}$ and $y = r\sin\theta = 2\sin(5\pi/6) = 2(1/2) = 1$.
The rectangular coordinates are $(-\sqrt{3}, 1)$.

5. With $x = 1$ and $y = 1$, find r from $r = \sqrt{x^2+y^2} = \sqrt{1^2+1^2} = \sqrt{2}$. Find θ from $\tan\theta = y/x = 1/1 = 1$. Thus, $\theta = \tan^{-1}(1) = \pi/4$. Since $(1,1)$ is in the first quadrant this is a correct θ. The polar coordinates are $(\sqrt{2}, \pi/4)$.

6. With $x = -1$ and $y = 0$, find $r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+0^2} = 1$. Find θ from $\tan\theta = y/x = 0/(-1) = 0$. Thus, $\theta = \tan^{-1}(0) = 0$. Since $(-1,0)$ is on the x-axis between the second and third quadrant, $\theta = \pi$. The polar coordinates are $(1, \pi)$.

7. With $x = \sqrt{6}$ and $y = -\sqrt{2}$, find $r = \sqrt{(\sqrt{6})^2+(-\sqrt{2})^2} = \sqrt{8} = 2\sqrt{2}$. Find θ from $\tan\theta = y/x = -\sqrt{2}/\sqrt{6} = -1/\sqrt{3}$. Thus, $\theta = \tan^{-1}(-1/\sqrt{3}) = -\pi/6$. Since $(\sqrt{6}, -\sqrt{2})$ is in the fourth quadrant, this is the correct θ. The polar coordinates are $(2\sqrt{2}, -\pi/6)$.

8. With $x = -\sqrt{3}$ and $y = 1$, find $r = \sqrt{(-\sqrt{3})^2+1^2} = \sqrt{4} = 2$. Find θ from $\tan\theta = y/x = 1/(-\sqrt{3})$. Thus, $\theta = \tan^{-1}(-1/\sqrt{3}) = -\pi/6$. Since $(-\sqrt{3}, 1)$ is in the second quadrant, $\theta = -\pi/6+\pi = 5\pi/6$. The polar coordinates are $(2, 5\pi/6)$.

9. (a) Table 8.4 contains values of $r = 1-\sin\theta$, both exact and rounded to one decimal.

Table 8.4

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π	$4\pi/3$	$3\pi/2$	$5\pi/3$	2π	$7\pi/3$	$5\pi/2$	$8\pi/3$
r	1	$1-\sqrt{3}/2$	0	$1-\sqrt{3}/2$	1	$1+\sqrt{3}/2$	2	$1+\sqrt{3}/2$	1	$1-\sqrt{3}/2$	0	$1-\sqrt{3}/2$
r	1	0.134	0	0.134	1	1.866	2	1.866	1	0.134	0	0.134

(b) See Figure 8.31.

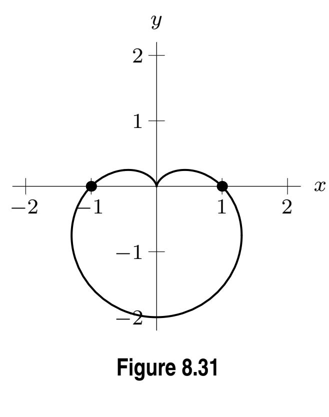


Figure 8.31

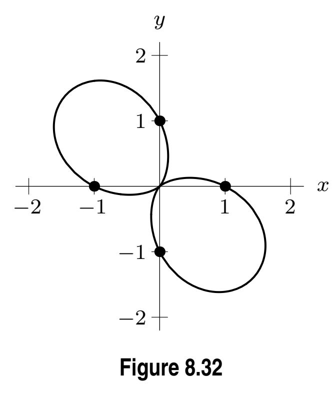


Figure 8.32

(c) The circle has equation $r = 1/2$. The cardioid is $r = 1-\sin\theta$. Solving these two simultaneously gives

$$1/2 = 1-\sin\theta,$$

or

$$\sin\theta = 1/2.$$

Thus, $\theta = \pi/6$ or $5\pi/6$. This gives the points $(x, y) = ((1/2)\cos\pi/6, (1/2)\sin\pi/6) = (\sqrt{3}/4, 1/4)$ and $(x, y) = ((1/2)\cos 5\pi/6, (1/2)\sin 5\pi/6) = (-\sqrt{3}/4, 1/4)$ as the location of intersection.

(d) The curve $r = 1 - \sin 2\theta$, pictured in Figure 8.32, has two regions instead of the one region that $r = 1 - \sin\theta$ has. This is because $1 - \sin 2\theta$ will be 0 twice for every 2π cycle in θ, as opposed to once for every 2π cycle in θ for $1 - \sin\theta$.

10. There will be n loops. See Figures 8.33-8.36.

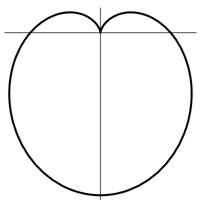


Figure 8.33: $n = 1$

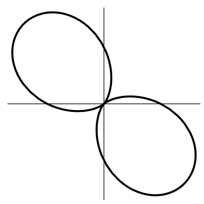


Figure 8.34: $n = 2$

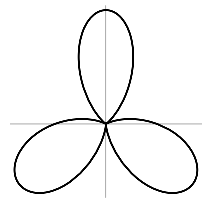


Figure 8.35: $n = 3$

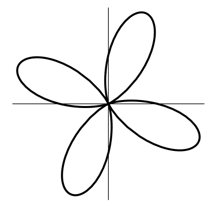


Figure 8.36: $n = 4$

11. The graph will begin to draw over itself for any $\theta \geq 2\pi$ so the graph will look the same in all three cases. See Figure 8.37.

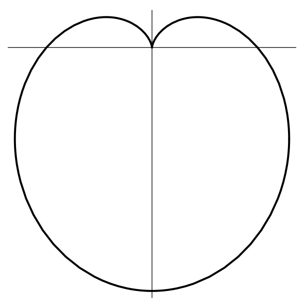


Figure 8.37

12. The curve will be a smaller loop inside a larger loop with an intersection point at the origin. Larger n values increase the size of the loops. See Figures 8.38-8.40.

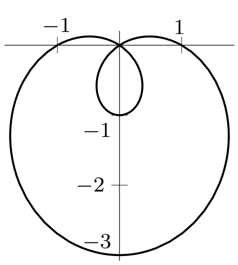


Figure 8.38: $n = 2$

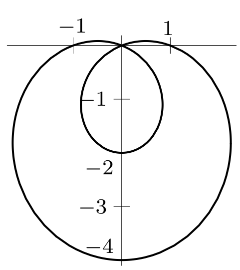


Figure 8.39: $n = 3$

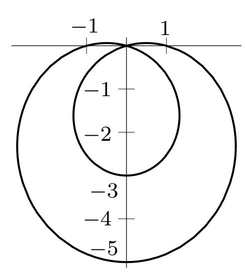


Figure 8.40: $n = 4$

13. See Figures 8.41 and 8.42. The first curve will be similar to the second curve, except the cardioid (heart) will be rotated clockwise by 90° ($\pi/2$ radians). This makes sense because of the identity $\sin\theta = \cos(\theta - \pi/2)$.

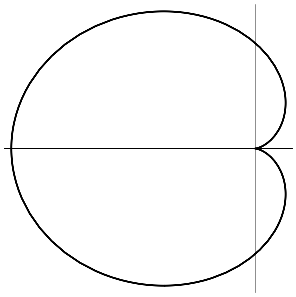


Figure 8.41: $r = 1 - \cos\theta$

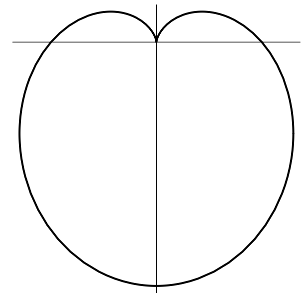


Figure 8.42: $r = 1 - \sin\theta$

14. Let $0 \le \theta \le 2\pi$ and $3/16 \le r \le 1/2$.

15. A loop starts and ends at the origin, that is, when $r = 0$. This happens first when $\theta = \pi/4$ and next when $\theta = 5\pi/4$. This can also be seen by using a trace mode on a calculator. Thus restricting θ so that $\pi/4 \le \theta \le 5\pi/4$ will graph the upper loop only. See Figure 8.43. To show only the other loop use $0 \le \theta \le \pi/4$ and $5\pi/4 \le \theta \le 2\pi$. See Figure 8.44.

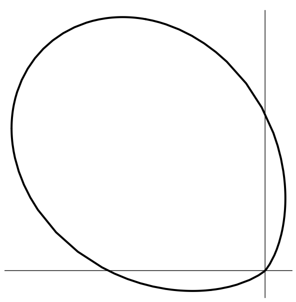


Figure 8.43: $\pi/4 \le \theta \le 5\pi/4$

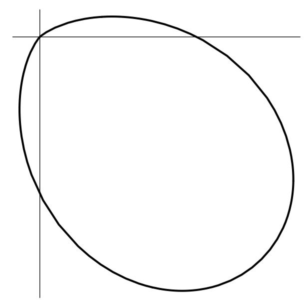


Figure 8.44: $0 \le \theta \le \pi/4$ and $5\pi/4 \le \theta \le 2\pi$

16. (a) Let $0 \le \theta \le \pi/4$ and $0 \le r \le 1$.
(b) Break the region into two pieces: one with $0 \le x \le \sqrt{2}/2$ and $0 \le y \le x$, the other with $\sqrt{2}/2 \le x \le 1$ and $0 \le y \le \sqrt{1-x^2}$.

17. The region is given by $\sqrt{8} \le r \le \sqrt{18}$ and $\pi/4 \le \theta \le \pi/2$.

18. The region is given by $0 \le r \le 2$ and $-\pi/6 \le \theta \le \pi/6$.

19. The circular arc has equation $r = 1$, for $0 \le \theta \le \pi/2$. the vertical line $x = 2$ has polar equation $r\cos\theta = 2$, or $r = 2/\cos\theta$. So the region is described by $0 \le \theta \le \pi/2$ and $1 \le r \le 2/\cos\theta$.

Problems

20. The formula for area is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$$

Therefore, since

$$A = \frac{1}{2}\int_{0}^{\pi/3} \sin^2(3\theta)\, d\theta = \frac{1}{2}\int_{0}^{\pi/3} (\sin 3\theta)^2\, d\theta,$$

we have $r = \sin 3\theta$. The integral represents the shaded area inside one petal of the three-petaled rose curve, $r = \sin 3\theta$, in Figure 8.45.

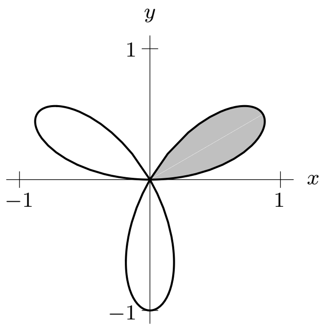


Figure 8.45: Graph of $r = \sin 3\theta$

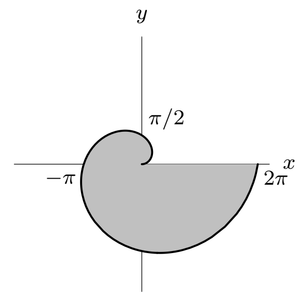


Figure 8.46: Spiral $r = \theta$

21. The spiral is shown in Figure 8.46.

$$\text{Area } = \frac{1}{2}\int_0^{2\pi} \theta^2\, d\theta = \frac{1}{6}\theta^3\Big|_0^{2\pi} = \frac{8\pi^3}{6}.$$

22. The region between the spirals is shaded in Figure 8.47.

$$\text{Area } = \frac{1}{2}\int_0^{2\pi} ((2\theta)^2 - \theta^2)\, d\theta = \frac{1}{2}\int_0^{2\pi} 3\theta^2\, d\theta = \frac{1}{2}\theta^3\Big|_0^{2\pi} = 4\pi^3.$$

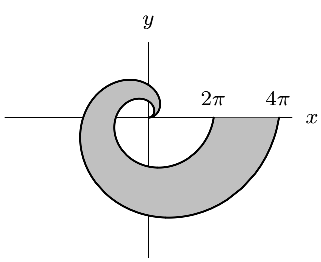


Figure 8.47: Region between the inner spiral, $r = \theta$, and the outer spiral, $r = 2\theta$

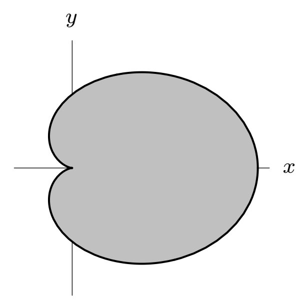


Figure 8.48: Cardioid $r = 1 + \cos\theta$

23. The cardioid is shown in Figure 8.48. The following integral can be evaluated using a calculator or by parts or using the table of integrals.

$$\begin{aligned}\text{Area } &= \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right)\, d\theta \\ &= \frac{1}{2}\left(\theta + 2\sin\theta + \frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right)\Big|_0^{2\pi} = \frac{1}{2}(2\pi + 0 + 0 + \pi) = \frac{3\pi}{2}.\end{aligned}$$

24. (a) See Figure 8.49. In polar coordinates, the line $x = 1$ is $r\cos\theta = 1$, so its equation is

$$r = \frac{1}{\cos\theta}.$$

The circle of radius 2 centered at the origin has equation

$$r = 2.$$

(b) The line and circle intersect where

$$\begin{aligned}\frac{1}{\cos\theta} &= 2\\ \cos\theta &= \frac{1}{2}\\ \theta &= -\frac{\pi}{3}, \frac{\pi}{3}.\end{aligned}$$

Thus,

$$\text{Area } = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left(2^2 - \left(\frac{1}{\cos\theta}\right)^2\right) d\theta.$$

(c) Evaluating gives

$$\text{Area } = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left(4 - \frac{1}{\cos^2\theta}\right) d\theta = \frac{1}{2}(4\theta - \tan\theta)\Big|_{-\pi/3}^{\pi/3} = \frac{4\pi}{3} - \sqrt{3}.$$

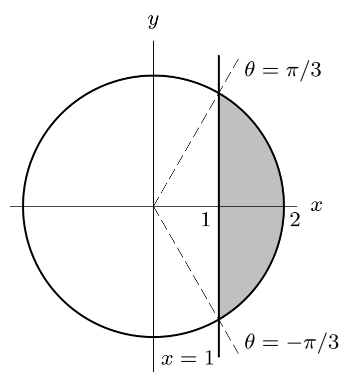


Figure 8.49

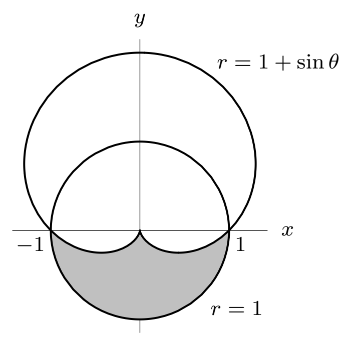


Figure 8.50

25. See Figure 8.50. Notice that the curves intersect at $(1, 0)$, where $\theta = 0, 2\pi$, and at $(-1, 0)$, where $\theta = \pi$, so

$$\text{Area } = \frac{1}{2}\int_{\pi}^{2\pi}(1^2 - (1+\sin\theta)^2)\, d\theta = \frac{1}{2}\int_{\pi}^{2\pi}(-2\sin\theta - \sin^2\theta)\, d\theta.$$

Using a calculator, integration by parts, or formula IV-17 in the integral table, we have

$$\text{Area } = \frac{1}{2}\left(2\cos\theta + \frac{1}{2}\sin\theta\cos\theta - \frac{1}{2}\theta\right)\Big|_{\pi}^{2\pi} = \frac{1}{2}\left(2\cdot 2 + 0 - \frac{1}{2}\pi\right) = 2 - \frac{\pi}{4}.$$

26. The two curves intersect where

$$\begin{aligned}1 - \sin\theta &= \frac{1}{2}\\ \sin\theta &= \frac{1}{2}\\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}.\end{aligned}$$

See Figure 8.51. We find the area of the right half and multiply that answer by 2 to get the entire area. The integrals can be computed numerically with a calculator or, as we show, using integration by parts or formula IV-17 in the integral tables.

$$\begin{aligned}\text{Area of right half } &= \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left((1-\sin\theta)^2 - \left(\frac{1}{2}\right)^2\right) d\theta\\ &= \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left(1 - 2\sin\theta + \sin^2\theta - \frac{1}{4}\right) d\theta\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\pi/2}^{\pi/6}\left(\frac{3}{4} - 2\sin\theta + \sin^2\theta\right)\,d\theta \\
&= \frac{1}{2}\left(\frac{3}{4}\theta + 2\cos\theta - \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\right)\Big|_{-\pi/2}^{\pi/6} \\
&= \frac{1}{2}\left(\frac{5\pi}{6} + \frac{7\sqrt{3}}{8}\right).
\end{aligned}
$$

Thus,

$$\text{Total area } = \frac{5\pi}{6} + \frac{7\sqrt{3}}{8}.$$

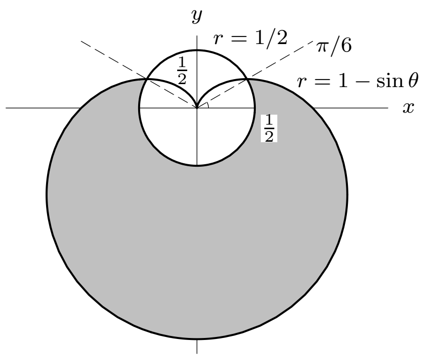


Figure 8.51

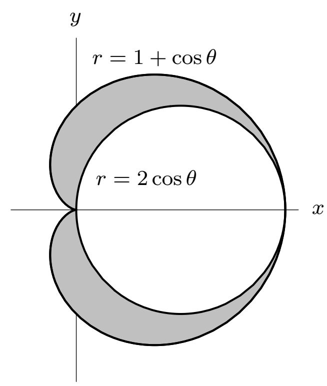


Figure 8.52

27. Figure 8.52 shows the curves which touch at $(2, 0)$ and the origin. However, the circle lies entirely inside the cardioid, so we find the area by subtracting the area of the circle from that of the cardioid. To find the areas, we take the integrals.

The cardioid, $r = 1 + \cos\theta$, starts at $(2, 0)$ when $\theta = 0$ and traces the top half, reaching the origin when $\theta = \pi$. Thus

$$\text{Area of cardioid } = 2\cdot\frac{1}{2}\int_0^{\pi}(1+\cos\theta)^2\,d\theta.$$

The circle starts at $(2, 0)$ when $\theta = 0$ and traces the top half, reaching the origin when $\theta = \pi/2$. Thus

$$\text{Area of circle } = 2\cdot\frac{1}{2}\int_0^{\pi/2}(2\cos\theta)^2\,d\theta.$$

The area, A, we want is therefore

$$
\begin{aligned}
\text{Area } &= 2\cdot\frac{1}{2}\int_0^{\pi}(1+\cos\theta)^2\,d\theta - 2\cdot\frac{1}{2}\int_0^{\pi/2}(2\cos\theta)^2\,d\theta \\
&= \int_0^{\pi}(1 + 2\cos\theta + \cos^2\theta)\,d\theta - \int_0^{\pi/2} 4\cos^2\theta\,d\theta \\
&= \left(\theta + 2\sin\theta + \frac{1}{2}(\sin\theta\cos\theta + \theta)\right)\Big|_0^{\pi} - \frac{4}{2}(\sin\theta\cos\theta + \theta)\Big|_0^{\pi/2} \\
&= \frac{3}{2}\pi - 2\cdot\frac{\pi}{2} = \frac{\pi}{2}.
\end{aligned}
$$

Alternatively, we could compute the area of the cardioid and subtract the area of the circle of radius 1 from it.

The integrals can be computed numerically using a calculator, or, as we show, using integration by parts or formula IV-18 from the integral tables.

28. (a) The graph of $r = 2\cos\theta$ is a circle of radius 1 centered at $(1, 0)$; the graph of $r = 2\sin\theta$ is a circle of radius 1 centered at $(0, 1)$. See Figure 8.53.

(b) The Cartesian coordinates of the points of intersection are at $(0, 0)$ and $(1, 1)$.

The origin corresponds to $\theta = \pi/2$ on $r = 2\cos\theta$ and to $\theta = 0$ on $r = 2\sin\theta$. The point $(1, 1)$ has polar coordinates $r = \sqrt{2}, \theta = \pi/4$.

We find the area below the line $\theta = \pi/4$ and above $r = 2\sin\theta$ and double it:

$$\text{Area } = 2 \cdot \frac{1}{2}\int_0^{\pi/4} (2\sin\theta)^2\, d\theta = 4\int_0^{\pi/4} \sin^2\theta\, d\theta.$$

Using a calculator, integration by parts or formula IV-17 from the integral tables,

$$\text{Area } = 4\left(-\frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2}\right)\Big|_0^{\pi/4} = -2\cdot\frac{1}{2} + 2\frac{\pi}{4} = \frac{\pi}{2} - 1.$$

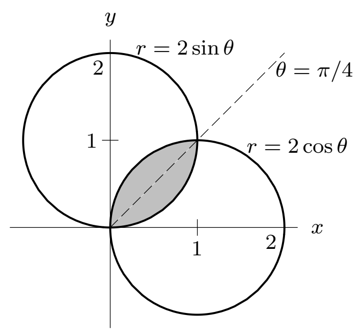


Figure 8.53

29. The area is

$$\begin{aligned} A = \frac{1}{2}\int_0^a r^2\, d\theta = \frac{1}{2}\int_0^a \theta^2\, d\theta &= 1 \\ \frac{1}{2}\left(\frac{\theta^3}{3}\right)\Big|_0^a &= 1 \\ \frac{a^3}{6} &= 1 \\ a^3 &= 6 \\ a &= \sqrt[3]{6}. \end{aligned}$$

30. (a) See Figure 8.54.

(b) The curves intersect when $r^2 = 2$

$$\begin{aligned} 4\cos 2\theta &= 2 \\ \cos 2\theta &= \frac{1}{2}. \end{aligned}$$

In the first quadrant:

$$2\theta = \frac{\pi}{3} \quad \text{so} \quad \theta = \frac{\pi}{6}.$$

Using symmetry, the area in the first quadrant can be multiplied by 4 to find the area of the total bounded region.

$$\begin{aligned} \text{Area } &= 4\left(\frac{1}{2}\right)\int_0^{\pi/6} (4\cos 2\theta - 2)\, d\theta \\ &= 2\left(\frac{4\sin 2\theta}{2} - 2\theta\right)\Big|_0^{\pi/6} \\ &= 4\sin\frac{\pi}{3} - \frac{2}{3}\pi \end{aligned}$$

$$= 4\frac{\sqrt{3}}{2} - \frac{2}{3}\pi$$
$$= 2\sqrt{3} - \frac{2}{3}\pi = 1.370.$$

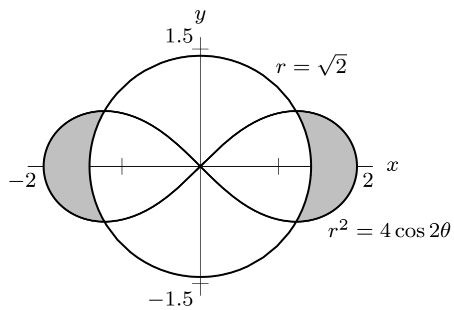


Figure 8.54

31. The slope of the tangent line at $\theta = \pi/3$ is $dy/dx = \sqrt{3}/5$. Since $x = 3\sin(2\theta)\cos\theta$ and $y = 3\sin(2\theta)\sin\theta$, when $\theta = \pi/3$, we have $x = 3\sqrt{3}/4$ and $y = 9/4$. Thus, the equation of the tangent line is

$$y - \frac{9}{4} = \frac{\sqrt{3}}{5}\left(x - \frac{3\sqrt{3}}{4}\right)$$
$$y = \frac{\sqrt{3}}{5}x - \frac{9}{20} + \frac{9}{4}$$
$$y = \frac{\sqrt{3}}{5}x + \frac{9}{5}.$$

32. We first find the points with horizontal and vertical tangents in the first quadrant and then use symmetry to obtain the points in other quadrants.

The slope of the tangent line is

$$\frac{dy}{dx} = \frac{6\cos(2\theta)\sin\theta + 3\sin(2\theta)\cos\theta}{6\cos(2\theta)\cos\theta - 3\sin(2\theta)\sin\theta}.$$

The curve has a horizontal tangent where

$$6\cos(2\theta)\sin\theta + 3\sin(2\theta)\cos\theta = 0.$$

Solving this equation numerically for $0 < \theta < \pi/2$, we have $\theta = 0.9553$; in addition $\theta = 0$ is a solution. Thus, there are horizontal tangents where $x = 1.633$ and $y = 2.309$ and where $x = 0$, $y = 0$. Thus, the five points with horizontal tangents are

$$(1.633, 2.309); \quad (-1.633, 2.309); \quad (-1.633, -2.309); \quad (1.633, -2.309); \quad (0, 0).$$

The curve has vertical tangents where

$$6\cos(2\theta)\cos\theta - 3\sin(2\theta)\sin\theta = 0.$$

Solving this equation numerically for $0 < \theta < \pi/2$, we have $\theta = 0.6155$; in addition $\theta = \pi/2$ is a solution. Thus, there are vertical tangents where $x = 2.309$, $y = 1.633$, and where $x = 0, y = 0$. Thus, there are five points with vertical tangents:

$$(2.309, 1.633); \quad (-2.309, 1.633); \quad (-2.309, -1.633); \quad (2.309, -1.633); \quad (0, 0).$$

33. We can express x and y in terms of θ as a parameter. Since $r = \theta$, we have

$$x = r\cos\theta = \theta\cos\theta \quad \text{and} \quad y = r\sin\theta = \theta\sin\theta.$$

Calculating the slope using the parametric formula,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta},$$

we have

$$\frac{dy}{dx} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}.$$

Horizontal tangents occur where $dy/dx = 0$, so

$$\begin{aligned} \sin\theta + \theta\cos\theta &= 0 \\ \theta &= -\tan\theta. \end{aligned}$$

Solving this equation numerically gives

$$\theta = 0, 2.029, 4.913.$$

Vertical tangents occur where dy/dx is undefined, so

$$\begin{aligned} \cos\theta - \theta\sin\theta &= 0 \\ \theta &= \frac{1}{\tan\theta} = \cot\theta. \end{aligned}$$

Solving this equation numerically gives

$$\theta = 0.860, 3.426.$$

34. (a) Expressing x and y parametrically in terms of θ, we have

$$x = r\cos\theta = \frac{\cos\theta}{\theta} \quad \text{and} \quad y = r\sin\theta = \frac{\sin\theta}{\theta}.$$

The slope of the tangent line is given by

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \left(\frac{\theta\cos\theta - \sin\theta}{\theta^2}\right) \Big/ \left(\frac{-\theta\sin\theta - \cos\theta}{\theta^2}\right) = \frac{\sin\theta - \theta\cos\theta}{\cos\theta + \theta\sin\theta}.$$

At $\theta = \pi/2$, we have

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \frac{1 - (\pi/2)0}{0 + (\pi/2)1} = \frac{2}{\pi}.$$

At $\theta = \pi/2$, we have $x = 0$, $y = 2/\pi$, so the equation of the tangent line is

$$y = \frac{2}{\pi}x + \frac{2}{\pi}.$$

(b) As $\theta \to 0$,

$$x = \frac{\cos\theta}{\theta} \to \infty \quad \text{and} \quad y = \frac{\sin\theta}{\theta} \to 1.$$

Thus, $y = 1$ is a horizontal asymptote. See Figure 8.55.

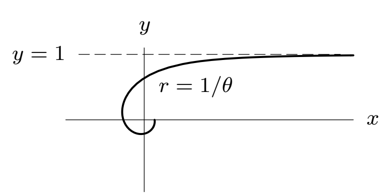


Figure 8.55

35. The limaçon is given by $r = 1+2\cos\theta$; see Figure 8.56. At $\theta = 0$, the graph is at $(3, 0)$; as θ increases, the graph sweeps out the top arc (on which the maximum value of y occurs), reaching the origin when

$$\begin{aligned} 1 + 2\cos\theta &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}. \end{aligned}$$

Thus, we want to find the maximum value of y on the interval $0 \le \theta \le 2\pi/3$. Since $y = r\sin\theta$, we want to find the maximum value of

$$y = (1 + 2\cos\theta)\sin\theta = \sin\theta + 2\cos\theta\sin\theta.$$

At a critical point

$$\begin{aligned} \frac{dy}{d\theta} &= \cos\theta - 2\sin^2\theta + 2\cos^2\theta = 0 \\ \cos\theta - 2(1-\cos^2\theta) + 2\cos^2\theta &= 0 \\ 4\cos^2\theta + \cos\theta - 2 &= 0 \\ \cos\theta &= \frac{-1 \pm \sqrt{33}}{8} = 0.593, -0.843. \end{aligned}$$

Thus, $\theta = \cos^{-1}(0.593) = 0.936$ and $\theta = \cos^{-1}(-0.843) = 2.574$ are the critical values. Since 2.574 is outside the interval $0 \le \theta \le 2\pi/3$, there is one critical point $\theta = 0.963$.

At the endpoints of the interval, $y = 0$. At $\theta = 0.936$, we have $y = 1.760$, which is the maximum value.

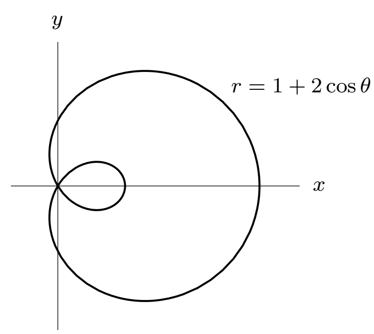


Figure 8.56: The inner loop has $r < 0$

36. Since $x = \theta\cos\theta$ and $y = \theta\sin\theta$, we have

$$\text{Arc length } = \int_0^{2\pi} \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{1+\theta^2}\, d\theta = 21.256.$$

37. Since $x = \cos\theta/\theta$ and $y = \sin\theta/\theta$, we have

$$\begin{aligned} \text{Arc length } &= \int_\pi^{2\pi} \sqrt{\left(\frac{-\theta\sin\theta - \cos\theta}{\theta^2}\right)^2 + \left(\frac{\theta\cos\theta - \sin\theta}{\theta^2}\right)^2}\, d\theta \\ &= \int_\pi^{2\pi} \sqrt{\frac{\theta^2+1}{\theta^4}}\, d\theta = 0.712. \end{aligned}$$

38. Parameterized by θ, the curve $r = f(\theta)$ is given by $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. Then

$$\begin{aligned} \text{Arc length } &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\, d\theta \end{aligned}$$

$$= \int_\alpha^\beta \sqrt{(f'(\theta))^2\cos^2\theta - 2f'(\theta)f(\theta)\cos\theta\sin\theta + (f(\theta))^2\sin^2\theta}$$
$$\overline{+(f'(\theta))^2\sin^2\theta + 2f'(\theta)f(\theta)\sin\theta\cos\theta + (f(\theta))^2\cos^2\theta}\,d\theta$$
$$= \int_\alpha^\beta \sqrt{(f'(\theta))^2(\cos^2\theta+\sin^2\theta) + (f(\theta))^2(\sin^2\theta+\cos^2\theta)}\,d\theta$$
$$= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\,d\theta.$$

Solutions for Section 8.4

Exercises

1. Since density is e^{-x} gm/cm,

$$\text{Mass } = \int_0^{10} e^{-x}\,dx = -e^{-x}\Big|_0^{10} = 1 - e^{-10} \text{ gm.}$$

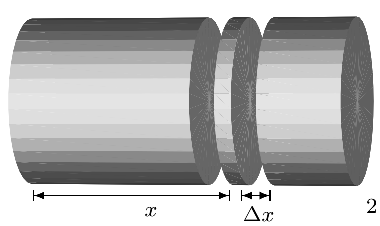
2. Strips perpendicular to the x-axis have length 3, area $3\Delta x$, and mass $5 \cdot 3\Delta x$ gm. Thus

$$\text{Mass } = \int_0^2 5\cdot 3\,dx = \int_0^2 15\,dx.$$

Strips perpendicular to the y-axis have length 2, area $2\Delta y$, and mass $5\cdot 2\Delta y$ gm. Thus

$$\text{Mass } = \int_0^3 5\cdot 2\,dy = \int_0^3 10\,dy.$$

3. (a) Suppose we choose an x, $0 \le x \le 2$. If Δx is a small fraction of a meter, then the density of the rod is approximately $\delta(x)$ anywhere from x to $x + \Delta x$ meters from the left end of the rod (see below). The mass of the rod from x to $x+\Delta x$ meters is therefore approximately $\delta(x)\Delta x = (2+6x)\Delta x$. If we slice the rod into N pieces, then a Riemann sum is $\displaystyle\sum_{i=1}^{N}(2+6xi)\Delta x$.



(b) The definite integral is

$$M = \int_0^2 \delta(x)\,dx = \int_0^2 (2+6x)\,dx = (2x+3x^2)\Big|_0^2 = 16 \text{ grams.}$$

4. We have

$$\text{Moment } = \int_0^2 x\delta(x)\,dx = \int_0^2 x(2+6x)\,dx$$
$$= \int_0^2 (6x^2+2x)\,dx = (2x^3+x^2)\Big|_0^2 = 20 \text{ gram-meters.}$$

Now, using this and Problem 3 (b), we have

$$\text{Center of mass} = \frac{\text{Moment}}{\text{Mass}} = \frac{20 \text{ gram-meters}}{16 \text{ grams}} = \frac{5}{4} \text{ meters (from its left end).}$$

5. (a) Figure 8.57 shows a graph of the density function.

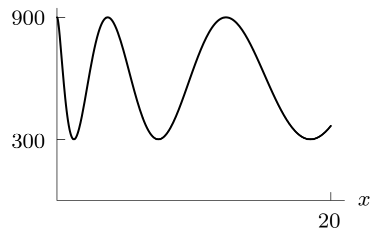


Figure 8.57

(b) Suppose we choose an x, $0 \le x \le 20$. We approximate the density of the number of the cars between x and $x + \Delta x$ miles as $\delta(x)$ cars per mile. Therefore, the number of cars between x and $x + \Delta x$ is approximately $\delta(x)\Delta x$. If we slice the 20 mile strip into N slices, we get that the total number of cars is

$$C \approx \sum_{i=1}^{N} \delta(x_i)\Delta x = \sum_{i=1}^{N} \left[600 + 300\sin(4\sqrt{x_i + 0.15})\right] \Delta x,$$

where $\Delta x = 20/N$. (This is a right-hand approximation; the corresponding left-hand approximation is $\sum_{i=0}^{N-1} \delta(x_i)\Delta x$.)

(c) As $N \to \infty$, the Riemann sum above approaches the integral

$$C = \int_0^{20} (600 + 300\sin 4\sqrt{x + 0.15})\, dx.$$

If we calculate the integral numerically, we find $C \approx 11513$. We can also find the integral exactly as follows:

$$\begin{aligned} C &= \int_0^{20} (600 + 300\sin 4\sqrt{x + 0.15})\, dx \\ &= \int_0^{20} 600\, dx + \int_0^{20} 300 \sin 4\sqrt{x + 0.15}\, dx \\ &= 12000 + 300 \int_0^{20} \sin 4\sqrt{x + 0.15}\, dx. \end{aligned}$$

Let $w = \sqrt{x + 0.15}$, so $x = w^2 - 0.15$ and $dx = 2w\, dw$. Then

$$\begin{aligned} \int_{x=0}^{x=20} \sin 4\sqrt{x + 0.15}\, dx &= 2\int_{w=\sqrt{0.15}}^{w=\sqrt{20.15}} w \sin 4w\, dw, \text{ (using integral table III-15)} \\ &= 2\left[-\frac{1}{4} w \cos 4w + \frac{1}{16}\sin 4w\right]\Bigg|_{\sqrt{0.15}}^{\sqrt{20.15}} \\ &\approx -1.624. \end{aligned}$$

Using this, we have $C \approx 12000 + 300(-1.624) \approx 11513$, which matches our numerical approximation.

6. (a) Orient the rectangle in the coordinate plane in such a way that the side referred to in the problem—call it S—lies on the y-axis from $y = 0$ to $y = 5$, as shown in Figure 8.58. We may subdivide the rectangle into strips of width Δx and length 5. If the left side of a given strip is a distance x away from S (i.e., the y-axis), its density 2 is $1/(1 + x^4)$. If Δx is small enough, the density of the strip is approximately constant—i.e., the density of the whole strip is about $1/(1 + x^4)$. The mass of the strip is just its density times its area, or $5\Delta x/(1 + x^4)$. Thus the mass of the whole rectangle is approximated by the Riemann sum

$$\sum \frac{5\Delta x}{1 + x^4},$$

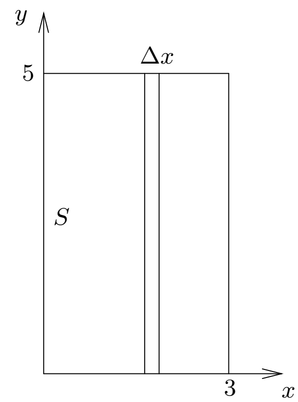


Figure 8.58

(b) The exact mass of the rectangle is obtained by letting $\Delta x \to 0$ in the Riemann sums above, giving us the integral

$$\int_0^3 \frac{5\,dx}{1+x^4}.$$

Since it is not easy to find an antiderivative of $5/(1+x^4)$, we evaluate this integral numerically, getting 5.5.

7. The total mass is 7 grams. The center of mass is given by

$$\overline{x} = \frac{2(-3)+5(4)}{7} = 2 \text{ cm to right of origin.}$$

8. The total mass is 9 gm, and so the center of mass is located at $\overline{x} = \frac{1}{9}(-10 \cdot 5 + 1 \cdot 3 + 2 \cdot 1) = -5$.

Problems

9. Since the density varies with x, the region must be sliced perpendicular to the x-axis. This has the effect of making the density approximately constant on each strip. See Figure 8.59. Since a strip is of height y, its area is approximately $y\Delta x$. The density on the strip is $\delta(x) = 1 + x$ gm/cm^2. Thus

$$\text{Mass of strip} \approx \text{Density} \cdot \text{Area} \approx (1+x)y\Delta x \text{ gm.}$$

Because the tops of the strips end on two different lines, one for $x \geq 0$ and the other for $x < 0$, the mass is calculated as the sum of two integrals. See Figure 8.59. For the left part of the region, $y = x + 1$, so

$$\begin{aligned}\text{Mass of left part} &= \lim_{\Delta x \to 0} \sum (1+x)y\Delta x = \int_{-1}^{0} (1+x)(x+1)\,dx \\ &= \int_{-1}^{0} (1+x)^2\,dx = \left.\frac{(x+1)^3}{3}\right|_{-1}^{0} = \frac{1}{3} \text{ gm.}\end{aligned}$$

From Figure 8.59, we see that for the right part of the region, $y = -x + 1$, so

$$\begin{aligned}\text{Mass of right part} &= \lim_{\Delta x \to 0} \sum (1+x)y\Delta x = \int_{0}^{1} (1+x)(-x+1)\,dx \\ &= \int_{0}^{1} (1-x^2)\,dx = x - \left.\frac{x^3}{3}\right|_{0}^{1} = \frac{2}{3} \text{ gm.}\end{aligned}$$

$$\text{Total mass} = \frac{1}{3} + \frac{2}{3} = 1 \text{ gm.}$$

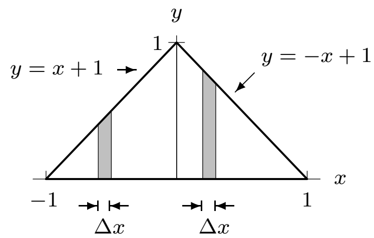


Figure 8.59

10. (a) Partition $[0, 10{,}000]$ into N subintervals of width Δr. The area in the i^{th} subinterval is $\approx 2\pi r_i \Delta r$. So the total mass in the slick $= M \approx \sum_{i=1}^{N} 2\pi r_i \left(\frac{50}{1+r_i}\right) \Delta r$.

(b) $M = \displaystyle\int_0^{10{,}000} 100\pi \frac{r}{1+r} dr$. We may rewrite $\dfrac{r}{1+r}$ as $\frac{1+r}{1+r} - \frac{1}{1+r} = 1 - \frac{1}{1+r}$, so that

$$M = \int_0^{10{,}000} 100\pi(1 - \frac{1}{1+r})\, dr = 100\pi \left(r - \ln|1+r| \Big|_0^{10{,}000} \right)$$
$$= 100\pi(10{,}000 - \ln(10{,}001)) \approx 3.14 \times 10^6 \text{ kg}.$$

(c) We wish to find an R such that

$$\int_0^R 100\pi \frac{r}{1+r} dr = \frac{1}{2} \int_0^{10{,}000} 100\pi \frac{r}{1+r} dr \approx 1.57 \times 10^6.$$

So $100\pi(R - \ln|R+1|) \approx 1.57 \times 10^6$; $R - \ln|R+1| \approx 5000$. By trial and error, we find $R \approx 5009$ meters.

11. (a) We form a Riemann sum by slicing the region into concentric rings of radius r and width Δr. Then the volume deposited on one ring will be the height $H(r)$ multiplied by the area of the ring. A ring of width Δr will have an area given by

$$\begin{aligned} \text{Area} &= \pi(r+\Delta r)^2 - \pi(r^2) \\ &= \pi(r^2 + 2r\Delta r + (\Delta r)^2 - r^2) \\ &= \pi(2r\Delta r + (\Delta r)^2). \end{aligned}$$

Since Δr is approaching zero, we can approximate

$$\text{Area of ring} \approx \pi(2r\Delta r + 0) = 2\pi r \Delta r.$$

From this, we have

$$\Delta V \approx H(r) \cdot 2\pi r \Delta r.$$

Thus, summing the contributions from all rings we have

$$V \approx \sum H(r) \cdot 2\pi r \Delta r.$$

Taking the limit as $\Delta r \to 0$, we get

$$V = \int_0^5 2\pi r \left(0.115 e^{-2r}\right) dr.$$

(b) We use integration by parts:

$$\begin{aligned} V &= 0.23\pi \int_0^5 \left(re^{-2r}\right) dr \\ &= 0.23\pi \left(\frac{re^{-2r}}{-2} - \frac{e^{-2r}}{4} \right) \Bigg|_0^5 \\ &\approx 0.181(\text{millimeters}) \cdot (\text{kilometers})^2 = 0.181 \cdot 10^{-3} \cdot 10^6 \text{ meters}^3 = 181 \text{ cubic meters}. \end{aligned}$$

12. Partition $a \leq x \leq b$ into N subintervals of width $\Delta x = \dfrac{(b-a)}{N}$; $a = x_0 < x_1 < \cdots < x_N = b$. The mass of the strip on the ith subinterval is approximately $m_i = \delta(x_i)[f(x_i) - g(x_i)]\Delta x$. If we use a right-hand Riemann sum, the approximation for the total mass is
$\displaystyle\sum_{i=1}^{N} \delta(x_i)[f(x_i) - g(x_i)]\Delta x$, and the exact mass is $M = \displaystyle\int_a^b \delta(x)[f(x) - g(x)]dx$.

13. (a) Use the formula for the volume of a cylinder:

$$\text{Volume} = \pi r^2 l.$$

Since it is only a half cylinder

$$\text{Volume of shed} = \frac{1}{2}\pi r^2 l.$$

(b) Set up the axes as shown in Figure 8.60. The density can be defined as

$$\text{Density} = ky.$$

Now slice the sawdust horizontally into slabs of thickness Δy as shown in Figure 8.61, and calculate

$$\text{Volume of slab} \approx 2xl\Delta y = 2l(\sqrt{r^2 - y^2})\Delta y.$$

$$\text{Mass of slab} = \text{Density} \cdot \text{Volume} \approx 2kly\sqrt{r^2 - y^2}\Delta y.$$

Finally, we compute the total mass of sawdust:

$$\text{Total mass of sawdust} = \int_0^r 2kly\sqrt{r^2 - y^2}\,dy = -\frac{2}{3}kl(r^2 - y^2)^{3/2}\Bigg|_0^r = \frac{2klr^3}{3}.$$

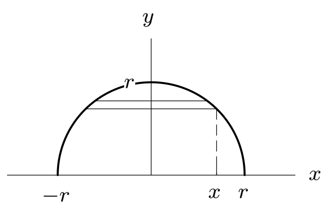


Figure 8.60

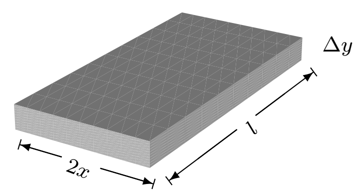


Figure 8.61

14. First we rewrite the chart, listing the density with the corresponding distance from the center of the earth (x km below the surface is equivalent to $6370 - x$ km from the center):

This gives us spherical shells whose volumes are $\frac{4}{3}\pi(r_i^3 - r_{i+1}^3)$ for any two consecutive distances from the origin. We will assume that the density of the earth is increasing with depth. Therefore, the average density of the i^{th} shell is between D_i and D_{i+1}, the densities at top and bottom of shell i. So $\frac{4}{3}\pi D_{i+1}(r_i^3 - r_{i+1}^3)$ and $\frac{4}{3}\pi D_i(r_i^3 - r_{i+1}^3)$ are upper and lower bounds for the mass of the shell.

Table 8.5

i	x_i	$r_i = 6370 - x_i$	D_i
0	0	6370	3.3
1	1000	5370	4.5
2	2000	4370	5.1
3	2900	3470	5.6
4	3000	3370	10.1
5	4000	2370	11.4
6	5000	1370	12.6
7	6000	370	13.0
8	6370	0	13.0

To get a rough approximation of the mass of the earth, we don't need to use all the data. Let's just use the densities at $x = 0$, 2900, 5000 and 6370 km. Calculating an upper bound on the mass,

$$M_U = \frac{4}{3}\pi[13.0(1370^3 - 0^3) + 12.6(3470^3 - 1370^3) + 5.6(6370^3 - 3470^3)] \cdot 10^{15} \approx 7.29 \times 10^{27} \text{ g}.$$

The factor of 10^{15} may appear unusual. Remember the radius is given in kilometers and the density is given in g/cm^3, so we must convert kilometers to centimeters: 1 km $= 10^5$ cm , so 1 km$^3 = 10^{15}$ cm^3.

The lower bound is

$$M_L = \frac{4}{3}\pi[12.6(1370^3 - 0^3) + 5.6(3470^3 - 1370^3) + 3.3(6370^3 - 3470^3)] \cdot 10^{15} \approx 4.05 \times 10^{27} \text{ g}.$$

Here, our upper bound is just under 2 times our lower bound.

Using all our data, we can find a more accurate estimate. The upper and lower bounds are

$$M_U = \frac{4}{3}\pi \sum_{i=0}^{7} D_{i+1}(r_i^3 - r_{i+1}^3) \cdot 10^{15} \text{ g}$$

and

$$M_L = \frac{4}{3}\pi \sum_{i=0}^{7} D_i(r_i^3 - r_{i+1}^3) \cdot 10^{15} \text{ g}.$$

We have

$$\begin{aligned} M_U = \frac{4}{3}\pi[&4.5(6370^3 - 5370^3) + 5.1(5370^3 - 4370^3) + 5.6(4370^3 - 3470^3) \\ &+ 10.1(3470^3 - 3370^3) + 11.4(3370^3 - 2370^3) + 12.6(2370^3 - 1370^3) \\ &+ 13.0(1370^3 - 370^3) + 13.0(370^3 - 0^3)] \cdot 10^{15} \text{ g} \\ &\approx 6.50 \times 10^{27} \text{ g} \end{aligned}$$

and

$$\begin{aligned} M_L = \frac{4}{3}\pi[&3.3(6370^3 - 5370^3) + 4.5(5370^3 - 4370^3) + 5.1(4370^3 - 3470^3) \\ &+ 5.6(3470^3 - 3370^3) + 10.1(3370^3 - 2370^3) + 11.4(2370^3 - 1370^3) \\ &+ 12.6(1370^3 - 370^3) + 13.0(370^3 - 0^3)] \cdot 10^{15} \text{ g} \\ &\approx 5.46 \times 10^{27} \text{ g}. \end{aligned}$$

15. We slice time into small intervals. Since t is given in seconds, we convert the minute to 60 seconds. We consider water loss over the time interval $0 \le t \le 60$. We also need to convert inches into feet since the velocity is given in ft/sec. Since 1 inch = $1/12$ foot, the square hole has area $1/144$ square feet. For water flowing through a hole with constant velocity v, the amount of water which has passed through in some time, Δt, can be pictured as the rectangular solid in Figure 8.62, which has volume

$$\text{Area} \cdot \text{Height} = \text{Area} \cdot \text{Velocity} \cdot \text{Time}.$$

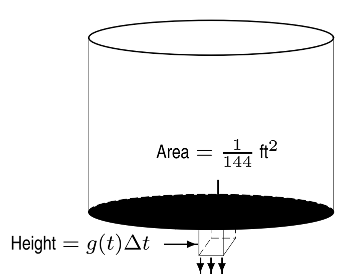


Figure 8.62: Volume of water passing through hole

Over a small time interval of length Δt, starting at time t, water flows with a nearly constant velocity $v = g(t)$ through a hole $1/144$ square feet in area. In Δt seconds, we know that

$$\text{Water lost } \approx \left(\frac{1}{144}\text{ ft}^2\right)(g(t)\text{ ft/sec})(\Delta t\text{ sec}) = \left(\frac{1}{144}\right) g(t)\,\Delta t\text{ ft}^3.$$

Adding the water from all subintervals gives

$$\text{Total water lost } \approx \sum \frac{1}{144} g(t)\,\Delta t\text{ ft}^3.$$

As $\Delta t \to 0$, the sum tends to the definite integral:

$$\text{Total water lost} = \int_0^{60} \frac{1}{144} g(t)\,dt\text{ ft}^3.$$

16. (a) Divide the atmosphere into spherical shells of thickness Δh. See Figure 8.63. The density on a typical shell, $\rho(h)$, is approximately constant. The volume of the shell is approximately the surface area of a sphere of radius $r_e + h$ meters times Δh, where $r_e = 6.4 \cdot 10^6$ meters is the radius of the earth,

$$\text{Volume of Shell } \approx 4\pi(r_e + h_i)^2 \Delta h.$$

A Riemann sum for the total mass is

$$\text{Mass } \approx \sum 4\pi(r_e + h)^2 \times 1.28e^{-0.000124 h_i}\,\Delta h\text{ kg}.$$

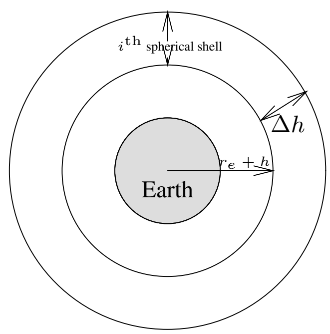


Figure 8.63

(b) This Riemann sum becomes the integral

$$\begin{aligned}\text{Mass } &= 4\pi \int_0^{100} (r_e + h)^2 \cdot 1.28e^{-0.000124h}\,dh \\ &= 4\pi \int_0^{100} (6.4 \cdot 10^6 + h)^2 \cdot 1.28e^{-0.000124h}\,dh.\end{aligned}$$

Evaluating the integral using numerical methods gives $M = 6.5 \cdot 10^{16}$ kg.

17. We need the numerator of $\overline{x}$, to be zero, i.e. $\sum x_i m_i = 0$. Since all of the masses are the same, we can factor them out and write $4\sum x_i = 0$. Thus the fourth mass needs to be placed so that all of the positions sum to zero. The first three positions sum to $(-6 + 1 + 3) = -2$, so the fourth mass needs to be placed at $x = 2$.

18. We have

$$\text{Total mass of the rod } = \int_0^3 (1 + x^2)\,dx = \left[x + \frac{x^3}{3}\right]\Bigg|_0^3 = 12\text{ grams}.$$

In addition,

$$\text{Moment } = \int_0^3 x(1 + x^2)\,dx = \left[\frac{x^2}{2} + \frac{x^4}{4}\right]\Bigg|_0^3 = \frac{99}{4}\text{ gram-meters}.$$

Thus, the center of mass is at the position $\bar{x} = \frac{99/4}{12} = 2.06$ meters.

19. The center of mass is
$$\bar{x} = \frac{\int_0^\pi x(2+\sin x)\,dx}{\int_0^\pi (2+\sin x)\,dx}.$$
The numerator is $\int_0^\pi (2x + x\sin x)\,dx = (x^2 - x\cos x + \sin x)\Big|_0^\pi = \pi^2 + \pi.$

The denominator is $\int_0^\pi (2+\sin x)\,dx = (2x - \cos x)\Big|_0^\pi = 2\pi + 2$. So the center of mass is at
$$\overline{x} = \frac{\pi^2+\pi}{2\pi+2} = \frac{\pi(\pi+1)}{2(\pi+1)} = \frac{\pi}{2}.$$

20. (a) We find that
$$\text{Moment } = \int_0^1 x(1+kx^2)\,dx = \left(\frac{x^2}{2} + \frac{kx^4}{4}\right)\Bigg|_0^1 = \frac{1}{2} + \frac{k}{4} \text{ gram-meters,}$$
and that
$$\text{Total mass } = \int_0^1 (1+kx^2)\,dx = \left(x + \frac{kx^3}{3}\right)\Bigg|_0^1 = 1 + \frac{k}{3} \text{ grams.}$$
Thus, the center of mass is
$$\bar{x} = \frac{\frac{1}{2}+\frac{k}{4}}{1+\frac{k}{3}} = \frac{3}{4}\left(\frac{2+k}{3+k}\right) \text{ meters.}$$
(b) Let $f(k) = \frac{3}{4}\left(\frac{2+k}{3+k}\right)$. Then $f'(k) = \frac{3}{4}\left(\frac{1}{(3+k)^2}\right)$, which is always positive, so f is an increasing function of k. Since $f(0) = 0.5$, this is the smallest value of f. As $k \to \infty$, $f(k) \to 3/4 = 0.75$. So $f(k)$ is always between 0.5 and 0.75.

21. (a) The density is minimum at $x = -1$ and increases as x increases, so more of the mass of the rod is in the right half of the rod. We thus expect the balancing point to be to the right of the origin.

(b) We need to compute
$$\int_{-1}^1 x(3-e^{-x})\,dx = \left(\frac{3}{2}x^2 + xe^{-x} + e^{-x}\right)\Bigg|_{-1}^1 \quad \text{(using integration by parts)}$$
$$= \frac{3}{2} + e^{-1} + e^{-1} - \left(\frac{3}{2} - e^1 + e^1\right) = \frac{2}{e}.$$
We must divide this result by the total mass, which is given by
$$\int_{-1}^1 (3-e^{-x})\,dx = (3x + e^{-x})\Big|_{-1}^1 = 6 - e + \frac{1}{e}.$$
We therefore have
$$\bar{x} = \frac{2/e}{6-e+(1/e)} = \frac{2}{1+6e-e^2} \approx 0.2.$$

22. Since the region is symmetric about the x-axis, $\overline{y} = 0$.

To find $\bar{x}$, we first find the density. The area of the disk is $\pi/2$ m^2, so it has density $3/(\pi/2) = 6/\pi$ kg/m^2. We find the mass of the small strip of width Δx in Figure 8.64. The height of the strip is $\sqrt{1-x^2}$, so
$$\text{Area of the small strip } \approx A_x(x)\Delta x = 2\cdot\sqrt{1-x^2}\Delta x \text{ m}^2.$$
When multiplied by the density $6/\pi$, we get
$$\text{Mass of the strip } \approx \frac{12}{\pi}\cdot\sqrt{1-x^2}\Delta x \text{ kg.}$$
We then sum the product of these masses with x, and take the limit as $\Delta x \to 0$ to get
$$\text{Moment } = \int_0^1 \frac{12}{\pi}x\sqrt{1-x^2}\,dx = -\frac{4}{\pi}(1-x^2)^{3/2}\Bigg|_0^1 = \frac{4}{\pi} \text{ meter.}$$

Finally, we divide by the total mass 3 kg to get the result $\bar{x} = 4/(3\pi)$ meters.

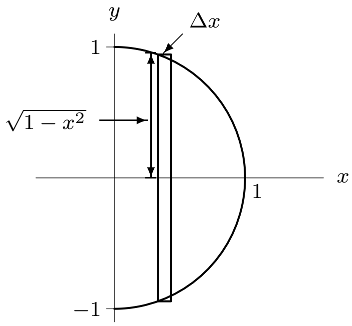


Figure 8.64: Area of a small strip

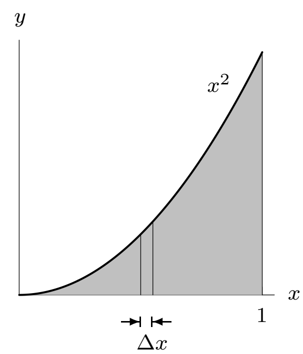


Figure 8.65

23. (a) Since the density is constant, the mass is the product of the area of the plate and its density.

$$\text{Area of the plate } = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}\text{ cm}^2.$$

Thus the mass of the plate is $2 \cdot 1/3 = 2/3$ gm.

(b) See Figure 8.65. Since the region is "fatter" closer to $x = 1$, $\bar{x}$ is greater than $1/2$.

(c) To find the center of mass along the x-axis, we slice the region into vertical strips of width Δx. See Figure 8.65. Then

$$\text{Area of strip } = A_x(x)\Delta x \approx x^2\Delta x$$

Then, since the density is 2 gm/cm^2, we have

$$\overline{x} = \frac{\int_0^1 2x^3\,dx}{2/3} = \frac{3}{2}\cdot\left.\frac{2x^4}{4}\right|_0^1 = 3\left(\frac{1}{4}\right) = \frac{3}{4}\text{cm}.$$

This is greater than $1/2$, as predicted in part (b).

24. (a) Since the density is constant, the mass is the product of the area of the plate and its density.

$$\text{Area of the plate } = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^1 = \frac{2}{3}\text{ cm}^2.$$

Thus the mass of the plate is $5 \cdot 2/3 = 10/3$ gm.

(b) To find $\bar{x}$, we slice the region into vertical strips of width Δx. See Figure 8.66. Then

$$\text{Area of strip } = A_x(x)\Delta x \approx \sqrt{x}\Delta x\text{ cm}^2.$$

Then, since the density is 5 gm/cm^2, we have

$$\overline{x} = \frac{\int x\delta A_x(x)\,dx}{\text{Mass}} = \frac{\int_0^1 5x^{3/2}\,dx}{10/3} = \frac{3}{10}\;2x^{5/2}\Big|_0^1 = \frac{3}{5}\text{ cm}.$$

To find $\bar{y}$, we slice the region into horizontal strips of width Δy

$$\text{Area of horizontal strip } = A_y(y)\Delta y \approx (1-x)\Delta y = (1-y^2)\Delta y\text{ cm}^2.$$

Then, since the density is 5 gm/cm^2, we have

$$\overline{y} = \frac{\int y\delta A_y(y)\,dy}{\text{Mass}} = \frac{\int_0^1 5(y-y^3)\,dy}{10/3} = \frac{3}{10}\,5\left(\frac{y^2}{2}-\frac{y^4}{4}\right)\Big|_0^1 = \frac{3}{10}\cdot\frac{5}{4} = \frac{3}{8}\text{ cm}.$$

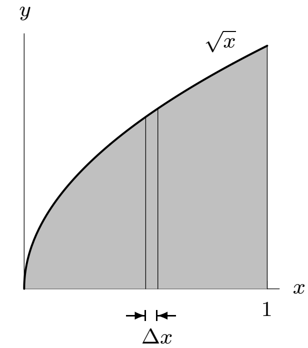


Figure 8.66

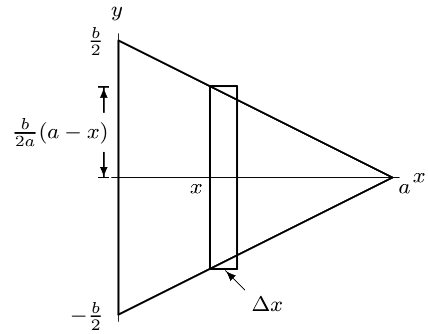


Figure 8.67

25. The triangle is symmetric about the x axis, so $\bar{y} = 0$.

To find $\bar{x}$, we first calculate the density. The area of the triangle is $ab/2$, so it has density $2m/(ab)$ where m is the total mass of the triangle. We need to find the mass of a small strip of width Δx located at x_i (see Figure 8.67).

$$\text{Area of the small strip } \approx A_x(x)\Delta x = 2 \cdot \frac{b(a-x)}{2a}\Delta x.$$

Multiplying by the density $2m/(ab)$ gives

$$\text{Mass of the strip } \approx 2m\frac{(a-x)}{a^2}\Delta x.$$

We then sum the product of these masses with x_i, and take the limit as $\Delta x \to 0$ to get

$$\text{Moment } = \int_0^a \frac{2mx(a-x)}{a^2}\,dx = \frac{2m}{a^2}\left(\frac{ax^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^a = \frac{2m}{a^2}\left(\frac{a^3}{2} - \frac{a^3}{3}\right) = \frac{ma}{3}.$$

Finally, we divide by the total mass m to get the desired result $\bar{x} = a/3$, which is independent of the length of the base b.

26. Stand the cone with the base horizontal, with center at the origin. Symmetry gives us that $\bar{x} = \bar{y} = 0$. Since the cone is fatter near its base we expect the center of mass to be nearer to the base.

Slice the cone into disks parallel to the xy-plane.

As we saw in Example 2 on page 369, a disk of thickness Δz at height z above the base has

$$\text{Volume of disk } = A_z(z)\Delta z \approx \pi(5-z)^2\Delta z \text{ cm}^3.$$

Thus, since the density is δ,

$$\overline{z} = \frac{\int z\delta A_z(z)\,dz}{\text{Mass}} = \frac{\int_0^5 z\cdot\delta\pi(5-z)^2\,dz}{\text{Mass}} \text{ cm.}$$

To evaluate the integral in the numerator, we factor out the constant density δ and π to get

$$\int_0^5 z\cdot\delta\pi(5-z)^2\,dz = \delta\pi\int_0^5 z(25-10z+z^2)\,dz = \delta\pi\left(\frac{25z^2}{2} - \frac{10z^3}{3} + \frac{z^4}{4}\right)\Bigg|_0^5 = \frac{625}{12}\delta\pi.$$

We divide this result by the total mass of the cone, which is $\left(\frac{1}{3}\pi 5^2\cdot 5\right)\delta$:

$$\overline{z} = \frac{\frac{625}{12}\delta\pi}{\frac{1}{3}\pi 5^3\delta} = \frac{5}{4} = 1.25\,\text{cm}.$$

As predicted, the center of mass is closer to the base of the cone than its top.

27. Since the density is constant, the total mass of the solid is the product of the volume of the solid and its density: $\delta\pi(1 - e^{-2})/2$. By symmetry, $\bar{y} = 0$. To find $\bar{x}$, we slice the solid into disks of width Δx, perpendicular to the x-axis. See Figure 8.68. A disk at x has radius $y = e^{-x}$, so

$$\text{Volume of disk } = A_x(x)\Delta x = \pi y^2\Delta x = \pi e^{-2x}\Delta x.$$

Since the density is δ, we have

$$\overline{x} = \frac{\int_0^1 x \cdot \delta\pi e^{-2x} dx}{\text{Total mass}} = \frac{\delta\pi \int_0^1 xe^{-2x} dx}{\delta\pi(1-e^{-2})/2} = \frac{2}{1-e^{-2}} \int_0^1 xe^{-2x}\, dx.$$

The integral $\int xe^{-2x}\, dx$ can be done by parts: let $u = x$ and $v' = e^{-2x}$. Then $u' = 1$ and $v = e^{-2x}/(-2)$. So

$$\int xe^{-2x}\, dx = \frac{xe^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}\, dx = \frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4}.$$

and then

$$\int_0^1 xe^{-2x}\, dx = \left(\frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4}\right)\Bigg|_0^1 = \left(\frac{e^{-2}}{-2} - \frac{e^{-2}}{4}\right) - \left(0 - \frac{1}{4}\right) = \frac{1-3e^{-2}}{4}.$$

The final result is:

$$\overline{x} = \frac{2}{1-e^{-2}} \cdot \frac{1-3e^{-2}}{4} = \frac{1-3e^{-2}}{2-2e^{-2}} \approx 0.343.$$

Notice that $\bar{x}$ is less that $1/2$, as we would expect from the fact that the solid is wider near the origin.

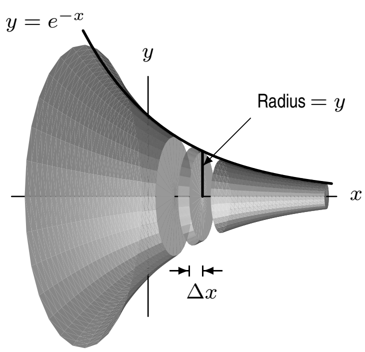


Figure 8.68

28. (a) Position the pyramid so that the center of its base lies at the origin on the xy-plane. Slice the pyramid into square slabs parallel to its base. We compute the mass of the pyramid by adding the masses of the slabs.

The mass of a slab is its volume multiplied by the density δ. To compute the volume of a slab, we need to get an expression for the side s of the slab in terms of its height z. Using the similar triangles in Figure 8.69, we see that

$$\frac{s}{40} = \frac{(10-z)}{10}.$$

Thus $s = 4(10-z)$. Since the area of the square slab's face is s^2,

$$\text{Volume of the slab } \approx A_z(z)\Delta z = s^2\Delta z = 16(10-z)^2\Delta z.$$

$$\text{Mass of slab } = 16\delta(10-z)^2\Delta z.$$

The mass of the pyramid can be found by summing all of the masses of the slabs, and letting the thickness Δz approach zero:

$$\text{Total mass } = \lim_{\Delta z \to 0} \sum 16\delta(10-z)^2\Delta z = \int_0^{10} 16\delta(10-z)^2\, dz = \frac{-16\delta(10-z)^3}{3}\Bigg|_0^{10} = \frac{16000\delta}{3}\text{gm}.$$

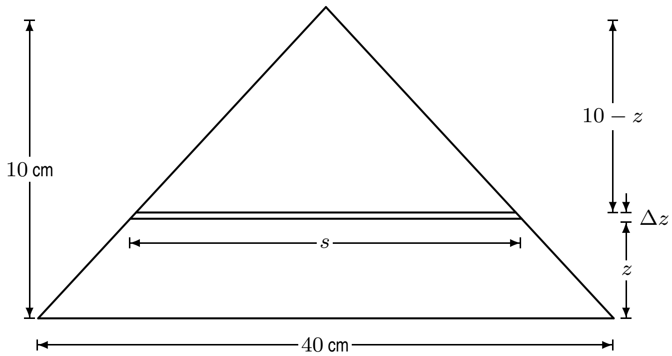


Figure 8.69

(b) From symmetry, we have $\bar{x} = \bar{y} = 0$. Since the pyramid is fatter near its base we expect the center of mass to be nearer to the base. Since

$$\text{Volume of slab } = A_z(z)\Delta z = 16(10-z)^2\Delta z,$$
$$\overline{z} = \frac{\int_0^{10} z \cdot 16\delta(10-z)^2\,dz}{\text{Total mass}}.$$

To evaluate the integral in the numerator, we factor out the constant 16δ and expand the integrand to get

$$16\delta\int_0^{10}(100z - 20z^2 + z^3)\,dz = 16\delta\left(50z^2 + \frac{-20z^3}{3} + \frac{z^4}{4}\right)\Bigg|_0^{10} = \frac{40000\delta}{3}.$$

We divide this result by the total mass $16000\delta/3$ of the pyramid

$$\overline{z} = \frac{40000\delta/3}{16000\delta/3} = \frac{40000}{16000} = 2.5\,\text{cm}.$$

As predicted, the center of mass is closer to the base of the pyramid than its top.

Solutions for Section 8.5

Exercises

1. The work done is given by

$$W = \int_1^2 3x\,dx = \frac{3}{2}x^2\Big|_1^2 = \frac{9}{2}\text{ joules.}$$

2. The work done is given by

$$W = \int_0^3 3x\,dx = \frac{3}{2}x^2\Big|_0^3 = \frac{27}{2}\text{ joules.}$$

3. (a) For compression from $x = 0$ to $x = 1$,

$$\text{Work } = \int_0^1 3x\,dx = \frac{3}{2}x^2\Big|_0^1 = \frac{3}{2} = 1.5\text{ joules.}$$

For compression from $x = 4$ to $x = 5$,

$$\text{Work } = \int_4^5 3x\,dx = \frac{3}{2}x^2\Big|_4^5 = \frac{3}{2}(25-16) = \frac{27}{2} = 13.5\text{ joules.}$$

(b) The second answer is larger. Since the force increases with x, for a given displacement, the work done is larger for larger x values. Thus, we expect more work to be done in moving from $x = 4$ to $x = 5$ than from $x = 0$ to $x = 1$.

4. Since the gravitational force is

$$F = \frac{4\cdot 10^{14}}{r^2} \text{ newtons}$$

and r varies between $6.4\cdot 10^6$ and $7.4\cdot 10^6$ meters,

$$\begin{aligned}\text{Work done } &= \int_{6.4\cdot 10^6}^{7.4\cdot 10^6} \frac{4\cdot 10^{14}}{r^2}\,dr = -4.10^{14}\frac{1}{r}\bigg|_{6.4\cdot 10^6}^{7.4\cdot 10^6} \\ &= 4\cdot 10^{14}\left(\frac{1}{6.4\cdot 10^6} - \frac{1}{7.4\cdot 10^6}\right) = 8.4\cdot 10^6 \text{ joules.}\end{aligned}$$

5. The force exerted on the satellite by the earth (and vice versa!) is GMm/r^2, where r is the distance from the center of the earth to the center of the satellite, m is the mass of the satellite, M is the mass of the earth, and G is the gravitational constant. So the total work done is

$$\int_{6.4\cdot 10^6}^{8.4\cdot 10^6} F\,dr = \int_{6.4\cdot 10^6}^{8.4\cdot 10^6} \frac{GMm}{r^2}\,dr = \left(\frac{-GMm}{r}\right)\bigg|_{6.4\cdot 10^6}^{8.4\cdot 10^6} \approx 1.489\cdot 10^{10} \text{ joules.}$$

Problems

6. Let x be the distance from ground to the bucket of cement. At height x, if the bucket is lifted by Δx, the work done is $500\Delta x + 0.5(75-x)\Delta x$. See Figure 8.70. The $500\Delta x$ term is due to the bucket of cement; the $0.5(75-x)\Delta x$ term is due to the remaining cable. So the total work, W, required to lift the bucket is

$$\begin{aligned}W &= \int_0^{30} 500dx + \int_0^{30} 0.5(75-x)dx \\ &= 500\cdot 30 + 0.5(75\cdot 30 - \frac{1}{2}30^2) \\ &= 15{,}900 \text{ ft-lb.}\end{aligned}$$

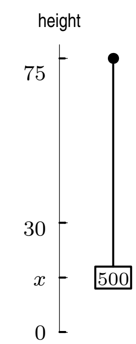


Figure 8.70

7. When the anchor has been lifted through h feet, the length of chain in the water is $25-h$ feet, so the total weight of the anchor and chain in the water is $50+3(25-h)$ lb. Then

$$\begin{aligned}\text{Work to lift the anchor and chain } \Delta h \text{ higher} &= \text{Weight}\cdot\text{Distance lifted} \\ &= (100+3(25-h))\Delta h.\end{aligned}$$

To find the total work, we integrate from $h=0$ to $h=25$:

$$W = \int_0^{25} (100+3(25-h))dh = \int_0^{25} (175-3h)dh = \left(175h - \frac{3h^2}{2}\right)\bigg|_0^{25} = 3437.5\text{ft-lbs.}$$

8. To lift the weight an additional height Δh off the ground from a height of h, we must do work on the weight and the amount of rope not yet pulled onto the roof. Since the roof is 30 ft off the ground, there will be $30-h$ feet remaining of rope, for a weight of $4(30-h)$. So the work required to raise the weight and the rope a height Δh will be $\Delta h(1000+4(30-h))$. To find the total work, we integrate this quantity from $h=0$ to $h=10$:

$$\begin{aligned}\text{Work} &= \int_0^{10} (1000+4(30-h))\,dh \\ &= \int_0^{10} (1120-4h)\,dh\end{aligned}$$

$$
\begin{aligned}
&= (1120h - 2h^2)\Big|_0^{10} \\
&= 11{,}200 - 200 \\
&= 11{,}000 \text{ ft-lbs.}
\end{aligned}
$$

9. The bucket moves upward at $40/10 = 4$ meters/minute. If time is in minutes, at time t the bucket is at a height of $x = 4t$ meters above the ground. See Figure 8.71.

The water drips out at a rate of $5/10 = 0.5$ kg/minute. Initially there is 20 kg of water in the bucket, so at time t minutes, the mass of water remaining is

$$m = 20 - 0.5t \text{ kg.}$$

Consider the time interval between t and $t + \Delta t$. During this time the bucket moves a distance $\Delta x = 4\Delta t$ meters. So, during this interval,

$$\text{Work done} \approx mg\Delta x = (20 - 0.5t)g4\Delta t \text{ joules.}$$

$$\text{Total work done} = \lim_{\Delta t \to 0} \sum (20 - 0.5t)g4\Delta t = 4g \int_0^{10} (20 - 0.5t)\,dt$$

$$= 4g(20t - 0.25t^2)\Big|_0^{10} = 700g = 700(9.8) = 6860 \text{ joules.}$$

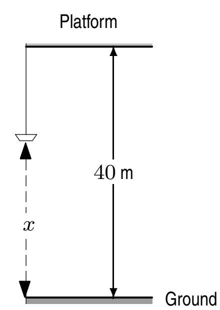


Figure 8.71

10. Consider lifting a rectangular slab of water h feet from the top up to the top. See Figure 8.72. The area of such a slab is $(10)(20) = 200$ square feet; if the thickness is dh, then the volume of such a slab is $200\,dh$ cubic feet. This much water weighs 62.4 pounds per ft^3, so the weight of such a slab is $(200\,dh)(62.4) = 12480\,dh$ pounds. To lift that much water h feet requires $12480h\,dh$ foot-pounds of work. To lift the whole tank, we lift one plate at a time; integrating over the slabs yields

$$\int_0^{15} 12480h\,dh = \frac{12480h^2}{2}\Big|_0^{15} = \frac{12480 \cdot 15^2}{2} = 1{,}404{,}000 \text{ foot-pounds.}$$

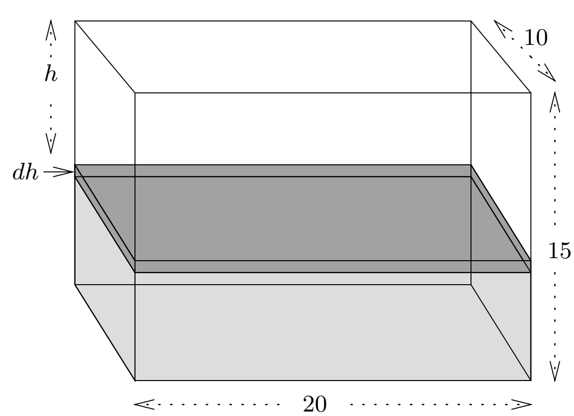


Figure 8.72

11. We slice the water horizontally and find the work required to pump each horizontal slice of water over the top. See Figure 8.73. At a distance h ft above the bottom, a slice of thickness Δh has

$$\text{Volume } \approx 50 \cdot 20\Delta h \text{ ft}^3.$$

Since the density of water is ρ lb/ft^3,

$$\text{Weight of the slice } \approx \rho(50 \cdot 20 \cdot \Delta h) \text{ lbs.}$$

The distance to lift the slice of water at height h ft is $10 - h$ ft, so

$$\begin{aligned}\text{Work to move one slice} &= \rho \cdot \text{Volume} \cdot \text{Distance lifted}\\ &\approx \rho(50 \cdot 20 \cdot \Delta h)(10 - h)\\ &= 100\rho(10 - h)\Delta h \text{ ft-lb.}\end{aligned}$$

The work done, W, to pump all the water is the sum of the work done on the pieces:

$$W \approx \sum 100\rho(10 - h)\Delta h.$$

As $\Delta h \to 0$, we obtain a definite integral. Since h varies from $h = 0$ to $h = 9$ and $\rho = 62.4$ lb/ft^3, the total work is:

$$W = \int_0^9 100\rho(10 - h)dh = 62400\left(10h - \frac{h^2}{2}\right)\Bigg|_0^9 = 62400(49.5) = 3{,}088{,}800.$$

The work to pump all the water out is 3,088,800 ft-lbs.

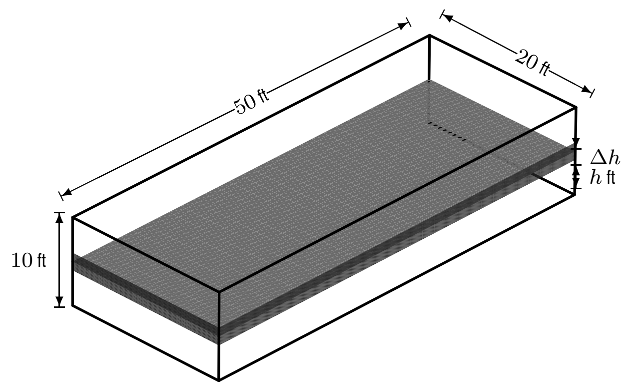


Figure 8.73

12. Let x be the distance measured from the bottom the tank. See Figure 8.74. To pump a layer of water of thickness Δx at x feet from the bottom, the work needed is

$$(62.4)\pi 6^2(20 - x)\Delta x.$$

Therefore, the total work is

$$\begin{aligned}W &= \int_0^{10} 36 \cdot (62.4)\pi(20 - x)dx\\ &= 36 \cdot (62.4)\pi(20x - \frac{1}{2}x^2)\Big|_0^{10}\\ &= 36 \cdot (62.4)\pi(200 - 50)\\ &\approx 1{,}058{,}591.1 \text{ ft-lb.}\end{aligned}$$

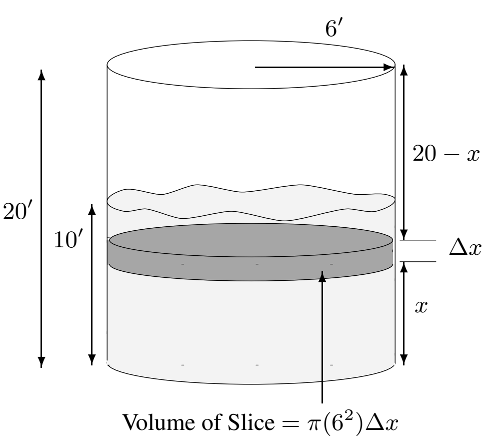


Figure 8.74

13. Let x be the distance from the bottom of the tank. See Figure 8.75. To pump a layer of water of thickness Δx at x feet from the bottom to 10 feet above the tank, the work done is $(62.4)\pi 6^2(30-x)\Delta x$. Thus the total work is

$$\int_0^{20} 36 \cdot (62.4)\pi(30-x)dx$$
$$= 36 \cdot (62.4)\pi \left(30x - \frac{1}{2}x^2\right)\Bigg|_0^{20}$$
$$= 36 \cdot (62.4)\pi(30(20) - \frac{1}{2}20^2)$$
$$\approx 2{,}822{,}909.50 \text{ ft-lb.}$$

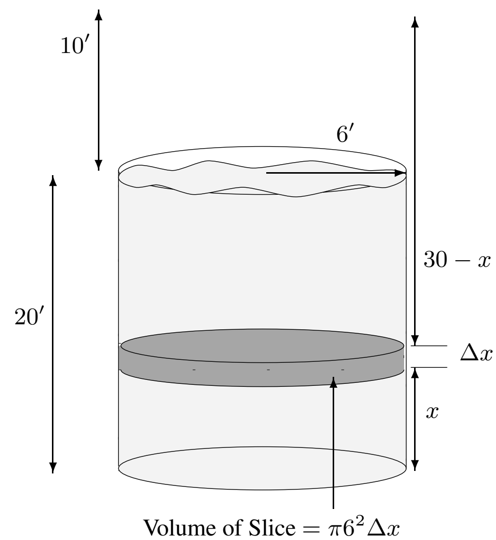


Figure 8.75

14. (a) We slice the water horizontally. Each slice is a cylindrical slab of radius 4 and thickness Δh, so

$$\text{Volume of each slab } \approx \pi 4^2 \Delta h \text{ ft}^3.$$

See Figure 8.76. The density of water is δ lb/ft^3, so

$$\text{Weight of slab } \approx \delta\pi 4^2 \Delta h \text{ lb.}$$

Water at a height of h ft must be lifted a distance of $10-h$ ft.

$$\text{Work to move one slice} = \delta \cdot \text{Volume} \cdot \text{Distance lifted}$$
$$\approx \delta(\pi(4)^2\Delta h)(10-h) \text{ ft-lb.}$$

Since the density of water is $\delta = 62.4$ lb/ft^3 and since h varies from $h=0$ to $h=10$, the total work, W, is:

$$W = \int_0^{10} \delta(\pi 4^2)(10-h)dh = 16\delta\pi \int_0^{10}(10-h)dh = 998.4\pi\left(10h - \frac{h^2}{2}\right)\Bigg|_0^{10} = 156{,}828 \text{ ft-lb.}$$

The total work required is 156,828 ft-lbs.

(b) This is the same as part (a) except the water must be lifted a distance of $15-h$ ft. The total work is:

$$W = \int_0^{10} \delta(\pi 4^2)(15-h)dh = 16\delta\pi \int_0^{10}(15-h)dh = 998.4\pi\left(15h - \frac{h^2}{2}\right)\Bigg|_0^{10} = 313{,}656 \text{ ft-lb.}$$

The total work required is 313,656 ft-lbs.

(c) This is the same as part (a) except that h varies from $h=0$ to $h=8$. The total work is:

$$W = \int_0^{8} \delta(\pi 4^2)(10-h)dh = 16\delta\pi \int_0^{8}(10-h)dh = 998.4\pi\left(10h - \frac{h^2}{2}\right)\Bigg|_0^{8} = 150{,}555.$$

The total work required is 150,555 ft-lbs.

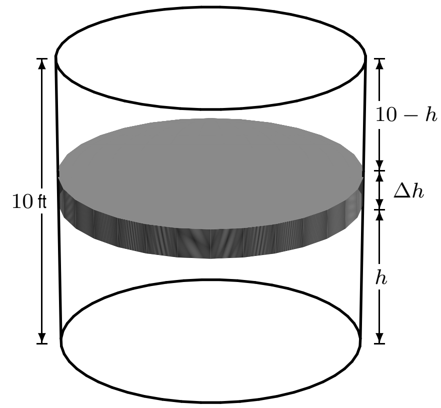


Figure 8.76

15. We begin by slicing the oil into slabs at a distance h below the surface with thickness Δh. We can then calculate the volume of the slab and the work needed to raise this slab to the surface, a distance of h.

$$\begin{aligned}\text{Volume of } \Delta h \text{ disk} &= \pi r^2 \Delta h = 25\pi\Delta h\\ \text{Weight of } \Delta h \text{ disk} &= (25\pi)(50)\Delta h\\ \text{Distance to raise} &= h\\ \text{Work to raise} &= (25\pi)(50)(h)\Delta h.\end{aligned}$$

Integrating the work over all such slabs, we have

$$\begin{aligned}\text{Work} &= \int_{19}^{25} (50)(25\pi)(h)\, dh\\ &= 625\pi h^2 \Big|_{19}^{25}\\ &= 390{,}625\pi - 225{,}625\pi\\ &\approx 518{,}363 \text{ ft-lbs.}\end{aligned}$$

A diagram of this tank is shown in Figure 8.77.

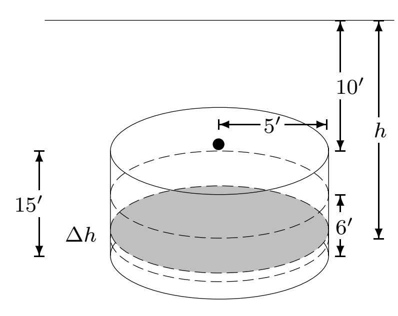


Figure 8.77

16. We slice the water horizontally as in Figure 8.78. We use similar triangles to find the radius r of the slice at height h in terms of h:

$$\frac{r}{h} = \frac{4}{12} \quad \text{so} \quad r = \frac{1}{3}h.$$

At height h,

$$\text{Volume of slice} \approx \pi r^2 \Delta h = \pi \left(\frac{1}{3}h\right)^2 \Delta h \text{ ft}^3.$$

The density of water is δ lb/ft^3, so

$$\text{Weight of slice } \approx \delta\pi \left(\frac{1}{3}h\right)^3 \Delta h \text{ lb}.$$

The water at height h must be lifted a distance of $12 - h$ ft, so

$$\begin{aligned}\text{Work to move slice} &= \delta \cdot \text{Volume} \cdot \text{Distance lifted}\\ &\approx \delta\left(\pi\left(\frac{1}{3}h\right)^2 \Delta h\right)(12-h) \text{ ft-lb}.\end{aligned}$$

The work done, W, to lift all the water is the sum of the work done on the pieces:

$$W \approx \sum \delta\pi(\frac{1}{3}h)^2 \Delta h(12-h) \text{ ft-lb}.$$

As $\Delta h \to 0$, we obtain a definite integral. Since h varies from $h = 0$ to $h = 9$, and $\delta = 62.4$, we have:

$$W = \int_0^9 \delta\pi(\frac{1}{3}h)^2(12-h)dh = \frac{62.4\pi}{9}\int_0^9 (12h^2 - h^3)dh = \frac{62.4\pi}{9}\left(4h^3 - \frac{h^4}{4}\right)\Bigg|_0^9 = 27{,}788 \text{ ft-lb}.$$

The work to pump all the water out is 27,788 ft-lbs.

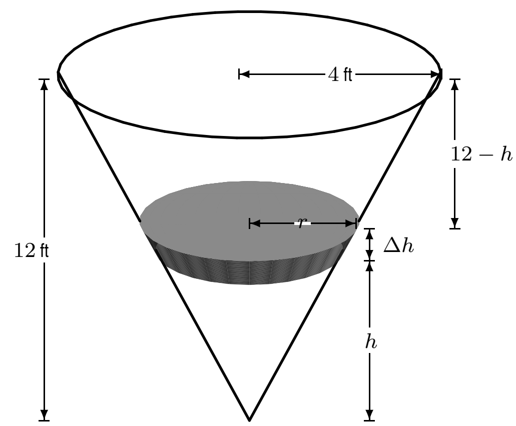


Figure 8.78

17. Let h represent distance below the surface in feet. We slice the tank up into horizontal slabs of thickness Δh. From looking at Figure 8.79, we can see that the slabs will be rectangular. The length of any slab is 12 feet. The width w of a slab h units below the ground will equal $2x$, where $(14-h)^2 + x^2 = 16$, so $w = 2\sqrt{4^2 - (14-h)^2}$. The volume of such a slab is therefore $12w\,\Delta h = 24\sqrt{16-(14-h)^2}\,\Delta h$ cubic feet; the slab weighs $42 \cdot 24\sqrt{16-(14-h)^2}\,\Delta h = 1008\sqrt{16-(14-h)^2}\,\Delta h$ pounds. So the total work done in pumping out all the gasoline is

$$\int_{10}^{18} 1008h\sqrt{16-(14-h)^2}\,dh = 1008\int_{10}^{18} h\sqrt{16-(14-h)^2}\,dh.$$

Substitute $s = 14 - h$, $ds = -dh$. We get

$$\begin{aligned}1008\int_{10}^{18} h\sqrt{16-(14-h)^2}\,dh &= -1008\int_4^{-4}(14-s)\sqrt{16-s^2}\,ds\\ &= 1008 \cdot 14\int_{-4}^{4}\sqrt{16-s^2}\,ds - 1008\int_{-4}^{4} s\sqrt{16-s^2}\,ds.\end{aligned}$$

The first integral represents the area of a semicircle of radius 4, which is 8π. The second is the integral of an odd function, over the interval $-4 \le s \le 4$, and is therefore 0. Hence, the total work is $1008 \cdot 14 \cdot 8\pi \approx 354{,}673$ foot-pounds.

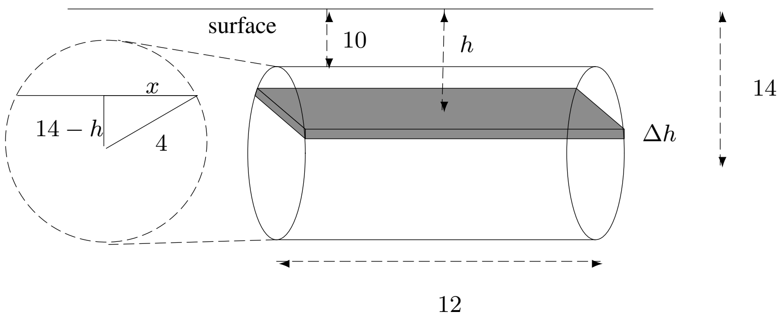


Figure 8.79

18. Divide the muddy water into horizontal slabs of thickness Δh. See Figure 8.80. Then for a typical slab

$$\begin{aligned}\text{Volume of slab} &= \pi(0.5)^2\Delta h \text{ m}^3\\ \text{Mass of slab} &\approx \delta(h)\pi(0.5)^2\Delta h = 0.25\pi(1+kh)\Delta h \text{ kg}\end{aligned}$$

The water in this slab is moved a distance of $h+0.3$ meters to the rim of the barrel. Now

$$\text{Work done} = \text{Mass} \cdot g \cdot \text{Distance moved},$$

and work is measured in newtons if mass is in kilograms and distance is in meters, so

$$\text{Work done in moving slab} \approx 0.25\pi(1+kh)g(h+0.3)\Delta h \text{ joules.}$$

Since the slices run from $h=0$ to $h=1.5$, we have

$$\begin{aligned}\text{Total work done} &= \int_0^{1.5} 0.25\pi(1+kh)g(h+0.3)\,dh\\ &= 0.366(k+1.077)g\pi \text{ joules}\end{aligned}$$

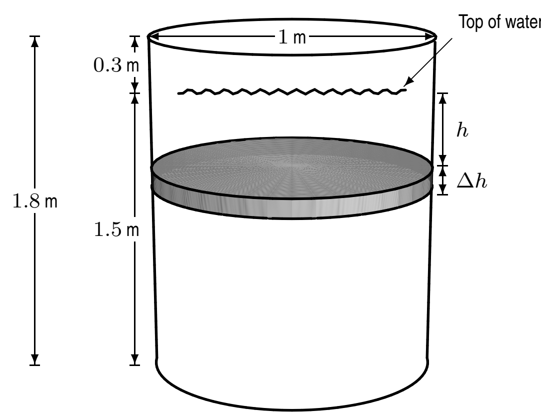


Figure 8.80

19. (a) We divide the triangular end into horizontal strips of thickness Δh. The length of a strip, s, depends on its height h from the bottom. See Figure 8.81. We use similar triangles to see that

$$\frac{s}{h} = \frac{2}{3} \quad \text{so} \quad s = \frac{2}{3}h.$$

Since each strip is approximately a rectangle, at height h,

$$\text{Area of strip } \approx \frac{2}{3}h\Delta h \text{ ft}^2.$$

Since the depth at height h is $3-h$, writing δ for the density of water, we have:

$$\begin{aligned}\text{Force on one strip} &= \delta \cdot \text{Depth} \cdot \text{Area}\\ &\approx \delta(3-h)\left(\frac{2}{3}h\Delta h\right) \text{ lb.}\end{aligned}$$

To find the total force, F, we integrate the force on a strip from $h=0$ to $h=3$, using $\delta = 62.4$ lb/ft^3:

$$F = \int_0^3 \delta(3-h)\frac{2}{3}hdh = \frac{2}{3}\delta\int_0^3 (3h-h^2)dh = \frac{2}{3}62.4\left(\frac{3h^2}{2}-\frac{h^3}{3}\right)\Bigg|_0^3 = 187.2 \text{ lbs.}$$

(b) To find the work, we slice the water horizontally. Each slice is a rectangular slab with thickness Δh, length 15 ft, and width s as in Figure 8.82. As we saw in part (a), at a height of h we have $s = \frac{2}{3}h$. At height h,

$$\text{Volume of slab } \approx 15\left(\frac{2}{3}h\right)\Delta h \text{ ft}^3.$$

The distance to lift the slice at height h is $3-h$, so if δ is the density of water, we have:

$$\begin{aligned}\text{Work to lift one slice} &= \delta \cdot \text{Volume} \cdot \text{Distance lifted}\\ &\approx \delta(15(\frac{2}{3}h)\Delta h)(3-h) \text{ ft-lb.}\end{aligned}$$

To find the total work, W, we integrate the work to lift a slice from $h=0$ to $h=3$, using $\delta = 62.4$ lb/ft^3.

$$W = \int_0^3 \delta 15\left(\frac{2}{3}h\right)(3-h)\,dh = 10\delta\int_0^3 (3h-h^2)dh = 10\cdot 62.4\left(\frac{3h^2}{2}-\frac{h^3}{3}\right)\Bigg|_0^3 = 10\delta(4.5) = 2808 \text{ ft-lbs.}$$

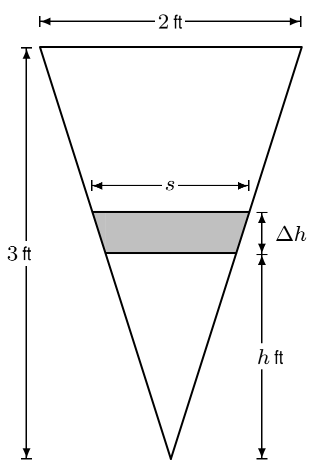


Figure 8.81

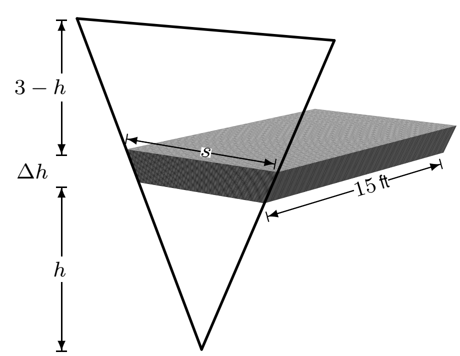


Figure 8.82

20. (a) Divide the wall into N horizontal strips, each of which is of height Δh. See Figure 8.83. The area of each strip is $1000\Delta h$, and the pressure at depth h_i is $62.4h_i$, so we approximate the force on the strip as $1000(62.4h_i)\Delta h$.

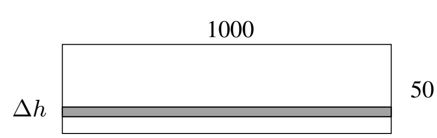


Figure 8.83

Therefore,

$$\text{Force on the Dam } \approx \sum_{i=0}^{N-1} 1000(62.4h_i)\Delta h.$$

(b) As $N \to \infty$, the Riemann sum becomes the integral, so the force on the dam is

$$\int_0^{50} (1000)(62.4h)\,dh = 62400\frac{h^2}{2}\bigg|_0^{50} = 78{,}000{,}000 \text{ pounds.}$$

21. See Figure 8.84.

For the bottom: The bottom of the tank is at constant depth 15 feet, and therefore is under constant pressure, $15 \cdot 62.4 = 936$ lb/ft^2. The area of the base is 200 ft^2, so

$$\text{Total force on bottom } = 200 \text{ ft}^2 \cdot 936 \text{ lb/ft}^2 = 187200 \text{ lb.}$$

For the 15×10 side: The area of a horizontal strip of width dh is $10\,dh$ square feet, and the pressure at height h is $62.4h$ pounds per square foot. Therefore, the force on such a strip is $62.4h(10\,dh)$ pounds. Hence,

$$\text{Total force on the } 15 \times 10 \text{ side } = \int_0^{15} (62.4h)(10)\,dh = 624\frac{h^2}{2}\bigg|_0^{15} = 70200 \text{ lbs.}$$

For the 15×20 side: Similarly,

$$\text{Total force on the } 15 \times 20 \text{ side } = \int_0^{15} (62.4h)(20)\,dh = 1248\frac{h^2}{2}\bigg|_0^{15} = 140400 \text{ lbs.}$$

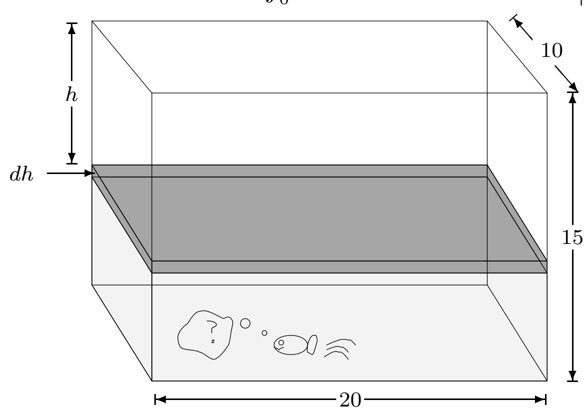


Figure 8.84

22. We divide the water against the dam into horizontal strips, each of thickness Δh and length 100.

$$\text{Area of each strip } \approx 100\Delta h \text{ ft}^2.$$

See Figure 8.85. The strip at height h ft from the bottom is at a water depth of $40 - h$, so, if δ lb/ft^3 is the density of water, we have:

$$\begin{aligned}\text{Force of one strip} &= \delta \cdot \text{Depth} \cdot \text{Area} \\ &\approx \delta(40-h)(100\Delta h) \text{ lb.}\end{aligned}$$

To find the total force, F, we integrate the force on a strip from $h = 0$ to $h = 40$, using $\delta = 62.4$ lb/ft^3:

$$F = \int_0^{40} \delta(40-h)100dh = 100 \cdot 62.4\left(40h - \frac{h^2}{2}\right)\bigg|_0^{40} = 6240(800) = 4{,}992{,}000\text{ft-lbs.}$$

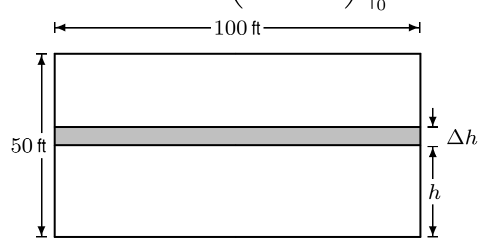


Figure 8.85

23. Bottom:

$$\text{Water force } = 62.4(2)(12) = 1497.6 \text{ lbs.}$$

Front and back:

$$\text{Water force } = (62.4)(4)\int_0^2 (2-x)\,dx = (62.4)(4)(2x - \frac{1}{2}x^2)\Big|_0^2$$
$$= (62.4)(4)(2) = 499.2 \text{ lbs.}$$

Both sides:

$$\text{Water force} = (62.4)(3)\int_0^2 (2-x)\,dx = (62.4)(3)(2) = 374.4 \text{ lbs.}$$

24. (a) Since the density of water is $\delta = 1000$ kg/m^3, at the base of the dam, water pressure $\delta g h = 1000 \cdot 9.8 \cdot 180 = 1.76 \cdot 10^6$ nt/m^2.

(b) To set up a definite integral giving the force, we divide the dam into horizontal strips. We use horizontal strips because the pressure along each strip is approximately constant, since each part is at approximately the same depth. See Figure 8.86.

$$\text{Area of strip } = 2000\Delta h \text{ m}^2.$$

Pressure at depth of h meters $= \delta g h = 9800h$ nt/m^2. Thus,

$$\text{Force on strip } \approx \text{ Pressure } \times \text{ Area } = 9800h \cdot 2000\Delta h = 1.96 \cdot 10^7 h\Delta h \text{ nt.}$$

Summing over all strips and letting $\Delta h \to 0$ gives:

$$\text{Total force } = \lim_{\Delta h \to 0} \sum 1.96 \cdot 10^7 h\Delta h = 1.96 \cdot 10^7 \int_0^{180} h\,dh \text{ newtons.}$$

Evaluating gives

$$\text{Total force } = 1.96 \cdot 10^7 \frac{h^2}{2}\Big|_0^{180} = 3.2 \cdot 10^{11} \text{ newtons.}$$

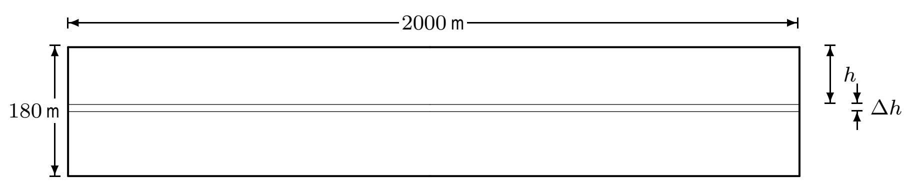


Figure 8.86

25. (a) At a depth of 350 feet,

$$\text{Pressure } = 62.4 \cdot 350 = 21{,}840 \text{ lb/ft}^2.$$

To imagine this pressure, we convert to pounds per square inch, giving a pressure of $21{,}840/144 = 151.7$ lb/in^2.

(b) (i) When the square is held horizontally, the pressure is constant at 21,840 lbs/ft^2, so

$$\text{Force } = \text{ Pressure } \cdot \text{ Area } = 21{,}840 \cdot 5^2 = 546{,}000 \text{ pounds.}$$

(ii) When the square is held vertically, only the bottom is at 350 feet. Dividing into horizontal strips, as in Figure 8.87, we have

$$\text{Area of strip } = 5\Delta h \text{ ft}^2.$$

Since the pressure on a strip at a depth of h feet is $62.4h$ lb/ft^2,

$$\text{Force on strip } \approx 62.4h \cdot 5\Delta h = 312h\Delta h \text{ pounds.}$$

Summing over all strips and taking the limit as $\Delta h \to 0$ gives a definite integral. The strips vary between a depth of 350 feet and 345 feet, so

$$\text{Total force } = \lim_{\Delta h \to 0} \sum 312h\Delta h = \int_{345}^{350} 312h\,dh \text{ pounds.}$$

Evaluating gives

$$\text{Total force } = 312\frac{h^2}{2}\bigg|_{345}^{350} = 156(350^2 - 345^2) = 542{,}100 \text{ pounds.}$$

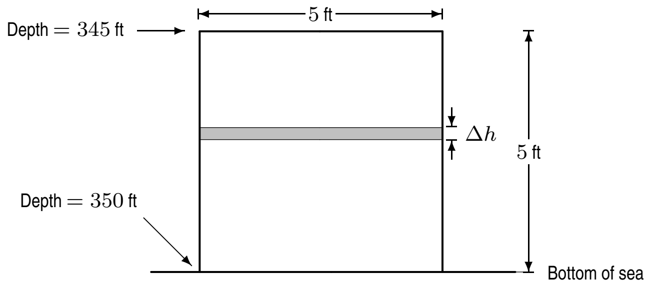


Figure 8.87

26. (a) Since water has density 62.4 lb/ft^3, at a depth of 12,500 feet,

$$\text{Pressure } = \text{ Density } \times \text{ Depth } = 62.4 \cdot 12{,}500 = 780{,}000 \text{ lb/square foot.}$$

To imagine this pressure, observe that it is equivalent to $780{,}000/144 \approx 5400$ pounds per square inch.

(b) To calculate the pressure on the porthole (window), we slice it into horizontal strips, as the pressure remains approximately constant along each one. See Figure 8.88. Since each strip is approximately rectangular

$$\text{Area of strip } \approx 2r\Delta h \text{ ft}^2.$$

To calculate r in terms of h, we use the Pythagorean Theorem:

$$\begin{aligned} r^2 + h^2 &= 9 \\ r &= \sqrt{9 - h^2}, \end{aligned}$$

so

$$\text{Area of strip } \approx 2\sqrt{9 - h^2}\Delta h \text{ ft}^2.$$

The center of the porthole is at a depth of 12,500 feet below the surface, so the strip shown in Figure 8.88 is at a depth of $(12{,}500 - h)$ feet. Thus, pressure on the strip is $62.4(12{,}500 - h)$ lb/ft^2, so

$$\begin{aligned} \text{Force on strip } = \text{ Pressure } \times \text{ Area } &\approx 62.4(12{,}500 - h)2\sqrt{9 - h^2}\Delta h \text{ lb} \\ &= 124.8(12{,}500 - h)\sqrt{9 - h^2}\Delta h \text{ lb.} \end{aligned}$$

To get the total force, we sum over all strips and take the limit as $\Delta h \to 0$. Since h ranges from -3 to 3, we get the integral

$$\begin{aligned} \text{Total force } &= \lim_{\Delta h \to 0} \sum 124.8(12{,}500 - h)\sqrt{9 - h^2}\Delta h \\ &= 124.8 \int_{-3}^{3} (12{,}500 - h)\sqrt{9 - h^2}\, dh \text{ lb.} \end{aligned}$$

Evaluating the integral numerically, we obtain a total force of $2.2 \cdot 10^7$ pounds.

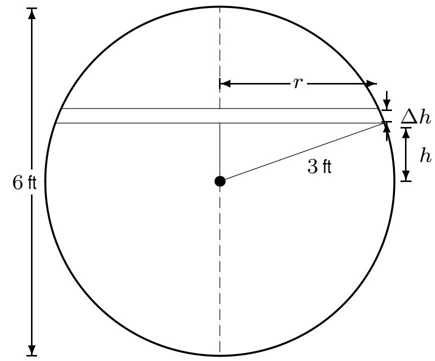


Figure 8.88: Center of circle is 12,500 ft below the surface of ocean

27. We divide the dam into horizontal strips since the pressure is then approximately constant on each one. See Figure 8.89.

$$\text{Area of strip } \approx w\Delta h \text{ m}^2.$$

Since w is a linear function of h, and $w = 3600$ when $h = 0$, and $w = 3000$ when $h = 100$, the function has slope $(3000 - 3600)/100 = -6$. Thus,

$$w = 3600 - 6h,$$

so

$$\text{Area of strip } \approx (3600 - 6h)\Delta h \text{ m}^2.$$

The density of water is $\delta = 1000$ kg/m^3, so the pressure at depth h meters $= \delta g h = 1000 \cdot 9.8h = 9800h$ nt/m^2. Thus,

$$\text{Total force } = \lim_{\Delta h \to 0} \sum 9800h(3600 - 6h)\Delta h = 9800 \int_0^{100} h(3600 - 6h)\, dh \text{ newtons.}$$

Evaluating the integral gives

$$\text{Total force } = 9800(1800h^2 - 2h^3)\bigg|_0^{100} = 1.6 \cdot 10^{11} \text{ newtons.}$$

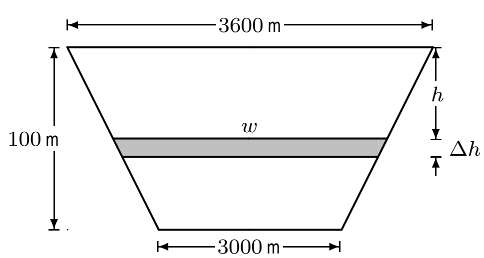


Figure 8.89

28. We need to divide the disk up into circular rings of charge and integrate their contributions to the potential (at P) from 0 to a. These rings, however, are not uniformly distant from the point P. A ring of radius z is $\sqrt{R^2 + z^2}$ away from point P (see Figure 8.90).

The ring has area $2\pi z\, \Delta z$, and charge $2\pi z\sigma\, \Delta z$. The potential of the ring is then $\dfrac{2\pi z\sigma\, \Delta z}{\sqrt{R^2 + z^2}}$ and the total potential at point P is

$$\int_0^a \frac{2\pi z\sigma\, dz}{\sqrt{R^2 + z^2}} = \pi\sigma \int_0^a \frac{2z\, dz}{\sqrt{R^2 + z^2}}.$$

We make the substitution $u = z^2$. Then $du = 2z\,dz$. We obtain

$$\pi\sigma \int_0^a \frac{2z\,dz}{\sqrt{R^2+z^2}} = \pi\sigma \int_0^{a^2} \frac{du}{\sqrt{R^2+u}} = \pi\sigma(2\sqrt{R^2+u})\Big|_0^{a^2}$$
$$= \pi\sigma(2\sqrt{R^2+z^2})\Big|_0^a = 2\pi\sigma(\sqrt{R^2+a^2} - R).$$

(The substitution $u = R^2 + z^2$ or $\sqrt{R^2+z^2}$ works also.)

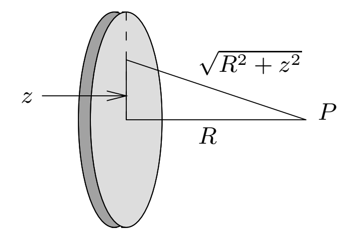


Figure 8.90

29. The density of the rod is $10\text{ kg}/6\text{ m} = \frac{5}{3}\frac{\text{kg}}{\text{m}}$. A little piece, dx m, of the rod thus has mass $5/3\,dx$ kg. If this piece has an angular velocity of 2 rad/sec, then its actual velocity is $2|x|\text{ m/sec}$. This is because a radian angle sweeps out an arc length equal to the radius of the circle, and in this case the little piece moves in circles about the origin of radius $|x|$. See Figure 8.91. The kinetic energy of the little piece is $mv^2/2 = (5/3\,dx)(2|x|)^2/2 = \frac{10}{3}x^2\,dx$.

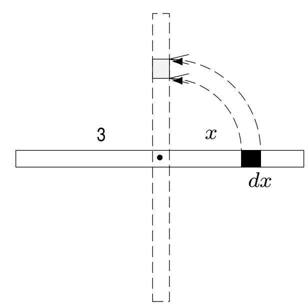


Figure 8.91

Therefore,

$$\text{Total Kinetic Energy} = \int_{-3}^{3} \frac{10x^2}{3}\,dx = \frac{20}{3}\left[\frac{x^3}{3}\right]\Bigg|_0^3 = 60\text{ kg}\cdot\text{m}^2/\text{sec}^2 = 60\text{ joules}.$$

30. We slice the record into rings in such a way that every point has approximately the same speed: use concentric circles around the hole. See Figure 8.92. We assume the record is a flat disk of uniform density: since its mass is 50 grams and its area is $\pi(10\text{cm})^2 = 100\pi\text{ cm}^2$, the record has density $\frac{50}{100\pi} = \frac{1}{2\pi}\frac{\text{gram}}{\text{cm}^2}$. So a ring of width dr, having area about $2\pi r\,dr\ \text{cm}^2$, has mass approximately $(2\pi r\,dr)(1/2\pi) = r\,dr$ gm. At radius r, the velocity of the ring is

$$33\tfrac{1}{3}\,\frac{\text{rev}}{\text{min}} \cdot \frac{1\text{ min}}{60\text{ sec}} \cdot \frac{2\pi r\text{ cm}}{1\text{ rev}} = \frac{10\pi r}{9}\,\frac{\text{cm}}{\text{sec}}.$$

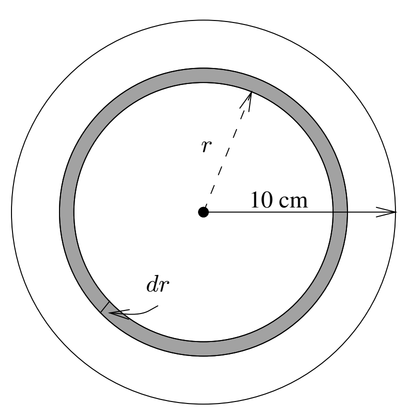


Figure 8.92

The kinetic energy of the ring is

$$\frac{1}{2}mv^2 = \frac{1}{2}(r\,dr \text{ grams})\left(\frac{10\pi r}{9}\ \frac{\text{cm}}{\text{sec}}\right)^2 = \frac{50\pi^2 r^3\,dr}{81}\ \frac{\text{gram}\cdot\text{cm}^2}{\text{sec}^2}.$$

So the kinetic energy of the record, summing the energies of all these rings, is

$$\int_0^{10} \frac{50\pi^2 r^3\,dr}{81} = \left.\frac{25\pi^2 r^4}{162}\right|_0^{10} \approx 15231\,\frac{\text{gram}\cdot\text{cm}^2}{\text{sec}^2} = 15231 \text{ ergs.}$$

31. The density of the rod, in mass per unit length, is M/l (see Figure 8.93). So a slice of size dr has mass $\frac{M\,dr}{l}$. It pulls the small mass m with force $Gm\frac{M\,dr}{l}/r^2 = \frac{GmM\,dr}{lr^2}$. So the total gravitational attraction between the rod and point is

$$\begin{aligned}\int_a^{a+l} \frac{GmM\,dr}{lr^2} &= \frac{GmM}{l}\left.\left(-\frac{1}{r}\right)\right|_a^{a+l} \\ &= \frac{GmM}{l}\left(\frac{1}{a} - \frac{1}{a+l}\right) \\ &= \frac{GmM}{l}\,\frac{l}{a(a+l)} = \frac{GmM}{a(a+l)}.\end{aligned}$$

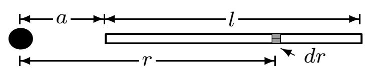


Figure 8.93

32. This time, let's split the second rod into small slices of length dr. See Figure 8.94. Each slice is of mass $\frac{M_2}{l_2}\,dr$, since the density of the second rod is $\frac{M_2}{l_2}$. Since the slice is small, we can treat it as a particle at distance r away from the end of the first rod, as in Problem 31. By that problem, the force of attraction between the first rod and particle is

$$\frac{GM_1\frac{M_2}{l_2}\,dr}{(r)(r+l_1)}.$$

So the total force of attraction between the rods is

$$\begin{aligned}\int_a^{a+l_2} \frac{GM_1\frac{M_2}{l_2}\,dr}{(r)(r+l_1)} &= \frac{GM_1M_2}{l_2}\int_a^{a+l_2} \frac{dr}{(r)(r+l_1)} \\ &= \frac{GM_1M_2}{l_2}\int_a^{a+l_2} \frac{1}{l_1}\left(\frac{1}{r} - \frac{1}{r+l_1}\right)\,dr. \\ &= \frac{GM_1M_2}{l_1l_2}\left.(\ln|r| - \ln|r+l_1|)\right|_a^{a+l_2} \\ &= \frac{GM_1M_2}{l_1l_2}\left[\ln|a+l_2| - \ln|a+l_1+l_2| - \ln|a| + \ln|a+l_1|\right] \\ &= \frac{GM_1M_2}{l_1l_2}\ln\left[\frac{(a+l_1)(a+l_2)}{a(a+l_1+l_2)}\right].\end{aligned}$$

This result is symmetric: if you switch l_1 and l_2 or M_1 and M_2, you get the same answer. That means it's not important which rod is "first," and which is "second."

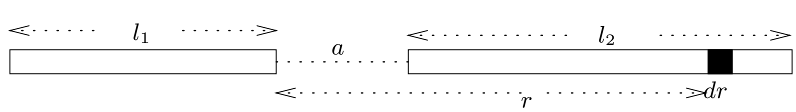


Figure 8.94

33. In Figure 8.95, consider a small piece of the ring of length Δl and mass

$$\Delta M = \frac{\Delta l M}{2\pi a}.$$

The gravitational force exerted by the small piece of the ring is along the line QP. As we sum over all pieces of the ring, the components perpendicular to the line OP cancel. The components of the force toward the point O are all in the same direction, so the net force is in this direction. The small piece of length Δl and mass $\Delta l M/2\pi a$ is at a distance of $\sqrt{a^2+y^2}$ from P, so

$$\text{Gravitational force from small piece} = \Delta F = \frac{G\frac{\Delta l M}{2\pi a}m}{(\sqrt{a^2+y^2})^2} = \frac{GMm\Delta l}{2\pi a(a^2+y^2)}.$$

Thus the force toward O exerted by the small piece is given by

$$\Delta F\cos\theta = \Delta F\frac{y}{\sqrt{a^2+y^2}} = \frac{GMm\Delta l}{2\pi a(a^2+y^2)}\frac{y}{\sqrt{a^2+y^2}} = \frac{GMmy\Delta l}{2\pi a(a^2+y^2)^{3/2}}.$$

The total force toward O is given by $F \approx \sum \Delta F\cos\theta$, so

$$F = \frac{GMmy\cdot\text{Total length}}{2\pi a(a^2+y^2)^{3/2}} = \frac{GMmy2\pi a}{2\pi a(a^2+y^2)^{3/2}} = \frac{GMmy}{(a^2+y^2)^{3/2}}.$$

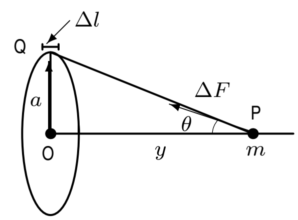


Figure 8.95

34. Divide the disk into rings of radius r, width Δr, as shown in Figure 8.96.
Then

$$\text{Area of ring} \approx 2\pi r\Delta r.$$

Since total area of disk is πa^2,

$$\text{Mass of ring} \approx \frac{2\pi r\Delta r}{\pi a^2}M = \frac{2rM}{a^2}\Delta r.$$

Thus, calculating the gravitational force due to the ring, we have

$$\begin{matrix}\text{Gravitational force}\\ \text{on } m \text{ due to ring}\end{matrix} = G\left(\frac{2rM}{a^2}\Delta r\right)\frac{my}{(r^2+y^2)^{3/2}} = \frac{2GMmyr}{a^2(r^2+y^2)^{3/2}}\Delta r.$$

Summing over all rings, we get

$$\begin{matrix}\text{Total gravitational force}\\ \text{on } m \text{ due to disk}\end{matrix} \approx \sum\frac{2GMmyr}{a^2(r^2+y^2)^{3/2}}\Delta r.$$

As $\Delta r \to 0$, we get

$$\begin{aligned}\begin{matrix}\text{Gravitational force}\\ \text{on } m \text{ due to disk}\end{matrix} &= \int_0^a \frac{2GMmyr}{a^2(r^2+y^2)^{3/2}}dr = \frac{2GMmy}{a^2}\cdot\left.\frac{-1}{(r^2+y^2)^{1/2}}\right|_0^a \\ &= \frac{2GMmy}{a^2}\left(\frac{1}{y}-\frac{1}{(a^2+y^2)^{1/2}}\right).\end{aligned}$$

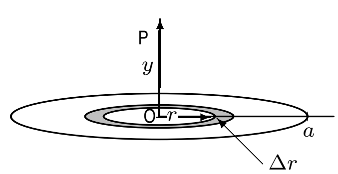


Figure 8.96

Solutions for Section 8.6

Exercises

1. At any time t, in a time interval Δt, an amount of $1000\Delta t$ is deposited into the account. This amount earns interest for $(10-t)$ years giving a future value of $1000e^{(0.08)(10-t)}$. Summing all such deposits, we have

$$\text{Future value} = \int_0^{10} 1000e^{0.08(10-t)}\,dt = \$15{,}319.30.$$

2.

$$\begin{aligned}\text{Future Value } &= \int_0^{15} 3000e^{0.06(15-t)}dt = 3000e^{0.9}\int_0^{15} e^{-0.06t}dt \\ &= 3000e^{0.9}\left(\frac{1}{-0.06}e^{-0.06t}\right)\Bigg|_0^{15} = 3000e^{0.9}\left(\frac{1}{-0.06}e^{-0.9}+\frac{1}{0.06}e^{0}\right) \\ &\approx \$72{,}980.16\end{aligned}$$

$$\begin{aligned}\text{Present Value } &= \int_0^{15} 3000e^{-0.06t}dt = 3000\left(-\frac{1}{0.06}\right)e^{-0.06t}\Bigg|_0^{15} \\ &\approx \$29{,}671.52.\end{aligned}$$

There's a quicker way to calculate the present value of the income stream, since the future value of the income stream is (as we've shown) \$72,980.16, the present value of the income stream must be the present value of \$72,980.16. Thus,

$$\begin{aligned}\text{Present Value} &= \$72{,}980.16(e^{-.06\cdot 15}) \\ &\approx \$29{,}671.52,\end{aligned}$$

which is what we got before.

3. We compute the future value first: we have

$$\text{Future value} = \int_0^5 2000e^{0.08(5-t)}dt = \$12,295.62.$$

We can compute the present value using an integral and the income stream or using the future value. We compute the present value, P, from the future value:

$$12295.62 = Pe^{0.08(5)} \quad \text{so} \quad P = 8242.00.$$

The future value of this income stream is $\$12,295.62$ and the present value of this income stream is $\$8,242.00$.

4. (a) We compute the future value of this income stream:

$$\text{Future value} = \int_0^{20} 1000e^{0.07(20-t)}dt = \$43,645.71.$$

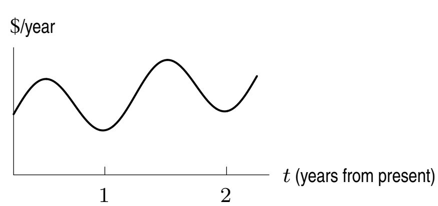
After 20 years, the account will contain $\$43,645.71$.

(b) The person has deposited \$1000 every year for 20 years, for a total of $\$20,000$.

(c) The total interest earned is $\$43,645.71 - \$20,000 = \$23,645.71$.

Problems

5.



The graph reaches a peak each summer, and a trough each winter. The graph shows sunscreen sales increasing from cycle to cycle. This gradual increase may be due in part to inflation and to population growth.

6. (a) The lump sum payment has a present value of 104 million dollars . We compute the present value of the other option in each case. An award of \$197 million paid out continuously over 26 years works out to an income stream of 7.576923 million dollars per year.

If the interest rate is 6%, compounded continuously, we have

$$\text{Present value at } 6\% = \int_0^{26} 7.576923e^{-0.06t}\,dt = 99.75.$$

The present value of this option is about 99.75 million dollars. Since this is less than the lump sum payment of 104 million dollars, the lump sum payment is preferable if the interest rate is 6%.

If the interest rate is 5%, we have

$$\text{Present value at } 5\% = \int_0^{26} 7.576923e^{-0.05t}\,dt = 110.24.$$

The present value of this option is about 110.24 million dollars. Since this is greater than the lump sum payment of 104 million dollars, taking payments continuously over 26 years is the better option if the interest rate is 5%.

(b) Since the winner chose the lump sum option, she was assuming that interest rates would be high (above about 5.5%).

7. (a) Solve for $P(t) = P$.

$$\begin{aligned}
100000 &= \int_0^{10} Pe^{0.10(10-t)}\,dt = Pe\int_0^{10} e^{-0.10t}\,dt \\
&= \frac{Pe}{-0.10}e^{-0.10t}\Big|_0^{10} = Pe(-3.678 + 10) \\
&= P \cdot 17.183.
\end{aligned}$$

So, $P \approx \$5820$ per year.

(b) To answer this, we'll calculate the present value of \$100,000:

$$\begin{aligned}
100000 &= Pe^{0.10(10)} \\
P &\approx \$36{,}787.94.
\end{aligned}$$

8. (a) Let L be the number of years for the balance to reach \$10,000. Since our income stream is \$1000 per year, the future value of this income stream should equal (in L years) \$10,000. Thus

$$\begin{aligned}
10000 &= \int_0^{L} 1000e^{0.05(L-t)}\,dt = 1000e^{0.05L}\int_0^{L} e^{-0.05t}\,dt \\
&= 1000e^{0.05L}\left(-\frac{1}{0.05}\right)e^{-0.05t}\Big|_0^{L} = 20000e^{0.05L}\left(1 - e^{-0.05L}\right) \\
&= 20000e^{0.05L} - 20000 \\
\text{so}\quad e^{0.05L} &= \frac{3}{2} \\
L &= 20\ln\left(\frac{3}{2}\right) \approx 8.11 \text{ years.}
\end{aligned}$$

(b) We want

$$10000 = 2000e^{0.05L} + \int_0^{L} 1000e^{0.05(L-t)}\,dt.$$

The first term on the right hand side is the future value of our initial balance. The second term is the future value of our income stream. We want this sum to equal \$10,000 in L years. We solve for L:

$$\begin{aligned}
10000 &= 2000e^{0.05L} + 1000e^{0.05L}\int_0^{L} e^{-0.05t}\,dt \\
&= 2000e^{0.05L} + 1000e^{0.05L}\left(\frac{1}{-0.05}\right)e^{-0.05t}\Big|_0^{L} \\
&= 2000e^{0.05L} + 20000e^{0.05L}\left(1 - e^{-0.05L}\right) \\
&= 2000e^{0.05L} + 20000e^{0.05L} - 20000.
\end{aligned}$$

So,

$$\begin{aligned}
22000e^{0.05L} &= 30000\\
e^{0.05L} &= \frac{30000}{22000}\\
L &= 20\ln\frac{15}{11} \approx 6.203 \text{ years.}
\end{aligned}$$

9. You should choose the payment which gives you the highest present value. The immediate lump-sum payment of \$2800 obviously has a present value of exactly \$2800, since you are getting it now. We can calculate the present value of the installment plan as:

$$\begin{aligned}
\text{PV} &= 1000e^{-0.06(0)} + 1000e^{-0.06(1)} + 1000e^{-0.06(2)}\\
&\approx \$2828.68.
\end{aligned}$$

Since the installment payments offer a (slightly) higher present value, you should accept this option.

10. (a) We calculate the future values of the two options:

$$\begin{aligned}
\text{FV}_1 &= 6e^{0.1(3)} + 2e^{0.1(2)} + 2e^{0.1(1)} + 2e^{0.1(0)}\\
&\approx 8.099 + 2.443 + 2.210 + 2\\
&= \$14.752 \text{ million.}
\end{aligned}$$

$$\begin{aligned}
\text{FV}_2 &= e^{0.1(3)} + 2e^{0.1(2)} + 4e^{0.1(1)} + 6e^{0.1(0)}\\
&\approx 1.350 + 2.443 + 4.421 + 6\\
&= \$14.214 \text{ million.}
\end{aligned}$$

As we can see, the first option gives a higher future value, so he should choose Option 1.

(b) From the future value we can easily derive the present value using the formula $\text{PV} = \text{FV}e^{-rt}$. So the present value is

$$\text{Option 1: PV} = 14.752e^{0.1(-3)} \approx \$10.929 \text{ million.}$$

$$\text{Option 2: PV} = 14.214e^{0.1(-3)} \approx \$10.530 \text{ million.}$$

11. At any time t, the company receives income of $s(t)$ per year. It will then invest this money for a length of $2-t$ years at 6% interest, giving it future value of $s(t)e^{(0.06)(2-t)}$ from this income. If we sum all such incomes over the two-year period, we can find the total value of the sales:

$$\begin{aligned}
\text{Value} &= \int_0^2 s(t)e^{(0.06)(2-t)}\,dt = \int_0^2 \left[50e^{-t}e^{(0.06)(2-t)}\right]\,dt\\
&= \int_0^2 \left[50e^{0.12-1.06t}\right]\,dt = \left(\frac{-53.1838}{e^{1.06t}}\right)\Bigg|_0^2 = \$46{,}800.
\end{aligned}$$

12. Price in future $= P(1+20\sqrt{t})$.
The present value V of price satisfies $V = P(1+20\sqrt{t})e^{-0.05t}$.
We want to maximize V. To do so, we find the critical points of $V(t)$ for $t \geq 0$. (Recall that $\sqrt{t}$ is nondifferentiable at $t=0$.)

$$\begin{aligned}
\frac{dV}{dt} &= P\left[\frac{20}{2\sqrt{t}}e^{-0.05t} + (1+20\sqrt{t})(-0.05e^{-0.05t})\right]\\
&= Pe^{-0.05t}\left[\frac{10}{\sqrt{t}} - 0.05 - \sqrt{t}\right].
\end{aligned}$$

Setting $\dfrac{dV}{dt} = 0$ gives $\dfrac{10}{\sqrt{t}} - 0.05 - \sqrt{t} = 0$. Using a calculator, we find $t \approx 10$ years. Since $V'(t) > 0$ for $0 < t < 10$ and $V'(t) < 0$ for $t > 10$, we confirm that this is a maximum. Thus, the best time to sell the wine is in 10 years.

13. (a) Suppose the oil extracted over the time period $[0, M]$ is S. (See Figure 8.97.) Since $q(t)$ is the rate of oil extraction, we have:

$$S = \int_0^M q(t)dt = \int_0^M (a - bt)dt = \int_0^M (10 - 0.1t)\,dt.$$

To calculate the time at which the oil is exhausted, set $S = 100$ and try different values of M. We find $M = 10.6$ gives

$$\int_0^{10.6} (10 - 0.1t)\,dt = 100,$$

so the oil is exhausted in 10.6 years.

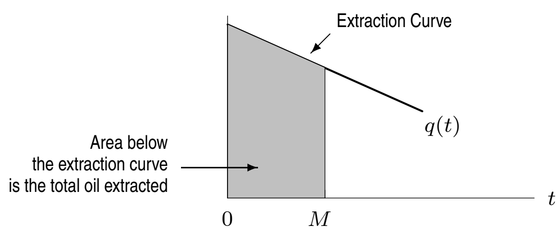


Figure 8.97

(b) Suppose p is the oil price, C is the extraction cost per barrel, and r is the interest rate. We have the present value of the profit as

$$\begin{aligned}\text{Present value of profit } &= \int_0^M (p - C)q(t)e^{-rt}dt \\ &= \int_0^{10.6} (20 - 10)(10 - 0.1t)e^{-0.1t}\,dt \\ &= 624.9 \text{ million dollars.}\end{aligned}$$

14. One good way to approach the problem is in terms of present values. In 1980, the present value of Germany's loan was 20 billion DM. Now let's figure out the rate that the Soviet Union would have to give money to Germany to pay off 10% interest on the loan by using the formula for the present value of a continuous stream. Since the Soviet Union sends gas at a constant rate, the rate of deposit, $P(t)$, is a constant c. Since they don't start sending the gas until after 5 years have passed, the present value of the loan is given by:

$$\text{Present Value} = \int_5^\infty P(t)e^{-rt}\,dt.$$

We want to find c so that

$$\begin{aligned}20{,}000{,}000{,}000 &= \int_5^\infty ce^{-rt}\,dt = c\int_5^\infty e^{-rt}\,dt \\ &= c\lim_{b\to\infty}(-10e^{-0.10t})\Big|_5^b = ce^{-0.10(5)} \\ &\approx 6.065c.\end{aligned}$$

Dividing, we see that c should be about 3.3 billion DM per year. At 0.10 DM per m^3 of natural gas, the Soviet Union must deliver gas at the constant, continuous rate of about 33 billion m^3 per year.

15. Measuring money in thousands of dollars, the equation of the line representing the demand curve passes through (50, 980) and (350, 560). Its slope is $(560 - 980)/(350 - 50) = -420/300$. See Figure 8.98. So the equation is $y - 560 = -\frac{420}{300}(x - 350)$, i.e. $y - 560 = -\frac{7}{5}x + 490$. Thus

$$\begin{aligned}\text{Consumer surplus } &= \int_0^{350}\left(-\frac{7}{5}x + 1050\right)dx - 350\cdot 560 = -\frac{7}{10}x^2 + 1050x\Big|_0^{350} - 196000 \\ &= 85{,}750.\end{aligned}$$

(Note that $85{,}750 = \frac{1}{2} \cdot 490 \cdot 350$, the area of the triangle in Figure 8.98. We could have used this instead of the integral to find the consumer surplus.)
Recalling that our unit measure for the price axis is \$1000/car, the consumer surplus is \$85,750,000.

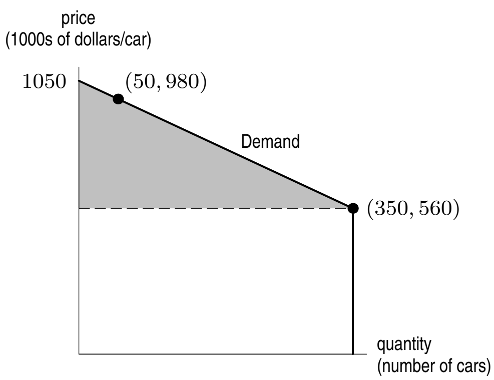


Figure 8.98

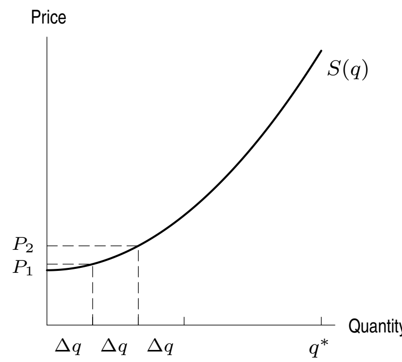


Figure 8.99

16. The supply curve, $S(q)$, represents the minimum price p per unit that the suppliers will be willing to supply some quantity q of the good for. See Figure 8.99. If the suppliers have q^* of the good and q^* is divided into subintervals of size Δq, then if the consumers could offer the suppliers for each Δq a price increase just sufficient to induce the suppliers to sell an additional Δq of the good, the consumers' total expenditure on q^* goods would be

$$p_1\Delta q + p_2\Delta q + \cdots = \sum p_i \Delta q.$$

As $\Delta q \to 0$ the Riemann sum becomes the integral $\displaystyle\int_0^{q^*} S(q)\,dq$. Thus $\displaystyle\int_0^{q^*} S(q)\,dq$ is the amount the consumers would pay if suppliers could be forced to sell at the lowest price they would be willing to accept.

17.

$$\begin{aligned}\int_0^{q^*} (p^* - S(q))\,dq &= \int_0^{q^*} p^*\,dq - \int_0^{q^*} S(q)\,dq \\ &= p^*q^* - \int_0^{q^*} S(q)\,dq.\end{aligned}$$

Using Problem 16, this integral is the extra amount consumers pay (i.e., suppliers earn over and above the minimum they would be willing to accept for supplying the good). It results from charging the equilibrium price.

18. (a) p^*q^* = the total amount paid for q^* of the good at equilibrium. See Figure 8.100.

(b) $\int_0^{q^*} D(q)\,dq$ = the maximum consumers would be willing to pay if they had to pay the highest price acceptable to them for each additional unit of the good. See Figure 8.101.

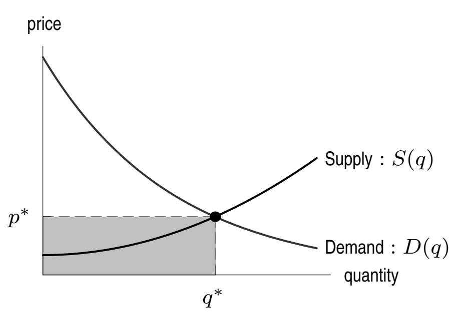


Figure 8.100

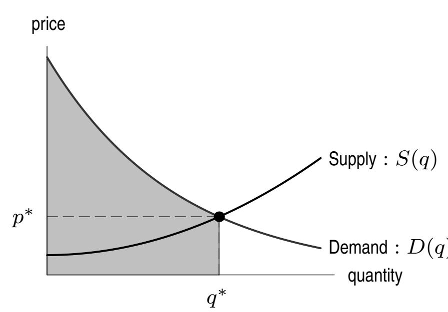


Figure 8.101

(c) $\int_0^{q^*} S(q)\,dq =$ the minimum suppliers would be willing to accept if they were paid the minimum price acceptable to them for each additional unit of the good. See Figure 8.102.

(d) $\int_0^{q^*} D(q)\,dq - p^*q^* =$ consumer surplus. See Figure 8.103.

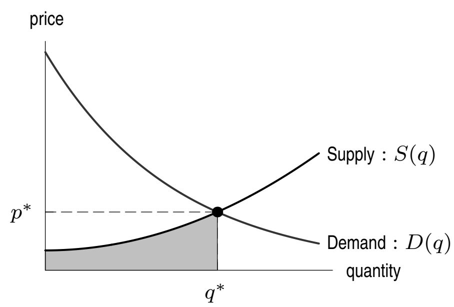


Figure 8.102

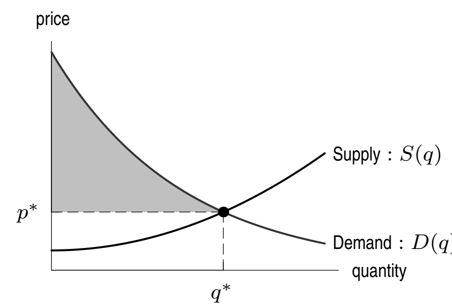


Figure 8.103

(e) $p^*q^* - \int_0^{q^*} S(q)\,dq =$ producer surplus. See Figure 8.104.

(f) $\int_0^{q^*} (D(q) - S(q))\,dq =$ producer surplus and consumer surplus. See Figure 8.105.

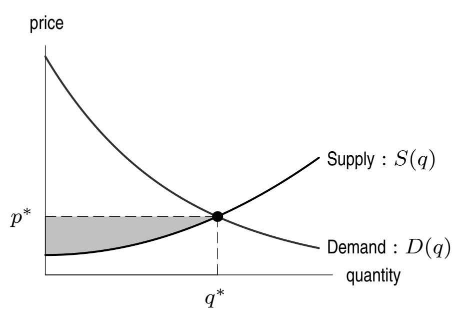


Figure 8.104

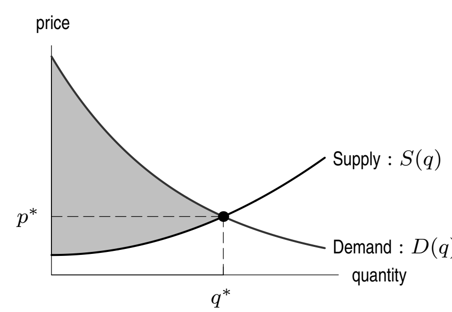


Figure 8.105

19.

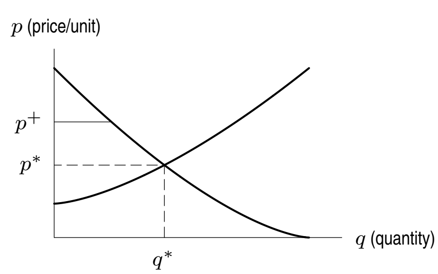


Figure 8.106: What effect does the artificially high price, p^+, have?

(a) A graph of possible demand and supply curves for the milk industry is given in Figure 8.106, with the equilibrium price and quantity labeled p^* and q^* respectively. Suppose that the price is fixed at the artificially high price labeled p^+ in Figure 8.106. Recall that the consumer surplus is the difference between the amount the consumers did pay (p^+) and the amount they would have been willing to pay (given on the demand curve). This is the area shaded in Figure 8.107(i). Notice that this consumer surplus is clearly less than the consumer surplus at the equilibrium price, shown in Figure 8.107(ii).

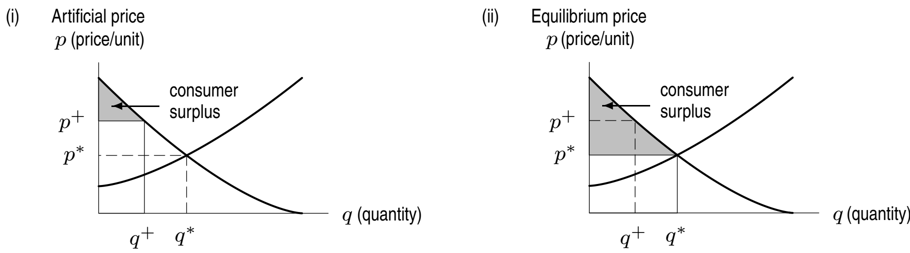


Figure 8.107: Consumer surplus for the milk industry

(b) At a price of p^+, the quantity sold, q^+, is less than it would have been at the equilibrium price. The producer surplus is the area between p^+ and the supply curve *at this reduced demand*. This area is shaded in Figure 8.108(i). Compare this producer surplus (at the artificially high price) to the producer surplus in Figure 8.108(ii) (at the equilibrium price). It appears that in this case, producer surplus is greater at the artificial price than at the equilibrium price. (Different supply and demand curves might have led to a different answer.)

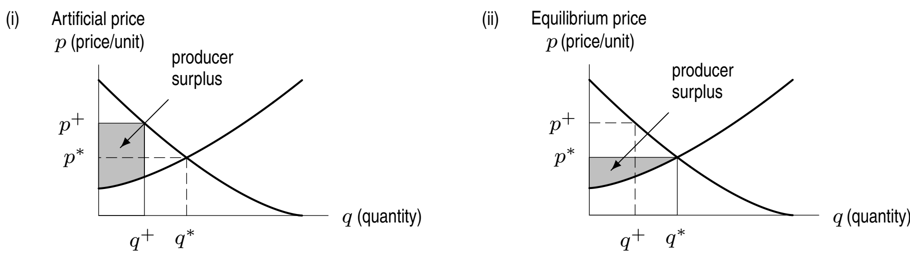


Figure 8.108: Producer surplus for the milk industry

(c) The total gains from trade (Consumer surplus + Producer surplus) at the artificially high price of p^+ is the area shaded in Figure 8.109(i). The total gains from trade at the equilibrium price of p^* is the area shaded in Figure 8.109(ii). It is clear that, under artificial price conditions, total gains from trade go down. The total financial effect of the artificially high price on all producers and consumers combined is a negative one.

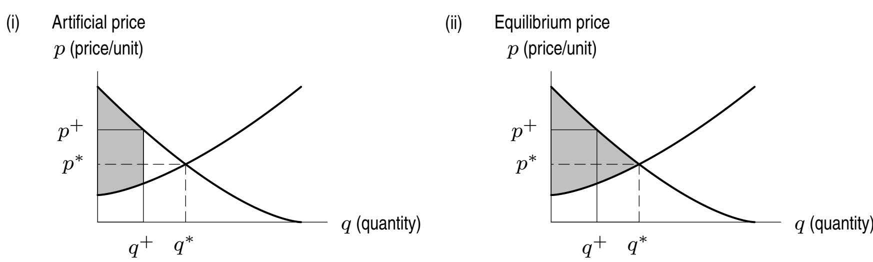
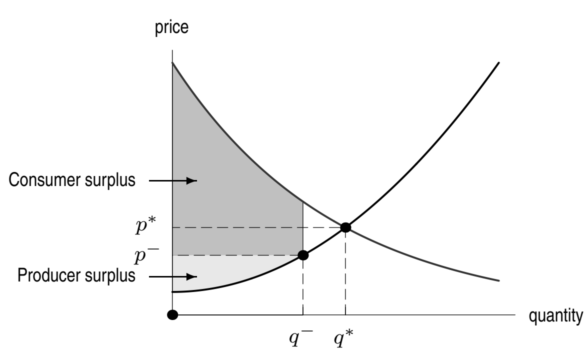


Figure 8.109: Total gains from trade

20.



(a) The producer surplus is the area on the graph between p^- and the supply function. Lowering the price also lowers the producer surplus.

(b) Note that the consumer surplus—the area between the line p^- and the supply curve—increases or decreases depending on the functions describing the supply and demand and on the lowered price. (For example, the consumer surplus seems to be increased in the graph above, but if the price were brought down to \$0 then the consumer surplus would be zero, and hence clearly less than the consumer surplus at equilibrium.)

(c) The graph above shows that the total gains from the trade are decreased.

Solutions for Section 8.7

Exercises

1.

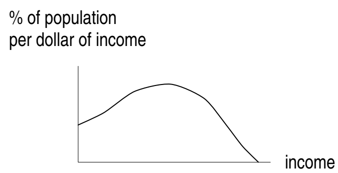


Figure 8.110: Density function

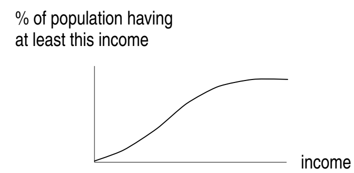


Figure 8.111: Cumulative distribution function

2.

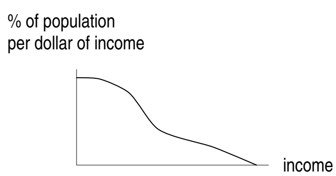


Figure 8.112: Density function

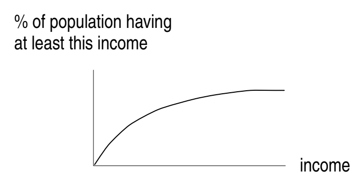


Figure 8.113: Cumulative distribution function

3.

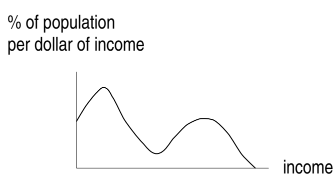


Figure 8.114: Density function

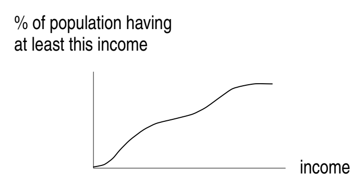


Figure 8.115: Cumulative distribution function

4. Since the function takes on the value of 4, it cannot be a cdf (whose maximum value is 1). In addition, the function decreases for $x > c$, which means that it is not a cdf. Thus, this function is a pdf. The area under a pdf is 1, so $4c = 1$ giving $c = \frac{1}{4}$. The pdf is $p(x) = 4$ for $0 \leq x \leq \frac{1}{4}$, so the cdf is given in Figure 8.116 by

$$P(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 4x & \text{for} \quad 0 \leq x \leq \dfrac{1}{4} \\ 1 & \text{for} \quad x > \dfrac{1}{4} \end{cases}$$

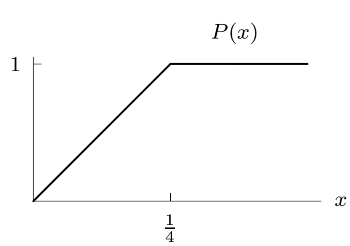


Figure 8.116

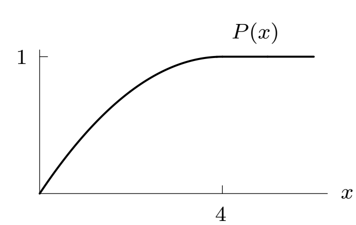


Figure 8.117

5. Since the function is decreasing, it cannot be a cdf (whose values never decrease). Thus, the function is a pdf.

The area under a pdf is 1, so, using the formula for the area of a triangle, we have

$$\frac{1}{2}4c = 1, \quad \text{giving} \quad c = \frac{1}{2}.$$

The pdf is

$$p(x) = \frac{1}{2} - \frac{1}{8}x \quad \text{for} \quad 0 \le x \le 4,$$

so the cdf is given in Figure 8.117 by

$$P(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \dfrac{x}{2} - \dfrac{x^2}{16} & \text{for} \quad 0 \le x \le 4 \\ 1 & \text{for} \quad x > 4. \end{cases}$$

6. Since the function levels off at the value of c, the area under the graph is not finite, so it is not 1. Thus, this function cannot be a pdf.

It is a cdf and $c = 1$. The cdf is given by

$$P(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \dfrac{x}{5} & \text{for} \quad 0 \le x \le 5 \\ 1 & \text{for} \quad x > 5. \end{cases}$$

The pdf in Figure 8.118 is given by

$$p(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1/5 & \text{for} \quad 0 \le x \le 5 \\ 0 & \text{for} \quad x > 5. \end{cases}$$

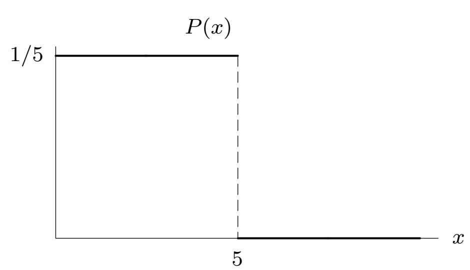


Figure 8.118

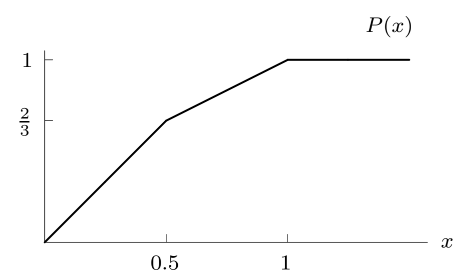


Figure 8.119

7. This function decreases, so it cannot be a cdf. Since the graph must represent a pdf, the area under it is 1. The region consists of two rectangles, each of base 0.5, and one of height $2c$ and one of height c, so

$$\begin{aligned} \text{Area} = 2c(0.5) + c(0.5) &= 1 \\ c &= \frac{1}{1.5} = \frac{2}{3} \end{aligned}$$

The pdf is therefore

$$p(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 4/3 & \text{for} \quad 0 \le x \le 0.5 \\ 2/3 & \text{for} \quad 0.5 < x \le 1 \\ 0 & \text{for} \quad x > 1. \end{cases}$$

The cdf $P(x)$ is the antiderivative of this function with $P(0) = 0$. See Figure 8.119. The formula for $P(x)$ is

$$P(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 4x/3 & \text{for} \quad 0 \le x \le 0.5 \\ 2/3 + (2/3)(x - 0.5) & \text{for} \quad 0.5 < x \le 1 \\ 1 & \text{for} \quad x > 1. \end{cases}$$

8. This function increases and levels off to c. The area under the curve is not finite, so it is not 1. Thus, the function must be a cdf, not a pdf, and $3c = 1$, so $c = 1/3$.

The pdf, $p(x)$ is the derivative, or slope, of the function shown, so, using $c = 1/3$,

$$p(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ (1/3 - 0)/(2 - 0) = 1/6 & \text{for} \quad 0 \le x \le 2 \\ (1 - 1/3)/(4 - 2) = 1/3 & \text{for} \quad 2 < x \le 4 \\ 0 & \text{for} \quad x > 4. \end{cases}$$

See Figure 8.120.

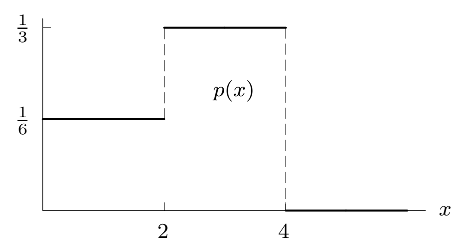


Figure 8.120

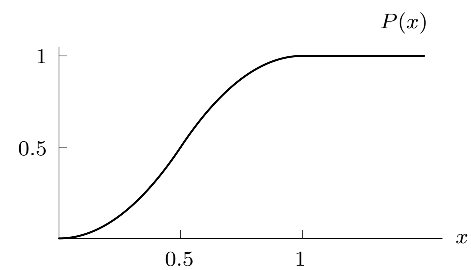


Figure 8.121

9. This function does not level off to 1, and it is not always increasing. Thus, the function is a pdf. Since the area under the curve must be 1, using the formula for the area of a triangle,

$$\frac{1}{2} \cdot c \cdot 1 = 1 \quad \text{so} \quad c = 2.$$

Thus, the pdf is given by

$$p(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 4x & \text{for} \quad 0 \le x \le 0.5 \\ 2 - 4(x - 0.5) = 4 - 4x & \text{for} \quad 0.5 < x \le 1 \\ 0 & \text{for} \quad x > 0. \end{cases}$$

To find the cdf, we integrate each part of the function separately, making sure that the constants of integration are arranged so that the cdf is continuous.

Since $\int 4x dx = 2x^2 + C$ and $P(0) = 0$, we have $2(0)^2 + C = 0$ so $C = 0$. Thus $P(x) = 2x^2$ on $0 \le x \le 0.5$. At $x = 0.5$, the cdf has value $P(0.5) = 2(0.5)^2 = 0.5$. Thus, we arrange that the integral of $4 - 4x$ goes through the point $(0.5, 0.5)$. Since $\int (4 - 4x)\, dx = 4x - 2x^2 + C$, we have

$$4(0.5) - 2(0.5)^2 + C = 0.5 \quad \text{giving} \quad C = -1.$$

Thus

$$P(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 2x^2 & \text{for} \quad 0 \le x \le 0.5 \\ 4x - 2x^2 - 1 & \text{for} \quad 0.5 < x \le 1 \\ 1 & \text{for} \quad x > 1. \end{cases}$$

See Figure 8.121.

Problems

10. No. Though the density function has its maximum value at 50, this does not mean that a large fraction of the population receives scores near 50. The value $p(50)$ can not be interpreted as a probability. Probability corresponds to *area* under the graph of a density function. Most of the area in this case is in the broad hump covering the range $0 \le x \le 40$, very little in the peak around $x = 50$. Most people score in the range $0 \le x \le 40$.

11. (a) Let $P(x)$ be the cumulative distribution function of the heights of the unfertilized plants. As do all cumulative distribution functions, $P(x)$ rises from 0 to 1 as x increases. The greatest number of plants will have heights in the range where $P(x)$ rises the most. The steepest rise appears to occur at about $x = 1$ m. Reading from the graph we see that $P(0.9) \approx 0.2$ and $P(1.1) \approx 0.8$, so that approximately $P(1.1) - P(0.9) = 0.8 - 0.2 = 0.6 = 60\%$ of the unfertilized plants grow to heights between 0.9 m and 1.1 m. Most of the plants grow to heights in the range 0.9 m to 1.1 m.

(b) Let $P_A(x)$ be the cumulative distribution function of the plants that were fertilized with A. Since $P_A(x)$ rises the most in the range $0.7 \text{ m} \le x \le 0.9$ m, many of the plants fertilized with A will have heights in the range 0.7 m to 0.9 m. Reading from the graph of P_A, we find that $P_A(0.7) \approx 0.2$ and $P_A(0.9) \approx 0.8$, so $P_A(0.9) - P_A(0.7) \approx 0.8 - 0.2 = 0.6 = 60\%$ of the plants fertilized with A have heights between 0.7 m and 0.9 m. Fertilizer A had the effect of stunting the growth of the plants.

On the other hand, the cumulative distribution function $P_B(x)$ of the heights of the plants fertilized with B rises the most in the range $1.1 \text{ m} \le x \le 1.3$ m, so most of these plants have heights in the range 1.1 m to 1.3 m. Fertilizer B caused the plants to grow about 0.2 m taller than they would have with no fertilizer.

12. (a) $F(7) = 0.6$ tells us that 60% of the trees in the forest have height 7 meters or less.

(b) $F(7) > F(6)$. There are more trees of height less than 7 meters than trees of height less than 6 meters because every tree of height ≤ 6 meters also has height ≤ 7 meters.

13. For a small interval Δx around 68, the fraction of the population of American men with heights in this interval is about $(0.2)\Delta x$. For example, taking $\Delta x = 0.1$, we can say that approximately $(0.2)(0.1) = 0.02 = 2\%$ of American men have heights between 68 and 68.1 inches.

14. We want to find the cumulative distribution function for the age density function. We see that $P(10)$ is equal to 0.15 since the table shows that 15% of the population is between 0 and 10 years of age. Also,

$$P(20) = \begin{array}{c}\text{Fraction of the population}\\ \text{between 0 and 20 years old}\end{array} = 0.15 + 0.14 = 0.29$$

and

$$P(30) = 0.15 + 0.14 + 0.14 = 0.43.$$

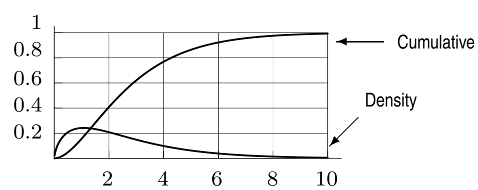
Continuing in this way, we obtain the values for $P(t)$ shown in Table 8.6.

Table 8.6 *Cumulative distribution function of ages in the US*

t	0	10	20	30	40	50	60	70	80	90	100
$P(t)$	0	0.15	0.29	0.43	0.60	0.74	0.84	0.92	0.97	0.99	1.00

15. (a) The two functions are shown below. The choice is based on the fact that the cumulative distribution does not decrease.

(b) The cumulative distribution levels off to 1, so the top mark on the vertical scale must be 1.



The total area under the density function must be 1. Since the area under the density function is about 2.5 boxes, each box must have area $1/2.5 = 0.4$. Since each box has a height of 0.2, the base must be 2.

16. (a) The area under the graph of the height density function $p(x)$ is concentrated in two humps centered at 0.5 m and 1.1 m. The plants can therefore be separated into two groups, those with heights in the range 0.3 m to 0.7 m, corresponding to the first hump, and those with heights in the range 0.9 m to 1.3 m, corresponding to the second hump. This grouping of the grasses according to height is probably close to the species grouping. Since the second hump contains more area than the first, there are more plants of the tall grass species in the meadow.

(b) As do all cumulative distribution functions, the cumulative distribution function $P(x)$ of grass heights rises from 0 to 1 as x increases. Most of this rise is achieved in two spurts, the first as x goes from 0.3 m to 0.7 m, and the second as x goes from 0.9 m to 1.3 m. The plants can therefore be separated into two groups, those with heights in the range 0.3 m to 0.7 m, corresponding to the first spurt, and those with heights in the range 0.9 m to 1.3 m, corresponding to the second spurt. This grouping of the grasses according to height is the same as the grouping we made in part (a), and is probably close to the species grouping.

(c) The fraction of grasses with height less than 0.7 m equals $P(0.7) = 0.25 = 25\%$. The remaining 75% are the tall grasses.

17. (a) The percentage of calls lasting from 1 to 2 minutes is given by the integral

$$\int_1^2 p(x)\,dx \int_1^2 0.4e^{-0.4x}\,dx = e^{-0.4} - e^{-0.8} \approx 22.1\%.$$

(b) A similar calculation (changing the limits of integration) gives the percentage of calls lasting 1 minute or less as

$$\int_0^1 p(x)\,dx = \int_0^1 0.4e^{-0.4x}\,dx = 1 - e^{-0.4} \approx 33.0\%.$$

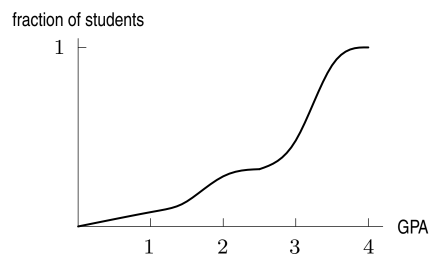
(c) The percentage of calls lasting 3 minutes or more is given by the improper integral

$$\int_3^\infty p(x)\,dx = \lim_{b\to\infty}\int_3^b 0.4e^{-0.4x}\,dx = \lim_{b\to\infty}(e^{-1.2} - e^{-0.4b}) = e^{-1.2} \approx 30.1\%.$$

(d) The cumulative distribution function is the integral of the probability density; thus,

$$C(h) = \int_0^h p(x)\,dx = \int_0^h 0.4e^{-0.4x}\,dx = 1 - e^{-0.4h}.$$

18. (a) The fraction of students passing is given by the area under the curve from 2 to 4 divided by the total area under the curve. This appears to be about $\frac{2}{3}$.
(b) The fraction with honor grades corresponds to the area under the curve from 3 to 4 divided by the total area. This is about $\frac{1}{3}$.
(c) The peak around 2 probably exists because many students work to get just a passing grade.
(d)



19. (a) Most of the earth's surface is below sea level. Much of the earth's surface is either around 3 miles below sea level or exactly at sea level. It appears that essentially all of the surface is between 4 miles below sea level and 2 miles above sea level. Very little of the surface is around 1 mile below sea level.
(b) The fraction below sea level corresponds to the area under the curve from -4 to 0 divided by the total area under the curve. This appears to be about $\frac{3}{4}$.

20. (a) We must have $\int_0^\infty f(t)dt = 1$, for even though it is possible that any given person survives the disease, everyone eventually dies. Therefore,

$$\int_0^\infty cte^{-kt}dt = 1.$$

Integrating by parts gives

$$\begin{aligned}\int_0^b cte^{-kt}dt &= -\frac{c}{k}te^{-kt}\Big|_0^b + \int_0^b \frac{c}{k}e^{-kt}dt \\ &= \left(-\frac{c}{k}te^{-kt} - \frac{c}{k^2}e^{-kt}\right)\Big|_0^b \\ &= \frac{c}{k^2} - \frac{c}{k}be^{-kb} - \frac{c}{k^2}e^{-kb}.\end{aligned}$$

As $b \to \infty$, we see

$$\int_0^\infty cte^{-kt}dt = \frac{c}{k^2} = 1 \quad \text{so} \quad c = k^2.$$

(b) We are told that $\int_0^5 f(t)dt = 0.4$, so using the fact that $c = k^2$ and the antiderivatives from part (a), we have

$$\int_0^5 k^2 te^{-kt} dt = \left(-\frac{k^2}{k} te^{-kt} - \frac{k^2}{k^2} e^{-kt} \right) \Big|_0^5$$
$$= 1 - 5ke^{-5k} - e^{-5k} = 0.4$$

so

$$5ke^{-5k} + e^{-5k} = 0.6.$$

Since this equation cannot be solved exactly, we use a calculator or computer to find $k = 0.275$. Since $c = k^2$, we have $c = (0.275)^2 = 0.076$.

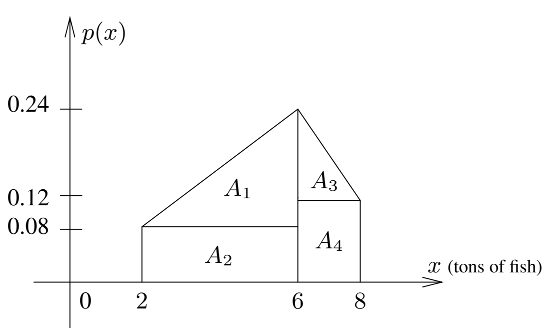
(c) The cumulative death distribution function, $C(t)$, represents the fraction of the population that have died up to time t. Thus,

$$C(t) = \int_0^t k^2 xe^{-kx} dx = \left(-kxe^{-kx} - e^{-kx} \right) \Big|_0^t$$
$$= 1 - kte^{-kt} - e^{-kt}.$$

Solutions for Section 8.8

Exercises

1.



Splitting the figure into four pieces, we see that

$$\begin{aligned} \text{Area under the curve } &= A_1 + A_2 + A_3 + A_4 \\ &= \frac{1}{2}(0.16)4 + 4(0.08) + \frac{1}{2}(0.12)2 + 2(0.12) \\ &= 1. \end{aligned}$$

We expect the area to be 1, since $\int_{-\infty}^{\infty} p(x)\,dx = 1$ for any probability density function, and $p(x)$ is 0 except when $2 \le x \le 8$.

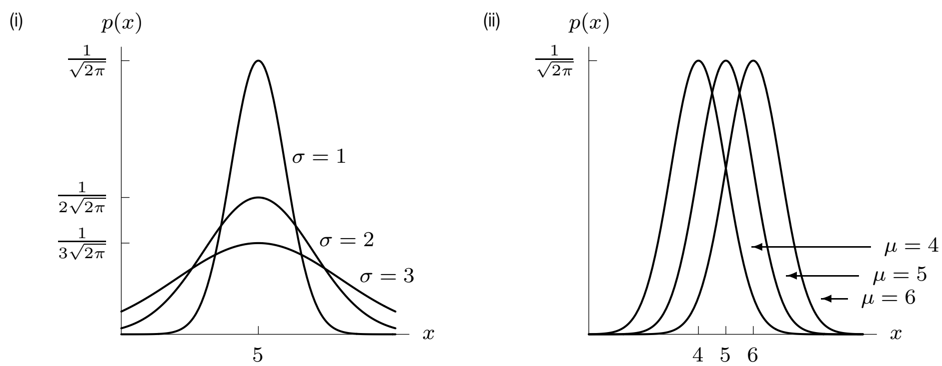
2. Recall that the mean is $\int_{-\infty}^{\infty} xp(x)\,dx$. In the fishing example, $p(x) = 0$ except when $2 \le x \le 8$, so the mean is

$$\int_2^8 xp(x)\,dx.$$

Using the equation for $p(x)$ from the graph,

$$\begin{aligned}\int_2^8 xp(x)\,dx &= \int_2^6 xp(x)\,dx + \int_6^8 xp(x)\,dx\\ &= \int_2^6 x(0.04x)\,dx + \int_6^8 x(-0.06x+0.6)\,dx\\ &= \frac{0.04x^3}{3}\bigg|_2^6 + \left(-0.02x^3+0.3x^2\right)\bigg|_6^8\\ &\approx 5.253 \text{ tons.}\end{aligned}$$

3. (a)



(b) Recall that the mean is the "balancing point." In other words, if the area under the curve was made of cardboard, we'd expect it to balance at the mean. All of the graphs are symmetric across the line $x = \mu$, so μ is the "balancing point" and hence the mean.
As the graphs also show, increasing σ flattens out the graph, in effect lessening the concentration of the data near the mean. Thus, the smaller the σ value, the more data is clustered around the mean.

Problems

4. (a) Since $d(e^{-ct})/dt = ce^{-ct}$, we have

$$c\int_0^6 e^{-ct}dt = -e^{-ct}\Big|_0^6 = 1 - e^{-6c} = 0.1,$$

so

$$c = -\frac{1}{6}\ln 0.9 \approx 0.0176.$$

(b) Similarly, with $c = 0.0176$, we have

$$\begin{aligned}c\int_6^{12} e^{-ct}dt &= -e^{-ct}\Big|_6^{12}\\ &= e^{-6c} - e^{-12c} = 0.9 - 0.81 = 0.09,\end{aligned}$$

so the probability is 9%.

5. (a) We can find the proportion of students by integrating the density $p(x)$ between $x = 1.5$ and $x = 2$:

$$\begin{aligned}P(2) - P(1.5) &= \int_{1.5}^2 \frac{x^3}{4}\,dx\\ &= \frac{x^4}{16}\bigg|_{1.5}^2\\ &= \frac{(2)^4}{16} - \frac{(1.5)^4}{16} = 0.684,\end{aligned}$$

so that the proportion is $0.684 : 1$ or 68.4%.

(b) We find the mean by integrating x times the density over the relevant range:

$$\begin{aligned}\text{Mean} &= \int_0^2 x\left(\frac{x^3}{4}\right)\,dx\\ &= \int_0^2 \frac{x^4}{4}\,dx\\ &= \frac{x^5}{20}\bigg|_0^2\\ &= \frac{2^5}{20} = 1.6 \text{ hours.}\end{aligned}$$

(c) The median will be the time T such that exactly half of the students are finished by time T, or in other words

$$\begin{aligned}\frac{1}{2} &= \int_0^T \frac{x^3}{4}\,dx\\ \frac{1}{2} &= \frac{x^4}{16}\bigg|_0^T\\ \frac{1}{2} &= \frac{T^4}{16}\\ T &= \sqrt[4]{8} = 1.682 \text{ hours.}\end{aligned}$$

6. (a) Since $\int_0^\infty p(x)\,dx = 1$, we have

$$\begin{aligned}1 &= \int_0^\infty ae^{-0.122x}\,dx\\ &= \frac{a}{-0.122}e^{-0.122x}\bigg|_0^\infty = \frac{a}{0.122}.\end{aligned}$$

So $a = 0.122$.

(b)

$$\begin{aligned}P(x) &= \int_0^x p(t)\,dt\\ &= \int_0^x 0.122e^{-0.122t}\,dt\\ &= -e^{0.122t}\Big|_0^x = 1 - e^{-0.122x}.\end{aligned}$$

(c) Median is the x such that

$$P(x) = 1 - e^{-0.122x} = 0.5.$$

So $e^{-0.122x} = 0.5$. Thus,

$$x = -\frac{\ln 0.5}{0.122} \approx 5.68 \text{ seconds}$$

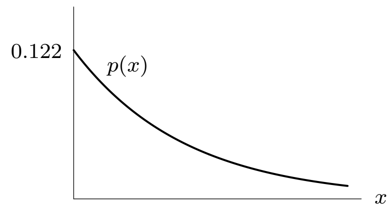
and

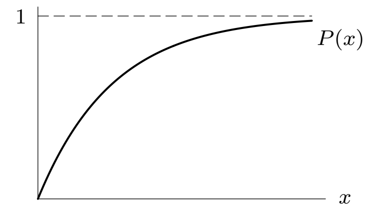
$$\text{Mean} = \int_0^\infty x(0.122)e^{-0.122x}\,dx = -\int_0^\infty x\left(-0.122e^{-0.122x}\right)\,dx.$$

We now use integration by parts. Let $u = -x$ and $v' = -0.122e^{-0.122x}$. Then $u' = -1$, and $v = e^{-0.122x}$. Therefore,

$$\text{Mean} = -xe^{-0.122x}\bigg|_0^\infty + \int_0^\infty e^{-0.122x}\,dx = \frac{1}{0.122} \approx 8.20 \text{ seconds.}$$

(d)





7. (a) The cumulative distribution function

$$\begin{aligned} P(t) = \int_0^t p(x)dx &= \text{Area under graph of density function } p(x) \text{ for } 0 \le x \le t \\ &= \text{Fraction of population who survive } t \text{ years or less after treatment} \\ &= \text{Fraction of population who survive up to } t \text{ years after treatment.} \end{aligned}$$

(b) The probability that a randomly selected person survives for at least t years is the probability that he lives t years or longer, so

$$\begin{aligned} S(t) &= \int_t^\infty p(x)\,dx = \lim_{b\to\infty} \int_t^b Ce^{-Ct}\,dx \\ &= \lim_{b\to\infty} -e^{-Ct}\Big|_t^b = \lim_{b\to\infty} -e^{-Cb} - (-e^{-Ct}) = e^{-Ct}, \end{aligned}$$

or equivalently,

$$S(t) = 1 - \int_0^t p(x)\,dx = 1 - \int_0^t Ce^{-Ct}\,dx = 1 + e^{-Ct}\Big|_0^t = 1 + (e^{-Ct} - 1) = e^{-Ct}.$$

(c) The probability of surviving at least two years is

$$S(2) = e^{-C(2)} = 0.70$$

so

$$\begin{aligned} \ln e^{-C(2)} &= \ln 0.70 \\ -2C &= \ln 0.7 \\ C &= -\frac{1}{2}\ln 0.7 \approx 0.178. \end{aligned}$$

8. (a) The probability you dropped the glove within a kilometer of home is given by

$$\int_0^1 2e^{-2x}dx = -e^{-2x}\Big|_0^1 = -e^{-2} + 1 \approx 0.865.$$

(b) Since the probability that the glove was dropped within y km $= \int_0^y p(x)dx = 1 - e^{-2y}$, we solve

$$\begin{aligned} 1 - e^{-2y} &= 0.95 \\ e^{-2y} &= 0.05 \\ y &= \frac{\ln 0.05}{-2} \approx 1.5 \text{ km.} \end{aligned}$$

9. (a) Since $\mu = 100$ and $\sigma = 15$:

$$p(x) = \frac{1}{15\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-100}{15}\right)^2}.$$

(b) The fraction of the population with IQ scores between 115 and 120 is (integrating numerically)

$$\begin{aligned} \int_{115}^{120} p(x)\,dx &= \int_{115}^{120} \frac{1}{15\sqrt{2\pi}} e^{-\frac{(x-100)^2}{450}}\,dx \\ &= \frac{1}{15\sqrt{2\pi}} \int_{115}^{120} e^{-\frac{(x-100)^2}{450}}\,dx \\ &\approx 0.067 = 6.7\% \text{ of the population.} \end{aligned}$$

10. (a) The normal distribution of car speeds with $\mu = 58$ and $\sigma = 4$ is shown in Figure 8.122.

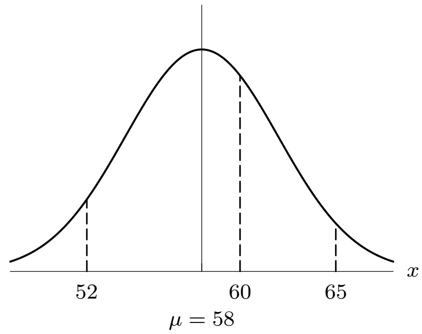


Figure 8.122

The probability that a randomly selected car is going between 60 and 65 is equal to the area under the curve from $x = 60$ to $x = 65$,

$$\text{Probability} = \frac{1}{4\sqrt{2\pi}} \int_{60}^{65} e^{-(x-58)^2/(2\cdot 4^2)}\, dx \approx 0.2685.$$

We obtain the value 0.2685 using a calculator or computer.

(b) To find the fraction of cars going under 52 km/hr, we evaluate the integral

$$\text{Fraction} = \frac{1}{4\sqrt{2\pi}} \int_{0}^{52} e^{-(x-58)^2/32} dx \approx 0.067.$$

Thus, approximately 6.7% of the cars are going less than 52 km/hr.

11. (a) First, we find the critical points of $p(x)$:

$$\begin{aligned} \frac{d}{dx}p(x) &= \frac{1}{\sigma\sqrt{2\pi}} \left[\frac{-2(x-\mu)}{2\sigma^2}\right] e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\ &= -\frac{(x-\mu)}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \end{aligned}$$

This implies $x = \mu$ is the only critical point of $p(x)$.

To confirm that $p(x)$ is maximized at $x = \mu$, we rely on the first derivative test. As $-\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is always negative, the sign of $p'(x)$ is the opposite of the sign of $(x-\mu)$; thus $p'(x) > 0$ when $x < \mu$, and $p'(x) < 0$ when $x > \mu$.

(b) To find the inflection points, we need to find where $p''(x)$ changes sign; that will happen only when $p''(x) = 0$. As

$$\frac{d^2}{dx^2}p(x) = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left[-\frac{(x-\mu)^2}{\sigma^2} + 1\right],$$

$p''(x)$ changes sign when $\left[-\frac{(x-\mu)^2}{\sigma^2} + 1\right]$ does, since the sign of the other factor is always negative. This occurs when

$$\begin{aligned} -\frac{(x-\mu)^2}{\sigma^2} + 1 &= 0, \\ -(x-\mu)^2 &= -\sigma^2, \\ x - \mu &= \pm\sigma. \end{aligned}$$

Thus, $x = \mu+\sigma$ or $x = \mu-\sigma$. Since $p''(x) > 0$ for $x < \mu-\sigma$ and $x > \mu+\sigma$ and $p''(x) < 0$ for $\mu-\sigma \le x \le \mu+\sigma$, these are in fact points of inflection.

(c) μ represents the mean of the distribution, while σ is the standard deviation. In other words, σ gives a measure of the "spread" of the distribution, i.e., how tightly the observations are clustered about the mean. A small σ tells us that most of the data are close to the mean; a large σ tells us that the data is spread out.

12. The fraction of the population within one standard deviation of the mean is given by

$$\text{Fraction within } \sigma \text{ of mean } = \int_{-\sigma}^{\sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)}\, dx.$$

Let us substitute $w = \dfrac{x}{\sigma}$ so that $dw = \dfrac{1}{\sigma}dx$, and when $x = \pm\sigma$, $w = \pm 1$. Then we have

$$\text{Fraction } = \int_{-\sigma}^{\sigma} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/(2\sigma^2)}\, dx = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}\sigma} e^{-w^2/2} \cdot \sigma dw = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw.$$

This integral is independent of σ. Evaluating the integral numerically gives 0.68, showing that about 68% of the population lies within one standard deviation of the mean.

13. It is not (a) since a probability density must be a non-negative function; not (c) since the total integral of a probability density must be 1; (b) and (d) are probability density functions, but (d) is not a good model. According to (d), the probability that the next customer comes after 4 minutes is 0. In real life there should be a positive probability of not having a customer in the next 4 minutes. So (b) is the best answer.

14. (a) P is the cumulative distribution function, so the percentage of the population that made between \$20,000 and \$50,000 is

$$P(50) - P(20) = 99\% - 75\% = 24\%.$$

Therefore $\frac{6}{25}$ of the population made between \$20,000 and \$50,000.

(b) The median income is the income such that half the people made less than this amount. Looking at the chart, we see that $P(12.6) = 50\%$, so the median must be \$12,600.

(c) The cumulative distribution function looks something like Figure 8.123. The density function is the derivative of the cumulative distribution. Qualitatively it looks like Figure 8.124.

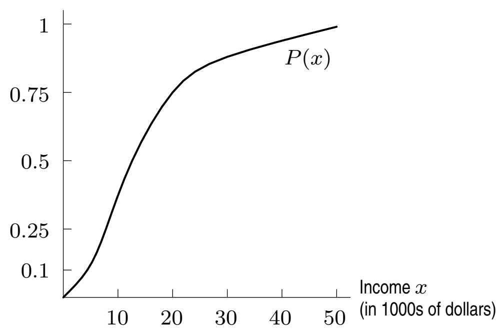


Figure 8.123: Cumulative distribution

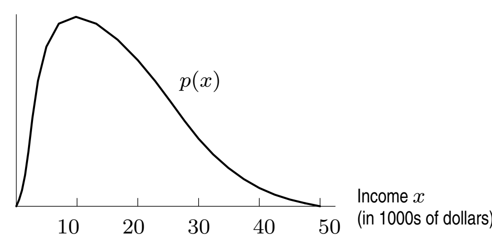


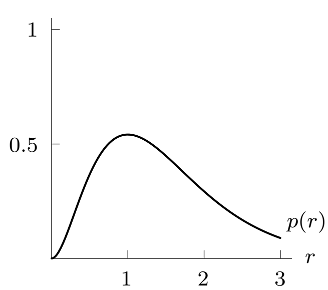
Figure 8.124: Density function

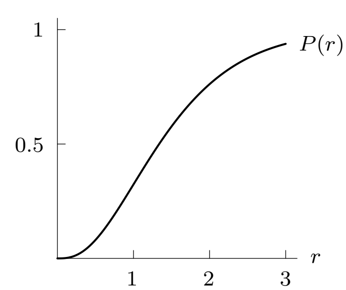
The density function has a maximum at about \$8000. This means that more people have incomes around \$8000 than around any other amount. On the density function, this is the highest point. On the cumulative distribution, this is the point of steepest slope (because $P' = p$), which is also the point of inflection.

15. (a) Let the $p(r)$ be the density function. Then $P(r) = \int_0^r p(x)\, dx$, and from the Fundamental Theorem of Calculus, $p(r) = \frac{d}{dr}P(r) = \frac{d}{dr}(1 - (2r^2 + 2r + 1)e^{-2r}) = -(4r+2)e^{-2r} + 2(2r^2 + 2r + 1)e^{-2r}$, or $p(r) = 4r^2e^{-2r}$.

We have that $p'(r) = 8r(e^{-2r}) - 8r^2e^{-2r} = e^{-2r} \cdot 8r(1-r)$, which is zero when $r = 0$ or $r = 1$, negative when $r > 1$, and positive when $r < 1$. Thus $p(1) = 4e^{-2} \approx 0.54$ is a relative maximum.

Here are sketches of $p(r)$ and the cumulative position $P(r)$:





(b) The median distance is the distance r such that $P(r) = 1 - (2r^2 + 2r + 1)e^{-2r} = 0.5$, or equivalently, $(2r^2 + 2r + 1)e^{-2r} = 0.5$.

By experimentation with a calculator, we find that $r \approx 1.33$ Bohr radii is the median distance.

The mean distance is equal to the value of the integral $\int_0^\infty rp(r)\,dr = \lim_{x\to\infty} \int_0^x rp(r)\,dr$. We have that $\int_0^x rp(r)\,dr = \int_0^x 4r^3e^{-2r}\,dr$. Using the integral table, we get

$$\int_0^x 4r^3e^{-2r}\,dr = \left[\left(-\frac{1}{2}\right)4r^3 - \frac{1}{4}(12r^2) - \frac{1}{8}(24r) - \frac{1}{16}(24)\right]e^{-2x}\Bigg|_0^x$$
$$= \frac{3}{2} - \left[2x^3 + 3x^2 + 3x + \frac{3}{2}\right]e^{-2x}.$$

Taking the limit of this expression as $x \to \infty$, we see that all terms involving (powers of x or constants) $\cdot\, e^{-2x}$ have limit 0, and thus the mean distance is 1.5 Bohr radii.

The most likely distance is obtained by maximizing $p(r) = 4r^2e^{-2r}$; as we have already seen this corresponds to $r = 1$ Bohr unit.

(c) Because it is the most likely distance of the electron from the nucleus.

Solutions for Chapter 8 Review

Exercises

1. Vertical slices are circular. Horizontal slices would be similar to ellipses in cross-section, or at least ovals (a word derived from *ovum*, the Latin word for egg).

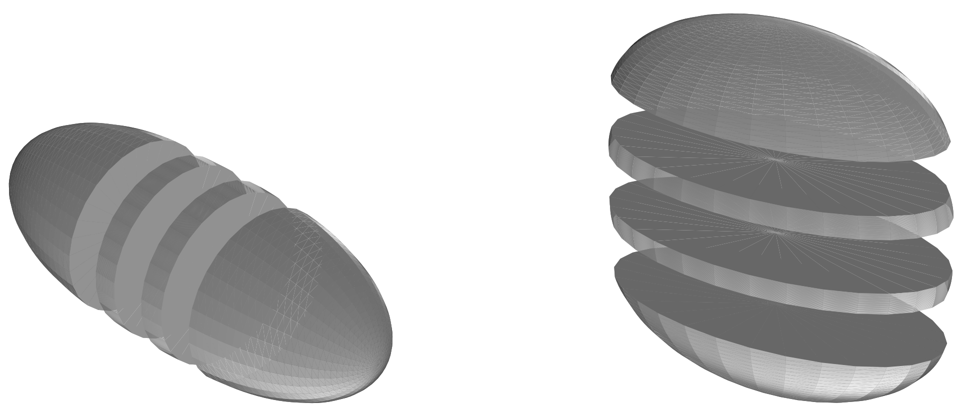
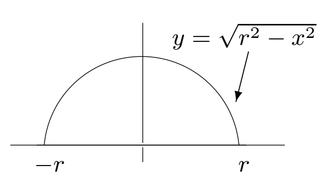


Figure 8.125

2. The limits of integration are 0 and b, and the rectangle represents the region under the curve $f(x) = h$ between these limits. Thus,

$$\text{Area of rectangle} = \int_0^b h\,dx = hx\Bigg|_0^b = hb.$$

3. The circle $x^2 + y^2 = r^2$ cannot be expressed as a function $y = f(x)$, since for every x with $-r < x < r$, there are two corresponding y values on the circle. However, if we consider the top half of the circle only, as shown below, we have $x^2 + y^2 = r^2$, or $y^2 = r^2 - x^2$, and taking the positive square root, we have that $y = \sqrt{r^2 - x^2}$ is the equation of the top semicircle.



Then

$$\text{Area of Circle} = 2(\text{Area of semicircle}) = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

We evaluate this using integral table formula 30.

$$\begin{aligned} 2\int_{x=-r}^{x=r} \sqrt{r^2 - x^2}\, dx &= 2\left[\frac{1}{2}\left(x\sqrt{r^2 - x^2} + r^2 \arcsin\frac{x}{r}\right)\right]\Bigg|_{-r}^{r} \\ &= r^2(\arcsin 1 - \arcsin(-1)) \\ &= r^2\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \pi r^2. \end{aligned}$$

4. Name the slanted line $y = f(x)$. Then the triangle is the region under the line $y = f(x)$ and between the lines $y = 0$ and $x = b$. Thus,

$$\text{Area of triangle} = \int_0^b f(x)\, dx.$$

Since $f(x)$ is a line of slope h/b which passes through the origin, its equation is $f(x) = hx/b$. Thus,

$$\text{Area of triangle} = \int_0^b \frac{hx}{b}\, dx = \frac{hx^2}{2b}\Bigg|_0^b = \frac{hb^2}{2b} = \frac{hb}{2}.$$

5. We slice the region vertically. Each rotated slice is approximately a cylinder with radius $y = x^2 + 1$ and thickness Δx. See Figure 8.126. The volume of a typical slice is $\pi(x^2+1)^2\Delta x$. The volume, V, of the object is the sum of the volumes of the slices:

$$V \approx \sum \pi(x^2+1)^2\Delta x.$$

As $\Delta x \to 0$ we obtain an integral.

$$V = \int_0^4 \pi(x^2+1)^2 dx = \pi\int_0^4 (x^4 + 2x^2 + 1)dx = \pi\left(\frac{x^5}{5} + \frac{2x^3}{3} + x\right)\Bigg|_0^4 = \frac{3772\pi}{15} = 790.006.$$

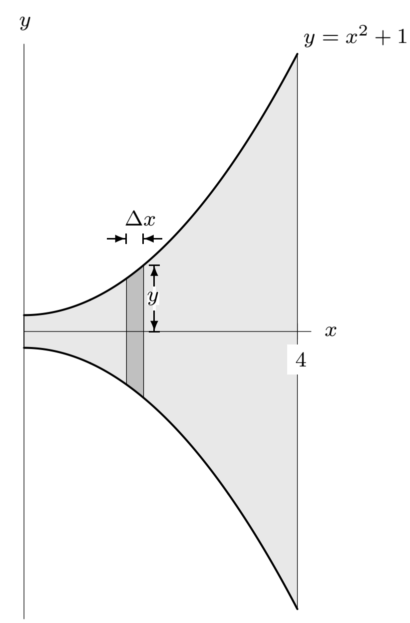


Figure 8.126

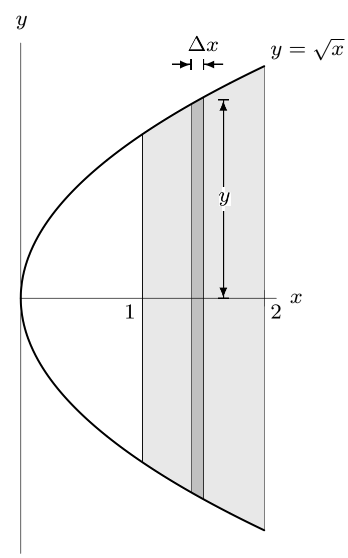


Figure 8.127

6. We slice the region vertically. Each rotated slice is approximately a cylinder with radius $y = \sqrt{x}$ and thickness Δx. See Figure 8.127. The volume of a typical slice is $\pi(\sqrt{x})^2\Delta x$. The volume, V, of the object is the sum of the volumes of the slices:

$$V \approx \sum \pi(\sqrt{x})^2\Delta x.$$

As $\Delta x \to 0$ we obtain an integral.

$$V = \int_1^2 \pi(\sqrt{x})^2 dx = \pi\int_1^2 x dx = \pi\left(\frac{x^2}{2}\right)\bigg|_1^2 = \frac{3\pi}{2} = 4.712.$$

7. We slice the region vertically. Each rotated slice is approximately a cylinder with radius $y = e^{-2x}$ and thickness Δx. See Figure 8.128. The volume of a typical slice is $\pi(e^{-2x})^2\Delta x$. The volume, V, of the object is the sum of the volumes of the slices:

$$V \approx \sum \pi(e^{-2x})^2\Delta x.$$

As $\Delta x \to 0$ we obtain an integral.

$$V = \int_0^1 \pi(e^{-2x})^2 dx = \pi\int_0^1 e^{-4x}dx = \pi\left(-\frac{1}{4}\right)(e^{-4x})\bigg|_0^1 = -\frac{\pi}{4}(e^{-4} - 1) = 0.771.$$

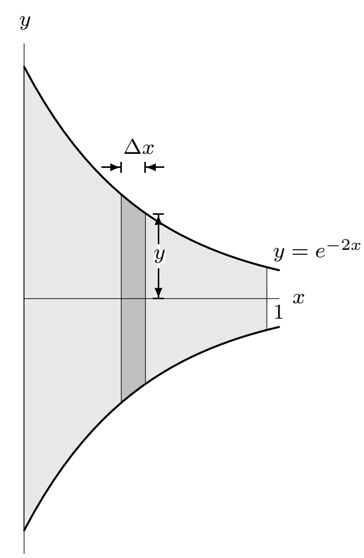


Figure 8.128

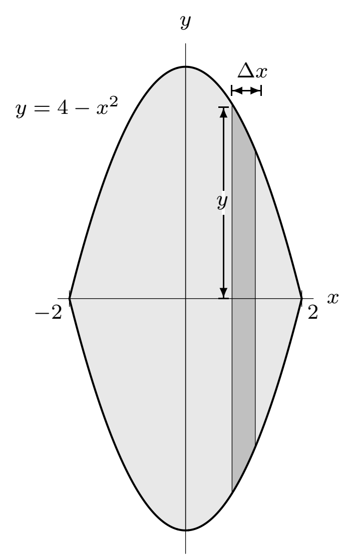


Figure 8.129

8. We slice the region vertically. Each rotated slice is approximately a cylinder with radius $y = 4 - x^2$ and thickness Δx. See Figure 8.129. The volume, V, of a typical slice is $\pi(4 - x^2)^2\Delta x$. The volume of the object is the sum of the volumes of the slices:

$$V \approx \sum \pi(4 - x^2)^2\Delta x.$$

As $\Delta x \to 0$ we obtain an integral. Since the region lies between $x = -2$ and $x = 2$, we have:

$$V = \int_{-2}^2 \pi(4 - x^2)^2 dx = \pi\int_{-2}^2 (16 - 8x^2 + x^4)dx = \pi\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\bigg|_{-2}^2 = \frac{512\pi}{15} = 107.233.$$

9. We divide the region into vertical strips of thickness Δx. As a slice is rotated about the x-axis, it creates a disk of radius r_{out} from which has been removed a smaller circular disk of inside radius r_{in}. We see in Figure 8.130 that $r_{\text{out}} = 2x$ and $r_{\text{in}} = x$. Thus,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta x - \pi(r_{\text{in}})^2\Delta x = \pi(2x)^2\Delta x - \pi(x)^2\Delta x.$$

To find the total volume, V, we integrate this quantity between $x = 0$ and $x = 3$:

$$V = \int_0^3 (\pi(2x)^2 - \pi(x)^2)dx = \pi \int_0^3 (4x^2 - x^2)\,dx = \pi \int_0^3 3x^2 dx = \pi x^3 \Big|_0^3 = 27\pi = 84.823.$$

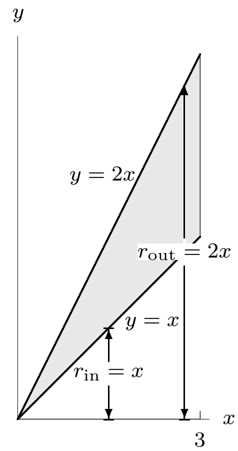


Figure 8.130

10. Each slice is a circular disk. The radius, r, of the disk increases with h and is given in the problem by $r = \sqrt{h}$. Thus

$$\text{Volume of slice } \approx \pi r^2 \Delta h = \pi h \Delta h.$$

Summing over all slices, we have

$$\text{Total volume } \approx \sum \pi h \Delta h.$$

Taking a limit as $\Delta h \to 0$, we get

$$\text{Total volume } = \lim_{\Delta h \to 0} \sum \pi h \Delta h = \int_0^{12} \pi h\,dh.$$

Evaluating gives

$$\text{Total volume } = \pi \frac{h^2}{2}\Big|_0^{12} = 72\pi.$$

11. We slice the cone horizontally into cylindrical disks with radius r and thickness Δh. See Figure 8.131. The volume of each disk is $\pi r^2 \Delta h$. We use the similar triangles in Figure 8.132 to write r as a function of h:

$$\frac{r}{h} = \frac{3}{12} \quad \text{so} \quad r = \frac{1}{4}h.$$

The volume of the disk at height h is $\pi(\frac{1}{4}h)^2 \Delta h$. To find the total volume, we integrate this quantity from $h = 0$ to $h = 12$.

$$V = \int_0^{12} \pi \left(\frac{1}{4}h\right)^2 dh = \frac{\pi}{16} \frac{h^3}{3}\Big|_0^{12} = 36\pi = 113.097 \text{ m}^3.$$

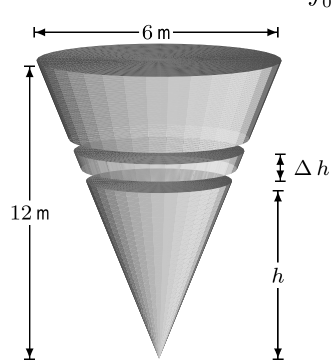


Figure 8.131

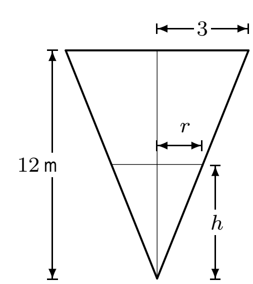


Figure 8.132

12. (a) We slice the pyramid horizontally. See Figure 8.133. Each slice is a square slab of thickness Δh, so the volume of a slice at height h is $s^2\Delta h$, where s is the length of a side. We use the similar triangles in Figure 8.134 to write s as a function of h:

$$\frac{s}{10-h} = \frac{8}{10} \quad \text{so} \quad s = 0.8(10-h).$$

The volume of the slice at height h is $(0.8(10-h))^2\Delta h$. To find the total volume, we integrate this quantity from $h = 0$ to $h = 10$.

$$V = \int_0^{10} (0.8(10-h))^2 dh = 0.64\int_0^{10} (h-10)^2 dh = \frac{16}{75}(h-10)^3\bigg|_0^{10} = \frac{640}{3} = 213.333 \text{ m}^3.$$

(b) As in part (a),

$$\text{Volume of a slice at height } h \approx s^2\Delta h = (0.8(10-h))^2\Delta h.$$

The height h ranges from $h = 0$ to $h = 6$. We have

$$V = \int_0^{6} (0.8(10-h))^2 dh = 0.64\int_0^{6} (h-10)^2\, dh = \frac{16}{75}(h-10)^3\bigg|_0^{6} = \frac{4992}{25} = 199.680 \text{ m}^3.$$

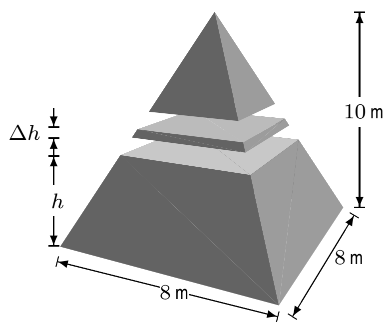


Figure 8.133

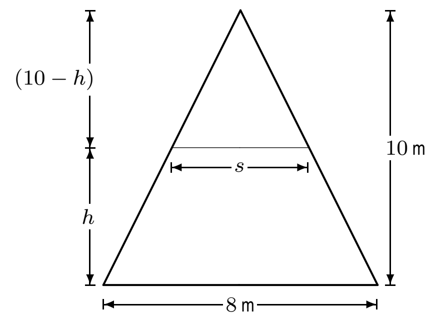


Figure 8.134

13. We slice the tank horizontally. There is an outside radius r_{out} and an inside radius r_{in} and, at height h,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta h - \pi(r_{\text{in}})^2\Delta h.$$

See Figure 8.135. We see that $r_{\text{out}} = 3$ for every slice. We use similar triangles to find r_{in} in terms of the height h:

$$\frac{r_{\text{in}}}{h} = \frac{3}{6} \quad \text{so} \quad r_{\text{in}} = \frac{1}{2}h.$$

At height h,

$$\text{Volume of slice } \approx \pi(3)^2\Delta h - \pi\left(\frac{1}{2}h\right)^2\Delta h.$$

To find the total volume, we integrate this quantity from $h = 0$ to $h = 6$.

$$V = \int_0^6 \left(\pi(3)^2 - \pi\left(\frac{1}{2}h\right)^2\right) dh = \pi\int_0^6 \left(9 - \frac{1}{4}h^2\right) dh = \pi\left(9h - \frac{h^3}{12}\right)\bigg|_0^6 = 36\pi = 113.097 \text{ m}^3.$$

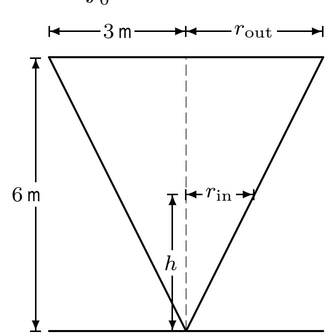


Figure 8.135

14. Since $f(x) = \sin x$, $f'(x) = \cos(x)$, so

$$\text{Arc Length} = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx.$$

15. We'll find the arc length of the top half of the ellipse, and multiply that by 2. In the top half of the ellipse, the equation $(x^2/a^2) + (y^2/b^2) = 1$ implies

$$y = +b\sqrt{1 - \frac{x^2}{a^2}}.$$

Differentiating $(x^2/a^2) + (y^2/b^2) = 1$ implicitly with respect to x gives us

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0,$$

so

$$\frac{dy}{dx} = \frac{\frac{-2x}{a^2}}{\frac{2y}{b^2}} = -\frac{b^2 x}{a^2 y}.$$

Substituting this into the arc length formula, we get

$$\begin{aligned}\text{Arc Length} &= \int_{-a}^{a} \sqrt{1 + \left(-\frac{b^2 x}{a^2 y}\right)^2}\, dx \\ &= \int_{-a}^{a} \sqrt{1 + \left(\frac{b^4 x^2}{a^4 (b^2)(1 - \frac{x^2}{a^2})}\right)}\, dx \\ &= \int_{-a}^{a} \sqrt{1 + \left(\frac{b^2 x^2}{a^2 (a^2 - x^2)}\right)}\, dx.\end{aligned}$$

Hence the arc length of the entire ellipse is

$$2\int_{-a}^{a} \sqrt{1 + \left(\frac{b^2 x^2}{a^2 (a^2 - x^2)}\right)}\, dx.$$

16. Since $f'(x) = \cos x$, we have

$$L = \int_0^3 \sqrt{1 + (f'(x))^2}\, dx = \int_0^3 \sqrt{1 + \cos^2 x}\, dx = 3.621.$$

We see in Figure 8.136 that the length of the curve is slightly longer than the length of the x-axis from $x = 0$ to $x = 3$, so the answer of 3.621 makes sense.

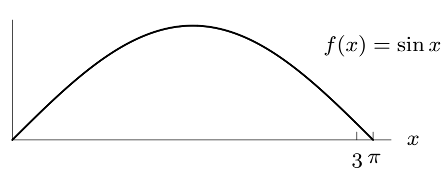


Figure 8.136

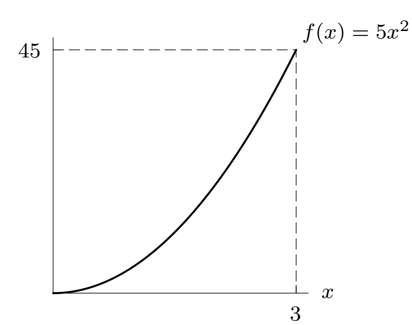


Figure 8.137

17. Since $f'(x) = 10x$, we have

$$L = \int_0^3 \sqrt{1 + (f'(x))^2}\, dx = \int_0^3 \sqrt{1 + (10x)^2}\, dx = \int_0^3 \sqrt{1 + 100x^2}\, dx = 45.230.$$

We see in Figure 8.137 that the length of the curve is definitely longer than 45 and slightly longer than $\sqrt{45^2 + 3^2} = 45.10$, so the answer of 45.230 is reasonable.

Problems

18. (a) The points of intersection are $x = 0$ to $x = 2$, so we have

$$\text{Area} = \int_0^2 (2x - x^2)dx = x^2 - \frac{x^3}{3}\bigg|_0^2 = \frac{4}{3} = 1.333.$$

(b) The outside radius is $2x$ and the inside radius is x^2, so we have

$$\text{Volume} = \int_0^2 (\pi(2x)^2 - \pi(x^2)^2)dx = \pi \int_0^2 (4x^2 - x^4)dx = \frac{\pi}{15}(20x^3 - 3x^5)\bigg|_0^2 = \frac{64\pi}{15} = 13.404.$$

(c) The length of the perimeter is equal to the length of the top plus the length of the bottom. Using the arclength formula, and the fact that the derivative of $2x$ is 2 and the derivative of x^2 is $2x$, we have

$$L = \int_0^2 \sqrt{1 + 2^2}dx + \int_0^2 \sqrt{1 + (2x)^2}dx = 4.4721 + 4.6468 = 9.119.$$

19. (a) See Figure 8.138

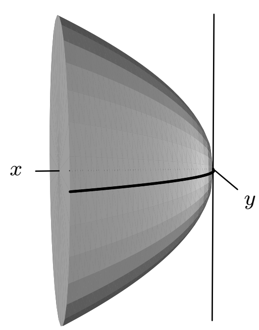


Figure 8.138: Rotated Region

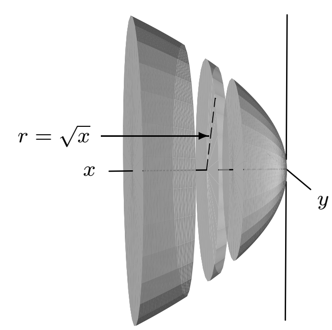


Figure 8.139: Cutaway View

(b) Divide [0,1] into N subintervals of width $\Delta x = \frac{1}{N}$. The volume of the i^{th} disc is $\pi(\sqrt{x_i})^2\Delta x = \pi x_i \Delta x$. So, $V \approx \sum_{i=1}^{N} \pi x_i \Delta x$. See Figure 8.139

(c)

$$\text{Volume} = \int_0^1 \pi x\, dx = \frac{\pi}{2}x^2\bigg|_0^1 = \frac{\pi}{2} \approx 1.57.$$

20. (a) See Figure 8.140.

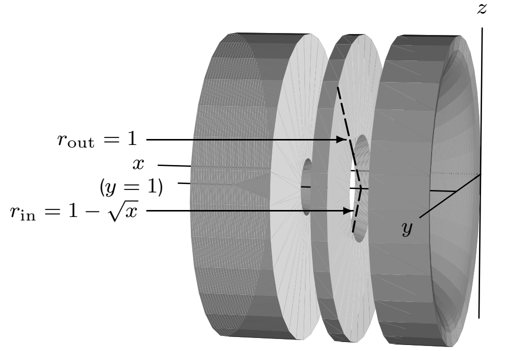


Figure 8.140

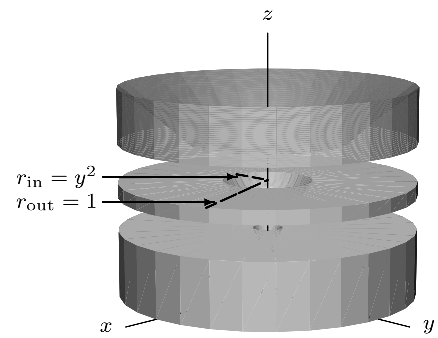


Figure 8.141

Slice the figure perpendicular to the x–axis. One gets washers of inner radius $1 - \sqrt{x}$ and outer radius 1. Therefore,

$$\begin{aligned} V &= \int_0^1 \left(\pi 1^2 - \pi(1-\sqrt{x})^2\right)\, dx \\ &= \pi \int_0^1 (1 - [1 - 2\sqrt{x} + x])\, dx \\ &= \pi \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2\right]_0^1 = \frac{5\pi}{6} \approx 2.62. \end{aligned}$$

(b) See Figure 8.141. Note that $x = y^2$. We now integrate over y instead of x, slicing perpendicular to the y–axis. This gives us washers of inner radius x and outer radius 1. So

$$\begin{aligned} V &= \int_{y=0}^{y=1} (\pi 1^2 - \pi x^2)\, dy \\ &= \int_0^1 \pi(1 - y^4) dy \\ &= \left(\pi y - \frac{\pi}{5}y^5\right)\Big|_0^1 = \pi - \frac{\pi}{5} = \frac{4\pi}{5} \approx 2.51. \end{aligned}$$

21. (a) Since $y = ax^2$ is non-negative, we integrate to find the area:

$$\text{Area} = \int_0^2 (ax^2) dx = a\frac{x^3}{3}\Big|_0^2 = \frac{8a}{3}.$$

(b) Each slice of the object is approximately a cylinder with radius ax^2 and thickness Δx. We have

$$\text{Volume} = \int_0^2 \pi(ax^2)^2 dx = \pi a^2 \frac{x^5}{5}\Big|_0^2 = \frac{32}{5}a^2\pi.$$

22. (a) Since $y = e^{-bx}$ is non-negative, we integrate to find the area:

$$\text{Area} = \int_0^1 (e^{-bx}) dx = \frac{-1}{b}e^{-bx}\Big|_0^1 = \frac{1}{b}(1 - e^{-b}).$$

(b) Each slice of the object is approximately a cylinder with radius e^{-bx} and thickness Δx. We have

$$\text{Volume} = \int_0^1 \pi(e^{-bx})^2 dx = \pi\int_0^1 e^{-2bx} dx = \frac{-\pi}{2b}e^{-2bx}\Big|_0^1 = \frac{\pi}{2b}(1 - e^{-2b}).$$

23. (a) We divide the region into vertical strips of thickness Δx. As a slice is rotated about the x-axis, it creates a disk of radius r_{out} from which has been removed a smaller circular disk of radius r_{in}. We see in Figure 8.142 that $r_{\text{out}} = \sin x$ and $r_{\text{in}} = 0.5x$. Thus,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta x - \pi(r_{\text{in}})^2\Delta x = \pi(\sin x)^2\Delta x - \pi(0.5x)^2\Delta x.$$

To find the total volume, we integrate this quantity between the points of intersection $x = 0$ and $x = 1.9$:

$$V = \int_0^{1.9} (\pi(\sin x)^2 - \pi(0.5x)^2)dx = \pi\left(-\frac{\sin x \cos x}{2} - \frac{x^3}{12} + x\right)\Bigg|_0^{1.9} = 1.669.$$

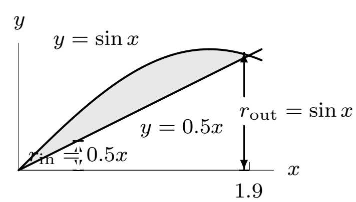


Figure 8.142

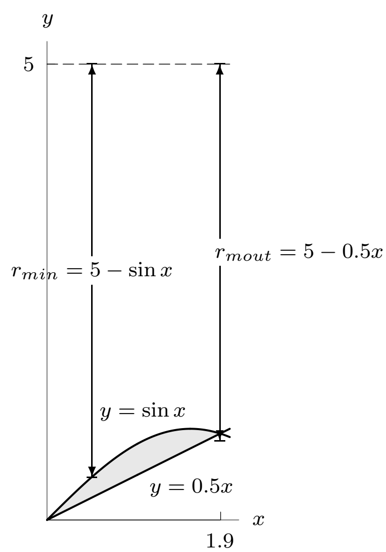


Figure 8.143

(b) We see in Figure 8.143 that $r_{\text{out}} = 5 - 0.5x$ and $r_{\text{in}} = 5 - \sin x$. Thus,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta x - \pi(r_{\text{in}})^2\Delta x = \pi(5 - 0.5x)^2\Delta x - \pi(5 - \sin x)^2\Delta x.$$

To find the total volume, V, we integrate this quantity between the points of intersection $x = 0$ and $x = 1.9$:

$$V = \int_0^{1.9} \left(\pi(5 - 0.5x)^2 - \pi(5 - \sin x)^2\right) dx = \frac{\pi}{12}(6(\sin x - 20)\cos x + x(x^2 - 30x - 6))\Bigg|_0^{1.9} = 11.550.$$

24. We divide the region into vertical strips of thickness Δx. As a slice is rotated about the x-axis, it creates a disk of radius r_{out} from which has been removed a disk of radius r_{in}. We see in Figure 8.144 that $r_{\text{out}} = 5 + 2x$ and $r_{\text{in}} = 5$. Thus,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta x - \pi(r_{\text{in}})^2\Delta x = \pi(5 + 2x)^2\Delta x - \pi(5)^2\Delta x.$$

To find the total volume, V, we integrate this quantity between $x = 0$ and $x = 4$:

$$V = \int_0^4 (\pi(5 + 2x)^2 - \pi(5)^2)dx = \pi\int_0^4 ((5 + 2x)^2 - 25)\, dx = \pi\left(\frac{4}{3}x^3 + 10x^2\right)\Bigg|_0^4 = \frac{736\pi}{3} = 770.737.$$

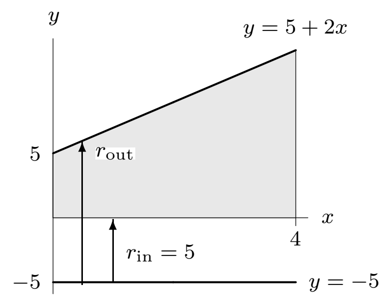


Figure 8.144

25. (a) We divide the region into vertical strips of thickness Δx. As a slice is rotated about the x-axis, it creates a disk of radius r_{out} from which has been removed a disk of radius r_{in}. We see in Figure 8.145 that $r_{\text{out}} = 2 + x^2$ and $r_{\text{in}} = 2$. Thus,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta x - \pi(r_{\text{in}})^2\Delta x = \pi(2+x^2)^2\Delta x - \pi(2)^2\Delta x.$$

To find the total volume, V, we integrate this quantity between $x = 0$ and $x = 3$:

$$V = \int_0^3 (\pi(2+x^2)^2 - \pi(2)^2)dx = \pi\int_0^3 ((2+x^2)^2 - 4)\,dx = \frac{\pi}{15}(3x^5 + 20x^3)\bigg|_0^3 = \frac{423\pi}{5} = 265.778.$$

(b) We see in Figure 8.146 that $r_{\text{out}} = 10$ and $r_{\text{in}} = 10 - x^2$. Thus,

$$\text{Volume of a slice } \approx \pi(r_{\text{out}})^2\Delta x - \pi(r_{\text{in}})^2\Delta x = \pi(10)^2\Delta x - \pi(10-x^2)^2\Delta x.$$

To find the total volume, V, we integrate this quantity between $x = 0$ and $x = 3$:

$$V = \int_0^3 (\pi(10)^2 - \pi(10-x^2)^2)dx = \pi\int_0^3 (100 - (10-x^2)^2)\,dx = \frac{\pi}{15}(100x^3 - 3x^5)\bigg|_0^3 = \frac{657\pi}{5} = 412.805.$$

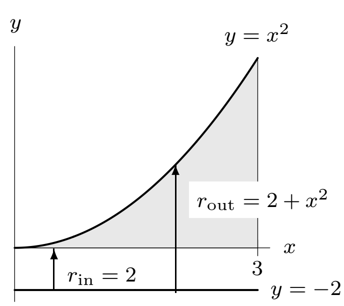


Figure 8.145

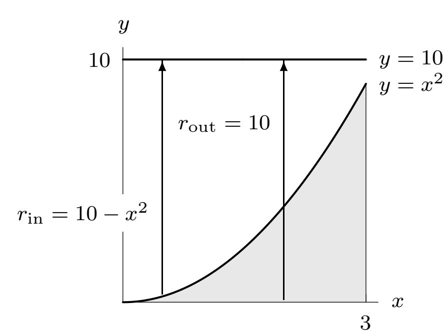


Figure 8.146

26. Slice the object into disks vertically, as in Figure 8.147. A typical disk has thickness Δx and radius $y = \sqrt{1-x^2}$. Thus

$$\text{Volume of disk } \approx \pi y^2\Delta x = \pi(1-x^2)\,\Delta x.$$

$$\text{Volume of solid } = \lim_{\Delta x\to 0}\sum \pi(1-x^2)\,\Delta x = \int_0^1 \pi(1-x^2)\,dx = \pi\left(x - \frac{x^3}{3}\right)\bigg|_0^1 = \frac{2\pi}{3}.$$

Note: As we expect, this is the volume of a half sphere.

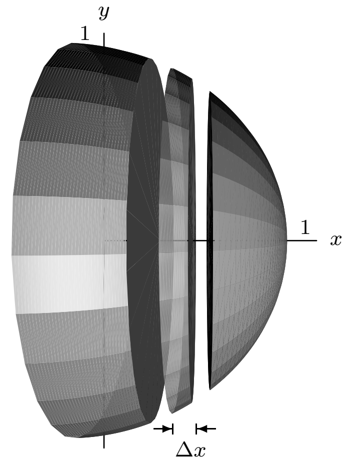


Figure 8.147

27. Slice the object into rings horizontally, as in Figure 8.148. A typical ring has thickness Δy, inner radius 2 and outer radius $2 + x = 2 + \sqrt{1-y^2}$. Thus,

$$\text{Volume of ring } \approx \pi(2+\sqrt{1-y^2})^2\Delta y - \pi 2^2 \Delta y = \pi(4\sqrt{1-y^2} + 1 - y^2)\,\Delta y.$$

$$\begin{aligned}
\text{Volume of solid } &= \int_0^1 \pi(4\sqrt{1-y^2} + 1 - y^2)\,dy \\
&= 4\pi\int_0^1 \sqrt{1-y^2}\,dy + \pi\int_0^1 1\,dy - \pi\int_0^1 y^2\,dy \\
&= 4\pi\left(\frac{1}{2}\left(y\sqrt{1-y^2}\bigg|_0^1 + \int_0^1 \frac{1}{\sqrt{1-y^2}}\,dy\right)\right) + \pi y\bigg|_0^1 - \frac{\pi y^3}{3}\bigg|_0^1 \\
&= 2\pi y\sqrt{1-y^2} + 2\pi \arcsin y + \pi y - \frac{\pi y^3}{3}\bigg|_0^1 \\
&= 0 + 2\pi\arcsin 1 + \pi - \frac{\pi}{3} - 0 - 2\pi\arcsin 0 - 0 + 0 \\
&= \pi^2 + \frac{2\pi}{3} = 11.964.
\end{aligned}$$

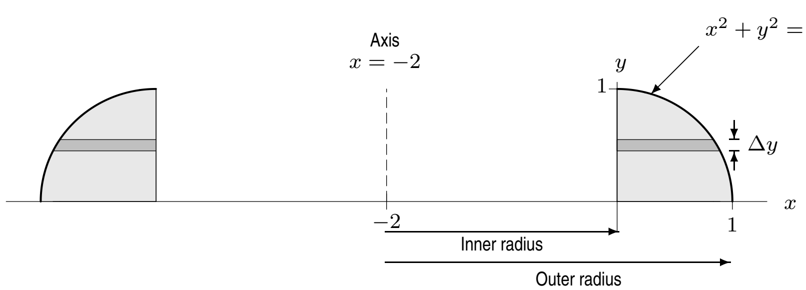


Figure 8.148: Cross-section of solid

28. Slice the object into rings horizontally, as in Figure 8.149. A typical ring has thickness Δy, outer radius 1, and inner radius $1 - x = 1 - \sqrt{1-y^2}$. Thus,

$$\text{Volume of ring } \approx \pi 1^2\Delta y - \pi(1-\sqrt{1-y^2})^2\Delta y = \pi(2\sqrt{1-y^2} - (1-y^2))\,\Delta y.$$

$$\begin{aligned}
\text{Volume of solid } &= \int_0^1 \pi(2\sqrt{1-y^2} - 1 + y^2)\,dy \\
&= 2\pi \int_0^1 \sqrt{1-y^2}\,dy - \pi \int_0^1 1\,dy + \pi \int_0^1 y^2\,dy \\
&= 2\pi \cdot \frac{1}{2}\left(y\sqrt{1-y^2}\Big|_0^1 + 1^2 \int_0^1 \frac{1}{\sqrt{1-y^2}}\,dy \right) - \pi y\Big|_0^1 + \frac{\pi y^3}{3}\Big|_0^1 \\
&= \pi y\sqrt{1-y^2} + \pi \arcsin y - \pi y + \frac{\pi y^3}{3}\Big|_0^1 \\
&= 0 + \frac{\pi^2}{2} - \pi + \frac{\pi}{3} - 0 - 0 + 0 - 0 \\
&= \frac{\pi^2}{2} - \frac{2\pi}{3} = 2.840.
\end{aligned}$$

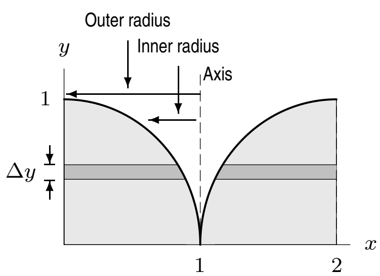


Figure 8.149: Cross-section of solid

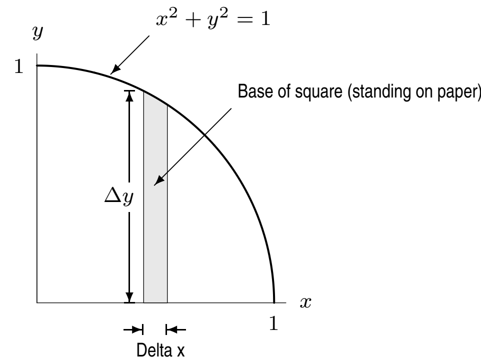


Figure 8.150: Base of solid

29. Slicing perpendicularly to the x-axis gives squares whose thickness is Δx and whose side is $y = \sqrt{1-x^2}$. See Figure 8.150. Thus,

$$\text{Volume of square slice } \approx (\sqrt{1-x^2})^2 \Delta x = (1-x^2)\,\Delta x.$$

$$\text{Volume of solid } = \int_0^1 (1-x^2)\,dx = x - \frac{x^3}{3}\Big|_0^1 = \frac{2}{3}.$$

30. Slicing perpendicularly to the y-axis gives semicircles whose thickness is Δy and whose diameter is $x = \sqrt{1-y^2}$. See Figure 8.151. Thus

$$\text{Volume of semicircular slice } \approx \pi \left(\frac{\sqrt{1-y^2}}{2} \right)^2 \Delta y = \frac{\pi}{4}(1-y^2)\,\Delta y.$$

$$\text{Volume of solid } = \int_0^1 \frac{\pi}{4}(1-y^2)\,dy = \frac{\pi}{4}\left(y - \frac{y^3}{3} \right)\Big|_0^1 = \frac{\pi}{4} \cdot \frac{2}{3} = \frac{\pi}{6}.$$

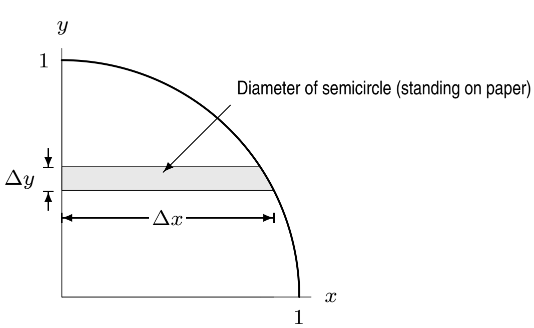


Figure 8.151: Base of Solid

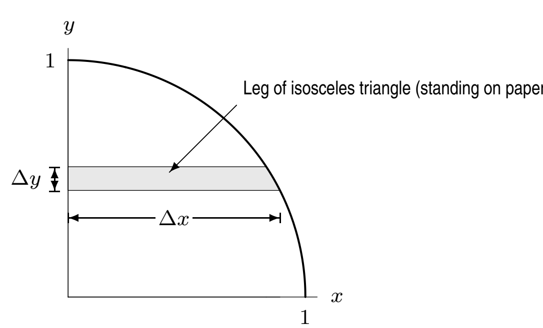


Figure 8.152: Base of solid

31. An isosceles triangle with legs of length s has

$$\text{Area } = \frac{1}{2}s^2.$$

Slicing perpendicularly to the y-axis gives isosceles triangles whose thickness is Δy and whose leg is $x = \sqrt{1-y^2}$. See Figure 8.152. Thus

$$\text{Volume of triangular slice } \approx \frac{1}{2}(\sqrt{1-y^2})\,\Delta y = \frac{1}{2}(1-y^2)\,\Delta y.$$

$$\text{Volume of solid } = \int_0^1 \frac{1}{2}(1-y^2)\,dy = \frac{1}{2}\left(y - \frac{y^3}{3}\right)\Bigg|_0^1 = \frac{1}{3}.$$

32. (a) Slice the headlight into N disks of height Δx by cutting perpendicular to the x–axis. The radius of each disk is y; the height is Δx. The volume of each disk is $\pi y^2 \Delta x$. Therefore, the Riemann sum approximating the volume of the headlight is

$$\sum_{i=1}^{N} \pi y_i^2 \Delta x = \sum_{i=1}^{N} \pi \frac{9x_i}{4} \Delta x.$$

(b)

$$\pi \int_0^4 \frac{9x}{4}\,dx = \pi \frac{9}{8}x^2\Bigg|_0^4 = 18\pi.$$

33. (a) The line $y = ax$ must pass through (l, b). Hence $b = al$, so $a = b/l$.

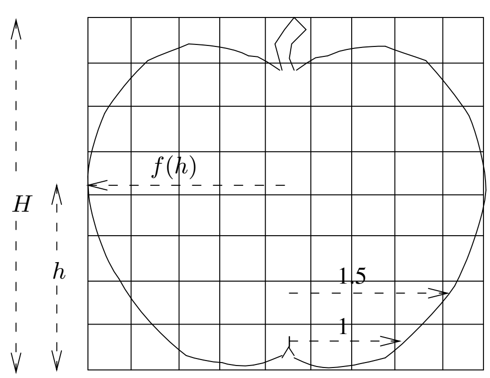
(b) Cut the cone into N slices, slicing perpendicular to the x–axis. Each piece is almost a cylinder. The radius of the ith cylinder is $r(x_i) = \dfrac{bx_i}{l}$, so the volume

$$V \approx \sum_{i=1}^{N} \pi \left(\frac{bx_i}{l}\right)^2 \Delta x.$$

Therefore, as $N \to \infty$, we get

$$\begin{aligned} V &= \int_0^l \pi b^2 l^{-2} x^2 dx \\ &= \pi \frac{b^2}{l^2}\left[\frac{x^3}{3}\right]_0^l = \left(\pi \frac{b^2}{l^2}\right)\left(\frac{l^3}{3}\right) = \frac{1}{3}\pi b^2 l. \end{aligned}$$

34. (a) If you slice the apple perpendicular to the core, you expect that the cross section will be approximately a circle.



If $f(h)$ is the radius of the apple at height h above the bottom, and H is the height of the apple, then

$$\text{Volume} = \int_0^H \pi f(h)^2\,dh.$$

Ignoring the stem, $H \approx 3.5$. Although we do not have a formula for $f(h)$, we can estimate it at various points. (Remember, we measure here from the bottom of the *apple*, which is not quite the bottom of the graph.)

h	0	0.5	1	1.5	2	2.5	3	3.5
$f(h)$	1	1.5	2	2.1	2.3	2.2	1.8	1.2

Now let $g(h) = \pi f(h)^2$, the area of the cross-section at height h. From our approximations above, we get the following table.

h	0	0.5	1	1.5	2	2.5	3	3.5
$g(h)$	3.14	7.07	12.57	13.85	16.62	13.85	10.18	4.52

We can now take left- and right-hand sum approximations. Note that $\Delta h = 0.5$ inches. Thus

$$\text{LEFT}(9) = (3.14 + 7.07 + 12.57 + 13.85 + 16.62 + 13.85 + 10.18)(0.5) = 38.64.$$
$$\text{RIGHT}(9) = (7.07 + 12.57 + 13.85 + 16.62 + 13.85 + 10.18 + 4.52)(0.5) = 39.33.$$

Thus the volume of the apple is ≈ 39 cu.in.

(b) The apple weighs $0.03 \times 39 \approx 1.17$ pounds, so it costs about 94¢.

35.

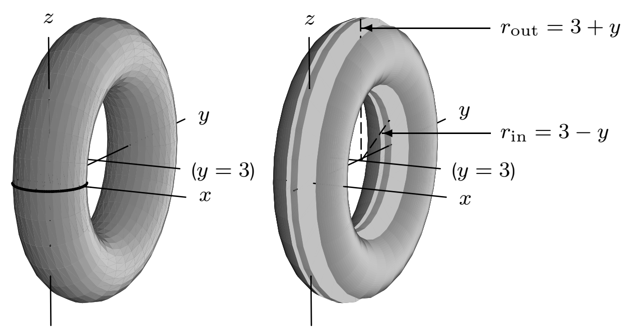


Figure 8.153: The Torus **Figure 8.154**: Slice of Torus

As shown in Figure 8.154, we slice the torus perpendicular to the line $y = 3$. We obtain washers with width dx, inner radius $r_{\text{in}} = 3 - y$, and outer radius $r_{\text{out}} = 3 + y$. Therefore, the area of the washer is $\pi r_{\text{out}}^2 - \pi r_{\text{in}}^2 = \pi[(3+y)^2 - (3-y)^2] = 12\pi y$. Since $y = \sqrt{1-x^2}$, the volume is gotten by summing up the volumes of the washers: we get

$$\int_{-1}^{1} 12\pi\sqrt{1-x^2}\,dx = 12\pi \int_{-1}^{1} \sqrt{1-x^2}\,dx.$$

But $\int_{-1}^{1} \sqrt{1-x^2}\,dx$ is the area of a semicircle of radius 1, which is $\frac{\pi}{2}$. So we get $12\pi \cdot \frac{\pi}{2} = 6\pi^2 \approx 59.22$. (Or, you could use

$$\int \sqrt{1-x^2}\,dx = \left[x\sqrt{1-x^2} + \arcsin(x)\right],$$

by VI-30 and VI-28.)

36. Multiplying $r = 2a\cos\theta$ by r, converting to Cartesian coordinates, and completing the square gives

$$\begin{aligned} r^2 &= 2ar\cos\theta \\ x^2 + y^2 &= 2ax \\ x^2 - 2ax + a^2 + y^2 &= a^2 \\ (x-a)^2 + y^2 &= a^2. \end{aligned}$$

This is the standard form of the equation of a circle with radius a and center $(x, y) = (a, 0)$.

To check the limits on θ note that the circle is in the right half plane, where $-\pi/2 \le \theta \le \pi/2$. Rays from the origin at all these angles meet the circle because the circle is tangent to the y-axis at the origin.

37. The area is given by

$$\int_{-\pi/2}^{\pi/2} \frac{1}{2} r^2\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(2a\cos\theta\right)^2\, d\theta = 2a^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = 2a^2\left(\frac{1}{2}\cos\theta\sin\theta + \frac{\theta}{2}\right)\Bigg|_{-\pi/2}^{\pi/2} = \pi a^2.$$

(We have used formula IV-18 from the integral table. The integral can also be done using a calculator or integration by parts.)

38. See Figure 8.155. The circles meet where

$$\begin{aligned} 2a\cos\theta &= a \\ \cos\theta &= \frac{1}{2} \\ \theta &= \pm\frac{\pi}{3}. \end{aligned}$$

The area is obtained by subtraction:

$$\begin{aligned} \text{Area } &= \int_{-\pi/3}^{\pi/3} \left(\frac{1}{2}(2a\cos\theta)^2 - \frac{1}{2}a^2\right) d\theta \\ &= \int_{-\pi/3}^{\pi/3} \left(2a^2\cos^2\theta - \frac{1}{2}a^2\right) d\theta \\ &= \left(2a^2\left(\frac{1}{2}\cos\theta\sin\theta + \frac{\theta}{2}\right) - \frac{a^2}{2}\theta\right)\Bigg|_{-\pi/3}^{\pi/3} \\ &= \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) a^2. \end{aligned}$$

Since

$$\frac{\left(\pi/3 + \sqrt{3}/2\right) a^2}{\pi a^2} = 61\%$$

the shaded region covers 61% of circle C.

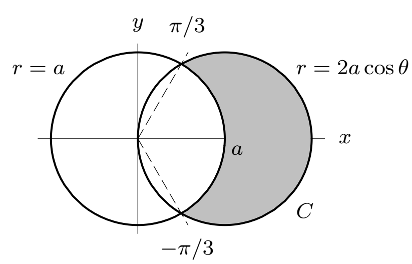


Figure 8.155

39. (a) Writing C in parametric form gives

$$x = 2a\cos^2\theta \quad \text{and} \quad y = 2a\cos\theta\sin\theta,$$

so the slope is given by

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2a\sin^2\theta + 2a\cos^2\theta}{-4a\cos\theta\sin\theta} = \frac{\sin^2\theta - \cos^2\theta}{2\cos\theta\sin\theta}.$$

(b) The maximum y-value occurs where $dy/dx = 0$, so

$$\begin{aligned} \sin^2\theta - \cos^2\theta &= 0 \\ \theta &= \pm\frac{\pi}{4}. \end{aligned}$$

The value $\theta = \pi/4$ gives the maximum y-value; $\theta = -\pi/4$ gives the minimum y-value.

40. Writing C in parametric form gives

$$x = 2a\cos^2\theta \quad \text{and} \quad y = 2a\cos\theta\sin\theta,$$

so

$$\begin{aligned}
\text{Arc length } &= \int_{-\pi/2}^{\pi/2} \sqrt{(-4a\cos\theta\sin\theta)^2 + (-2a\sin^2\theta + 2a\cos^2\theta)^2}\, d\theta \\
&= 2a\int_{-\pi/2}^{\pi/2} \sqrt{4\cos^2\theta\sin^2\theta + \sin^4\theta - 2\sin^2\theta\cos^2\theta + \cos^4\theta}\, d\theta \\
&= 2a\int_{-\pi/2}^{\pi/2} \sqrt{\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta}\, d\theta \\
&= 2a\int_{-\pi/2}^{\pi/2} \sqrt{(\sin^2\theta + \cos^2\theta)^2}\, d\theta \\
&= 2a\int_{-\pi/2}^{\pi/2} d\theta = 2\pi a.
\end{aligned}$$

41. The total mass is 12 gm, so the center of mass is located at $\overline{x} = \frac{1}{12}(-5\cdot 3 - 3\cdot 3 + 2\cdot 3 + 7\cdot 3) = \frac{1}{4}$.

42. (a) Since the density is constant, the mass is the product of the area of the plate and its density.

$$\text{Area of the plate } = \int_0^1 (\sqrt{x} - x^2)\, dx = \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Bigg|_0^1 = \frac{1}{3}\text{ cm}^2.$$

Thus the mass of the plate is $2\cdot 1/3 = 2/3$ gm.

(b) See Figure 8.156. Since the region is "fatter" closer to the origin, $\bar{x}$ is less than $1/2$.

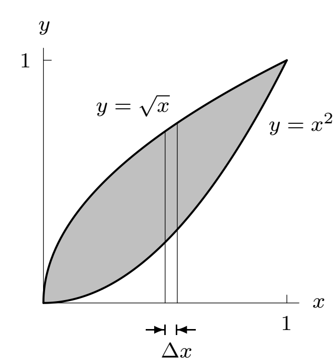


Figure 8.156

(c) To find $\bar{x}$, we slice the region into vertical strips of width Δx. See Figure 8.156.

$$\text{Area of strip } = A_x(x)\Delta x \approx (\sqrt{x} - x^2)\Delta x \text{ cm}^2.$$

Then we have

$$\overline{x} = \frac{\int x\delta A_x(x)\, dx}{\text{Mass}} = \frac{\int_0^1 2x(\sqrt{x} - x^2)\, dx}{2/3} = \frac{3}{2}\int_0^1 2(x^{3/2} - x^3)\, dx = \frac{3}{2}\cdot 2\left(\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right)\Bigg|_0^1 = \frac{9}{20}\text{ cm}.$$

This is less than $1/2$, as predicted in part (b). So $\bar{x} = \bar{y} = 9/20$ cm.

43. Let x be the height from ground to the weight. It follows that $0 \leq x \leq 20$. At height x, to lift the weight Δx more, the work needed is $200\Delta x + 2(20 - x)\Delta x = (240 - 2x)\Delta x$. So the total work is

$$\begin{aligned} W &= \int_0^{20} (240 - 2x)dx \\ &= (240x - x^2)\Big|_0^{20} \\ &= 240(20) - 20^2 = 4400 \text{ ft-lb.} \end{aligned}$$

44. Let x be the distance from the bucket to the surface of the water. It follows that $0 \leq x \leq 40$. At x feet, the bucket weighs $(30 - \frac{1}{4}x)$, where the $\frac{1}{4}x$ term is due to the leak. When the bucket is x feet from the surface of the water, the work done by raising it Δx feet is $(30 - \frac{1}{4}x)\,\Delta x$. So the total work required to raise the bucket to the top is

$$\begin{aligned} W &= \int_0^{40} (30 - \frac{1}{4}x)dx \\ &= \left(30x - \frac{1}{8}x^2\right)\Big|_0^{40} \\ &= 30(40) - \frac{1}{8}40^2 = 1000 \text{ ft-lb.} \end{aligned}$$

45.

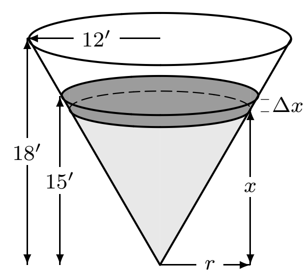


Figure 8.157

Let x be the depth of the water measured from the bottom of the tank. See Figure 8.157. It follows that $0 \leq x \leq 15$. Let r be the radius of the section of the cone with height x. By similar triangles, $\frac{r}{x} = \frac{12}{18}$, so $r = \frac{2}{3}x$. Then the work required to pump a layer of water with thickness of Δx at depth x over the top of the tank is $62.4\pi\left(\frac{2}{3}x\right)^2 \Delta x(18 - x)$. So the total work done by pumping the water over the top of the tank is

$$\begin{aligned} W &= \int_0^{15} 62.4\pi\left(\frac{2}{3}x\right)^2 (18 - x)dx \\ &= \frac{4}{9}\,62.4\pi \int_0^{15} x^2(18 - x)dx \\ &= \frac{4}{9}\,62.4\pi\left(6x^3 - \frac{1}{4}x^4\right)\Big|_0^{15} \\ &= \frac{4}{9}\,62.4\pi(7593.75) \approx 661{,}619.41 \text{ ft-lb.} \end{aligned}$$

46. We slice the gasoline horizontally. At a distance h feet below the surface, the horizontal slab is a cylinder with radius r and thickness Δh, so

$$\text{Volume of one slab } \approx \pi r^2 \Delta h.$$

To find the radius r at a depth h from the top as in Figure 8.158, we note that $h^2 + r^2 = 5^2$, so $r = \sqrt{25 - h^2}$. At depth h

$$\text{Volume of one slice} \approx \pi(\sqrt{25 - h^2})^2 \Delta h = \pi(25 - h^2)\Delta h \text{ ft}^3.$$

The gasoline at depth h must be lifted a distance of h ft, so

$$\begin{aligned} \text{Work to move one slice} &= \rho \cdot \text{Volume} \cdot \text{Distance lifted} \\ &\approx \rho(\pi(25 - h^2)\Delta h)(h) \text{ ft-lb.} \end{aligned}$$

The work done, W, to lift all the gasoline is the sum of the work done on the pieces:

$$W \approx \sum \rho(\pi(25 - h^2)\Delta h)h \text{ ft-lb..}$$

As $\Delta h \to 0$, we obtain a definite integral. Since h varies from $h = 0$ to $h = 5$ and $\rho = 42$, we have:

$$W = \int_0^5 \rho\pi(25h - h^3)dh = 42\pi\left(25\frac{h^2}{2} - \frac{h^4}{4}\right)\Bigg|_0^5 = \frac{13125\pi}{2} = 20{,}617 \text{ ft-lb.}$$

The work to pump all the gasoline out is 20,617 ft-lbs.

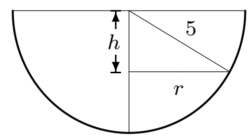


Figure 8.158

47. Let h be height above the bottom of the dam. Then

$$\begin{aligned}
\text{Water force } &= \int_0^{25} (62.4)(25 - h)(60)\, dh \\
&= (62.4)(60)\left(25h - \frac{h^2}{2}\right)\Bigg|_0^{25} \\
&= (62.4)(60)(625 - 312.5) \\
&= (62.4)(60)(312.5) \\
&= 1{,}170{,}000 \text{ lbs.}
\end{aligned}$$

48. If the weight of the chain were negligible, the work required would be $1000 \cdot 20 = 20{,}000$ ft-lbs. Because of the chain, the total work is slightly more than 20,000 ft-lbs. When the object is h ft off the ground, the length of chain is $50 - h$ so the total weight being lifted is $1000 + 2(50 - h)$ lb. See Figure 8.159. Thus

$$\begin{aligned}
\text{Work to lift the weight an addition } \Delta h \text{ higher } &= \text{Weight} \cdot \text{Distance lifted} \\
&\approx (1000 + 2(50 - h))\Delta h \text{ ft-lb.}
\end{aligned}$$

To find the total work, we integrate this quantity from $h = 0$ to $h = 20$:

$$W = \int_0^{20} (1000 + 2(50 - h))dh = \int_0^{20} (1100 - 2h)dh = (1100h - h^2)\Bigg|_0^{20} = 21{,}600\text{ft-lbs.}$$

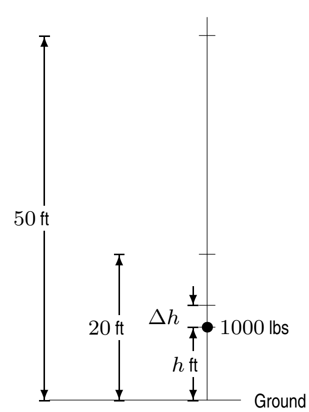


Figure 8.159

49. (a)

$$\begin{aligned}
\text{Future Value } &= \int_0^{20} 100e^{0.10(20-t)}dt \\
&= 100\int_0^{20} e^2 e^{-0.10t}dt \\
&= \frac{100e^2}{-0.10}e^{-0.10t}\Big|_0^{20} \\
&= \frac{100e^2}{0.10}(1-e^{-0.10(20)}) \approx \$6389.06.
\end{aligned}$$

The present value of the income stream is

$$\begin{aligned}
\int_0^{20} 100e^{-0.10t}dt &= 100\left(\frac{1}{-0.10}\right)e^{-0.10t}\Big|_0^{20} \\
&= 1000\left(1-e^{-2}\right) = \$864.66.
\end{aligned}$$

Note that this is also the present value of the sum \$6389.06.

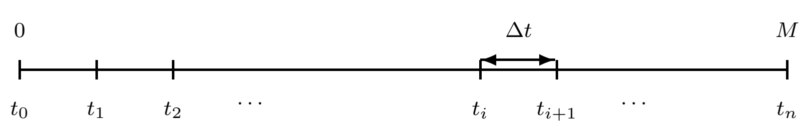
(b) Let T be the number of years for the balance to reach \$5000. Then

$$\begin{aligned}
5000 &= \int_0^{T} 100e^{0.10(T-t)}dt \\
50 &= e^{0.10T}\int_0^{T} e^{-0.10t}dt \\
&= \frac{e^{0.10T}}{-0.10}e^{-0.10t}\Big|_0^{T} \\
&= 10e^{0.10T}(1-e^{-0.10T}) = 10e^{0.10}T - 10.
\end{aligned}$$

So, $60 = 10e^{0.10T}$, and $T = 10\ln 6 \approx 17.92$ years.

50. (a) Let's split the time interval into n parts, each of length Δt. During the interval from t_i to t_{i+1}, profit is earned at a rate of approximately $(2 - 0.1t_i)$ thousand dollars per year, or $(2000 - 100t_i)$ dollars per year. Thus during this period, a total profit of $(2000-100t_i)\Delta t$ dollars is earned. Since this profit is earned t_i years in the future, its present value is $(2000 - 100t_i)\Delta t e^{-0.1t_i}$ dollars. Thus

$$\text{Total present value} \approx \sum_{i=0}^{n-1}(2000-100t_i)e^{-0.1t_i}\Delta t.$$



(b) The Riemann sum corresponds to the integral:

$$\text{Present value} = \int_0^{M} e^{-0.10t}(2000-100t)\,dt.$$

(c) To find where the present value is maximized, we take the derivative of

$$P(M) = \int_0^{M} e^{-0.10t}(2000-100t)\,dt,$$

with respect to M, and obtain

$$P'(M) = e^{-0.10M}(2000 - 100M).$$

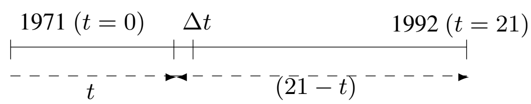
This is 0 when $2000 - 100M = 0$, that is, when $M = 20$ years. The value $M = 20$ maximizes $P(M)$, since $P'(M) > 0$ for $M < 20$, and $P'(M) < 0$ for $M > 20$. To determine what the maximum is, we evaluate the integral representation for $P(20)$ by III-14 in the integral table:

$$\begin{aligned} P(20) &= \int_0^{20} e^{-0.10t}(2000 - 100t)\, dt \\ &= \left[\frac{(2000 - 100t)}{-0.10} e^{-0.10t} + 10000e^{-0.10t}\right]\Bigg|_0^{20} \approx \$11353.35. \end{aligned}$$

51. We divide up time between 1971 and 1992 into intervals of length Δt, and calculate how much of the strontium-90 produced during that time interval is still around.

Strontium-90 decays exponentially, so if a quantity S_0 was produced t years ago, and S is the quantity around today, $S = S_0 e^{-kt}$. Since the half-life is 28 years, $\frac{1}{2} = e^{-k(28)}$, giving $k = -\ln(1/2)/28 \approx 0.025$.

We measure t in years from 1971, so that 1992 is $t = 21$.



Since strontium-90 is produced at a rate of 3 kg/year, during the interval Δt, a quantity $3\Delta t$ kg was produced. Since this was $(21 - t)$ years ago, the quantity remaining now is $(3\Delta t)e^{-0.025(21-t)}$. Summing over all such intervals gives

$$\text{Strontium remaining in 1992} \approx \int_0^{21} 3e^{-0.025(21-t)}\, dt = \frac{3e^{-0.025(21-t)}}{0.025}\Bigg|_0^{21} = 49 \text{ kg}.$$

[Note: This is like a future value problem from economics, but with a negative interest rate.]

52. (a) Slice the mountain horizontally into N cylinders of height Δh. The sum of the volumes of the cylinders will be

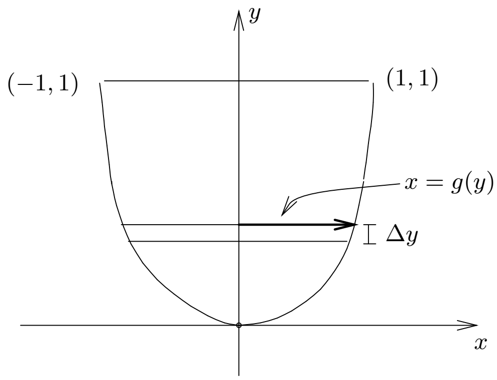
$$\sum_{i=1}^{N} \pi r^2 \Delta h = \sum_{i=1}^{N} \pi \left(\frac{3.5 \cdot 10^5}{\sqrt{h + 600}}\right)^2 \Delta h.$$

(b)

$$\begin{aligned} \text{Volume} &= \int_{400}^{14400} \pi \left(\frac{3.5 \cdot 10^5}{\sqrt{h + 600}}\right)^2 dh \\ &= 1.23 \cdot 10^{11}\pi \int_{400}^{14400} \frac{1}{(h + 600)}\, dh \\ &= 1.23 \cdot 10^{11}\pi \ln(h + 600)\Bigg|_{400}^{14400} dh \\ &= 1.23 \cdot 10^{11}\pi\, [\ln 15000 - \ln 1000] \\ &= 1.23 \cdot 10^{11}\pi \ln(15000/1000) \\ &= 1.23 \cdot 10^{11}\pi \ln 15 \approx 1.05 \cdot 10^{12} \text{ cubic feet.} \end{aligned}$$

53. Look at the disc-shaped slab of water at height y and of thickness Δy. The rate at which water is flowing out when it is at depth y is $k\sqrt{y}$ (Torricelli's Law, with k constant). Then, if $x = g(y)$, we have

$$\Delta t = \begin{pmatrix}\text{Time for water to} \\ \text{drop by this amount}\end{pmatrix} = \frac{\text{Volume}}{\text{Rate}} = \frac{\pi(g(y))^2 \Delta y}{k\sqrt{y}}.$$



If the rate at which the depth of the water is dropping is constant, then dy/dt is constant, so we want

$$\frac{\pi(g(y))^2}{k\sqrt{y}} = \text{constant,}$$

so $g(y) = c\sqrt[4]{y}$, for some constant c. Since $x = 1$ when $y = 1$, we have $c = 1$ and so $x = \sqrt[4]{y}$, or $y = x^4$.

54. For a given energy E, Figure 8.160 shows that the area under the graph to the right of E is larger for graph B than it is for graph A. Therefore graph B has more molecules at higher kinetic energies, so it is the hotter gas. So graph A corresponds to 300 kelvins and graph B corresponds to 500 kelvins.

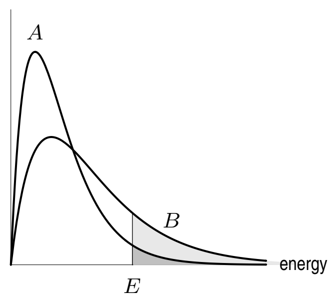


Figure 8.160

55. Graph B is more spread out to the right, and so it represents a gas in which more of the molecules are moving at faster velocities. Thus the average velocity in gas B is larger.

56. Every photon which falls a given distance from the center of the detector has the same probability of being detected. This suggests that we divide the plate up into concentric rings of thickness Δr. Consider one such ring having inner radius r and outer radius $r + \Delta r$. For this ring,

$$\text{Number of photons hitting ring per unit time } \approx N \cdot \text{ Area of ring } \approx N \cdot 2\pi r \Delta r.$$

Then,

$$\text{Number of photons detected on ring per unit time } \approx \text{ Number hitting } \cdot S(r) \approx N \cdot 2\pi r \Delta r \cdot S(r).$$

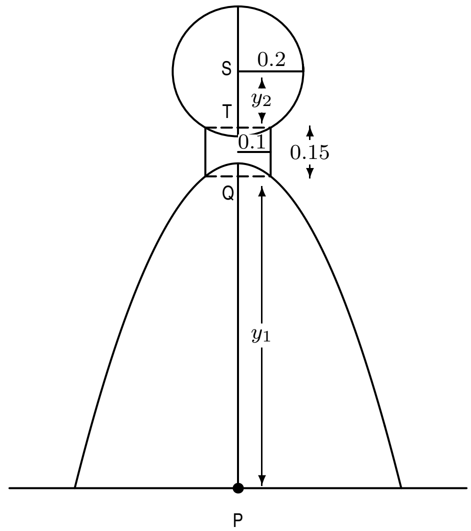
Summing over all rings gives us

$$\text{Total number of photons detected per unit time } \approx \sum 2\pi N r S(r) \Delta r.$$

Taking the limit as $\Delta r \to 0$ gives

$$\text{Total number of photons detected per unit time } = \int_0^R 2\pi N r S(r)\, dr.$$

57. First we find the volume of the body up to the horizontal line through Q.



We put the origin at P, the x-axis horizontal and the y-axis pointing upward, and compute the volume obtained by rotating the curve $y = 1 - 4x^2$ around the y-axis up to Q. At Q, we have $x = 0.1$, so

$$y_1 = 1 - 4(0.1^2) = 0.96.$$

Slicing the body horizontally into disks of radius x, thickness Δy, we have

$$\text{Volume of disk in body} \approx \pi x^2 \Delta y = \frac{\pi}{4}(1-y)\Delta y.$$

Thus,

$$\text{Volume of body up to Q} = \int_0^{0.96} \frac{\pi}{4}(1-y)dy = \frac{\pi}{4}\left(y - \frac{y^2}{2}\right)\Bigg|_0^{0.96} = 0.3921.$$

To find the volume of the head, it is easiest to consider the origin at S, the x-axis horizontal, and the y-axis pointed upward. Then think of the head as the volume obtained by rotating the circle $x^2 + y^2 = (0.2)^2$ about the y-axis. We compute the volume of the head down to the horizontal line through T, at which point $x = 0.1$. Thus

$$(0.1)^2 + {y_2}^2 = (0.2)^2.$$

So

$$y_2 = -\sqrt{0.03} = -0.1732.$$

Slicing the head into circular disks, we have

$$\text{Volume of disk in head} \approx \pi x^2 \Delta y = \pi(0.2^2 - y^2)\Delta y.$$

Thus,

$$\begin{aligned}\text{Volume of head down to T} &= \int_{-0.1732}^{0.2} \pi(0.2^2 - y^2)dy = \pi(0.2^2 y - \frac{y^3}{3})\Bigg|_{-0.1732}^{0.2} \\ &= 0.0331.\end{aligned}$$

The neck is exactly cylindrical, with

$$\text{Volume of neck} = \pi(0.1^2)0.15 = 0.0047.$$

Thus,

$$\begin{aligned}\text{Total volume} &= \text{Vol body} + \text{Vol head} + \text{Vol neck} \\ &= 0.3921 + 0.0331 + 0.0047 \\ &= 0.4299 \approx 0.43\text{m}^3.\end{aligned}$$

58. (a) Divide the cross-section of the blood into rings of radius r, width Δr. See Figure 8.161.

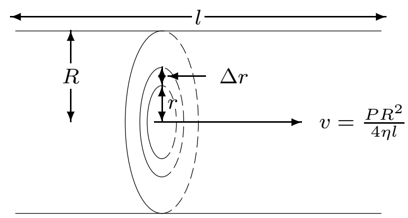


Figure 8.161

Then

$$\text{Area of ring} \approx 2\pi r\Delta r.$$

The velocity of the blood is approximately constant throughout the ring, so

$$\begin{aligned}\text{Rate blood flows through ring} &\approx \text{Velocity} \cdot \text{Area}\\ &= \frac{P}{4\eta l}(R^2 - r^2) \cdot 2\pi r\Delta r.\end{aligned}$$

Thus, summing over all rings, we find the total blood flow:

$$\text{Rate blood flowing through blood vessel} \approx \sum \frac{P}{4\eta l}(R^2 - r^2)2\pi r\Delta r.$$

Taking the limit as $\Delta r \to 0$, we get

$$\begin{aligned}\text{Rate blood flowing through blood vessel} &= \int_0^R \frac{\pi P}{2\eta l}(R^2 r - r^3)dr\\ &= \frac{\pi P}{2\eta l}\left(\frac{R^2r^2}{2} - \frac{r^4}{4}\right)\bigg|_0^R = \frac{\pi PR^4}{8\eta l}.\end{aligned}$$

(b) Since

$$\text{Rate of blood flow} = \frac{\pi PR^4}{8\eta l},$$

if we take $k = \pi P/(8\eta l)$, then we have

$$\text{Rate of blood flow} = kR^4,$$

that is, rate of blood flow is proportional to R^4, in accordance with Poiseuille's Law.

59. Pick a small interval of time Δt which takes place at time t. Fuel is consumed at a rate of $(25 + 0.1v)^{-1}$ gallons per mile. In the time Δt, the car moves $v\,\Delta t$ miles, so it consumes $v\,\Delta t/(25 + 0.1v)$ gallons during the instant Δt. Since $v = 50\frac{t}{t+1}$, the car consumes

$$\frac{v\,\Delta t}{25 + 0.1v} = \frac{50\frac{t}{t+1}\,\Delta t}{25 + 0.1\left(50\frac{t}{t+1}\right)} = \frac{50t\,\Delta t}{25(t+1) + 5t} = \frac{10t\,\Delta t}{6t + 5}$$

gallons of gas, in terms of the time t at which the instant occurs. To find the total gas consumed, sum up the instants in an integral:

$$\text{Gas consumed } = \int_2^3 \frac{10t}{6t+5}dt \approx 1.25 \text{ gallons}.$$

60. (a) Slicing horizontally, as shown in Figure 8.162, we see that the volume of one disk-shaped slab is

$$\Delta V \approx \pi x^2 \Delta y = \frac{\pi y}{a}\Delta y.$$

Thus, the volume of the water is given by

$$V = \int_0^h \frac{\pi}{a} y dy = \frac{\pi}{a}\frac{y^2}{2}\bigg|_0^h = \frac{\pi h^2}{2a}.$$

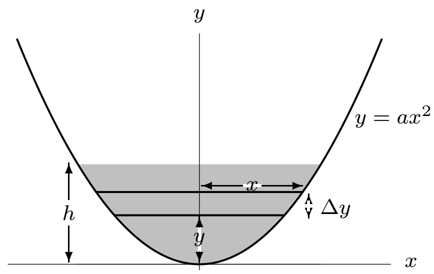


Figure 8.162

(b) The surface of the water is a circle of radius x. Since at the surface, $y = h$, we have $h = ax^2$. Thus, at the surface, $x = \sqrt{(h/a)}$. Therefore the area of the surface of water is given by

$$A = \pi x^2 = \frac{\pi h}{a}.$$

(c) If the rate at which water is evaporating is proportional to the surface area, we have

$$\frac{dV}{dt} = -kA.$$

(The negative sign is included because the volume is decreasing.) By the chain rule, $\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$. We know $\frac{dV}{dh} = \frac{\pi h}{a}$ and $A = \frac{\pi h}{a}$ so

$$\frac{\pi h}{a}\frac{dh}{dt} = -k\frac{\pi h}{a} \quad \text{giving} \quad \frac{dh}{dt} = -k.$$

(d) Integrating gives

$$h = -kt + h_0.$$

Solving for t when $h = 0$ gives

$$t = \frac{h_0}{k}.$$

61. (a) The volume of water in the centrifuge is $\pi(1^2)\cdot 1 = \pi$ cubic meters. The centrifuge has total volume 2π cubic meters, so the volume of the air in the centrifuge is π cubic meters. Now suppose the equation of the parabola is $y = h + bx^2$. We know that the volume of air in the centrifuge is the volume of the top part (a cylinder) plus the volume of the middle part (shaped like a bowl). See Figure 8.163.

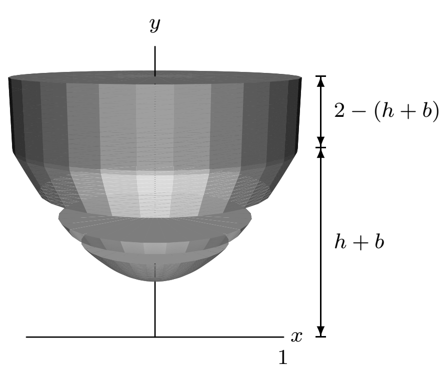


Figure 8.163: The Volume of Air

To find the volume of the cylinder of air, we find the maximum water depth. If $x = 1$, then $y = h + b$. Therefore the height of the water at the edge of the bowl, 1 meter away from the center, is $h + b$. The volume of the cylinder of air is therefore $[2 - (h + b)] \cdot \pi \cdot (1)^2 = [2 - h - b]\pi$.

To find the volume of the bowl of air, we note that the bowl is a volume of rotation with radius x at height y, where $y = h + bx^2$. Solving for x^2 gives $x^2 = (y - h)/b$. Hence, slicing horizontally as shown in the picture:

$$\text{Bowl Volume} = \int_h^{h+b} \pi x^2\,dy = \int_h^{h+b} \pi \frac{y-h}{b}\,dy = \left.\frac{\pi(y-h)^2}{2b}\right|_h^{h+b} = \frac{b\pi}{2}.$$

So the volume of both pieces together is $[2 - h - b]\pi + b\pi/2 = (2 - h - b/2)\pi$. But we know the volume of air should be π, so $(2 - h - b/2)\pi = \pi$, hence $h + b/2 = 1$ and $b = 2 - 2h$. Therefore, the equation of the parabolic cross-section is $y = h + (2 - 2h)x^2$.

(b) The water spills out the top when $h + b = h + (2 - 2h) = 2$, or when $h = 0$. The bottom is exposed when $h = 0$. Therefore, the two events happen simultaneously.

62. Any small piece of mass ΔM on either of the two spheres has kinetic energy $\frac{1}{2}v^2\Delta M$. Since the angular velocity of the two spheres is the same, the actual velocity of the piece ΔM will depend on how far away it is from the axis of revolution. The further away a piece is from the axis, the faster it must be moving and the larger its velocity v. This is because if ΔM is at a distance r from the axis, in one revolution it must trace out a circular path of length $2\pi r$ about the axis. Since every piece in either sphere takes 1 minute to make 1 revolution, pieces farther from the axis must move faster, as they travel a greater distance.

Thus, since the thin spherical shell has more of its mass concentrated farther from the axis of rotation than does the solid sphere, the bulk of it is traveling faster than the bulk of the solid sphere. So, it has the higher kinetic energy.

63. Any small piece of mass ΔM on either of the two hoops has kinetic energy $\frac{1}{2}v^2\Delta M$. Since the angular velocity of the two hoops is the same, the actual velocity of the piece ΔM will depend on how far away it is from the axis of revolution. The further away a piece is from the axis, the faster it must be moving and the larger its velocity v. This is because if ΔM is at a distance r from the axis, in one revolution it must trace out a circular path of length $2\pi r$ about the axis. Since every piece in either hoop takes 1 minute to make 1 revolution, pieces farther from the axis must move faster, as they travel a greater distance.

The hoop rotating about the cylindrical axis has all of its mass at a distance R from the axis, whereas the other hoop has a good bit of its mass close (or on) the axis of rotation. So, since the bulk of the hoop rotating about the cylindrical axis is traveling faster than the bulk of the other hoop, it must have the higher kinetic energy.

CAS Challenge Problems

64. (a) We need to check that the point with the given coordinates is on the curve, i.e., that

$$x = a\sin^2 t, \quad y = \frac{a\sin^3 t}{\cos t}$$

satisfies the equation

$$y = \sqrt{\frac{x^3}{a-x}}.$$

This can be done by substituting into the computer algebra system and asking it to simplify the difference between the two sides, or by hand calculation:

$$\begin{aligned}\text{Right-hand side} &= \sqrt{\frac{(a\sin^2 t)^3}{a - a\sin^2 t}} = \sqrt{\frac{a^3\sin^6 t}{a(1-\sin^2 t)}}\\ &= \sqrt{\frac{a^3\sin^6 t}{a\cos^2 t}} = \sqrt{\frac{a^2\sin^6 t}{\cos^2 t}}\\ &= \frac{a\sin^3 t}{\cos t} = y = \text{Left-hand side.}\end{aligned}$$

We chose the positive square root because both $\sin t$ and $\cos t$ are nonnegative for $0 \le t \le \pi/2$. Thus the point always lies on the curve. In addition, when $t = 0$, $x = 0$ and $y = 0$, so the point starts at $x = 0$. As t approaches $\pi/2$, the value of $x = a\sin^2 t$ approaches a and the value of $y = a\sin^3 t/\cos t$ increases without bound (or approaches ∞), so the point on the curve approaches the vertical asymptote at $x = a$.

(b) We calculate the volume using horizontal slices. See the graph of $y = \sqrt{x^3/(a-x)}$ in Figure 8.164.

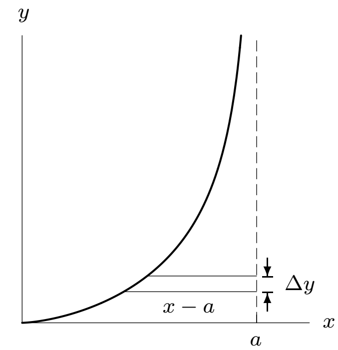


Figure 8.164

The slice at y is a disk of thickness Δy and radius $x - a$, hence it has volume $\pi(x-a)^2\Delta y$. So the volume is given by the improper integral

$$\text{Volume} = \int_0^{\infty} \pi(x-a)^2\,dy.$$

(c) We substitute

$$x = a\sin^2 t, \quad y = \frac{a\sin^3 t}{\cos t}$$

and

$$dy = \frac{d}{dt}\left(\frac{a\sin^3 t}{\cos t}\right)dt = a\left(3\sin^2 t + \frac{\sin^4 t}{\cos^2 t}\right)dt.$$

Since $t = 0$ where $y = 0$ and $t = \pi/2$ at the asymptote where $y \to \infty$, we get

$$\begin{aligned}\text{Volume} &= \int_0^{\pi/2} \pi(a\sin^2 t - a)^2 a\left(3\sin^2 t + \frac{\sin^4 t}{\cos^2 t}\right)dt \\ &= \pi a^3 \int_0^{\pi/2} (3\sin^2 t\cos^4 t + \sin^4 t\cos^2 t)\,dt = \frac{\pi^2 a^3}{8}.\end{aligned}$$

You can use a CAS to calculate this integral; it can also be done using trigonometric identities.

65. (a) The expression for arc length in terms of a definite integral gives

$$A(t) = \int_0^t \sqrt{1+4x^2}\,dx = \frac{2\,t\sqrt{1+4\,t^2} + \operatorname{arcsinh}(2\,t)}{4}.$$

The integral was evaluated using a computer algebra system; different systems may give the answer in different forms. Here arcsinh is the inverse function of the hyperbolic sine function.

(b) Figure 8.165 shows that the graphs of $A(t)$ and t^2 look very similar. This suggests that $A(t) \approx t^2$.

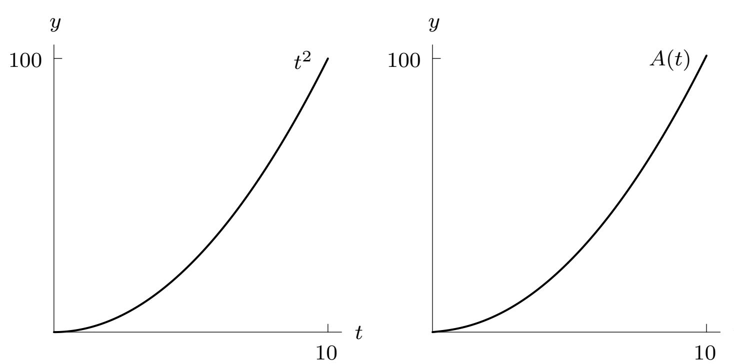


Figure 8.165

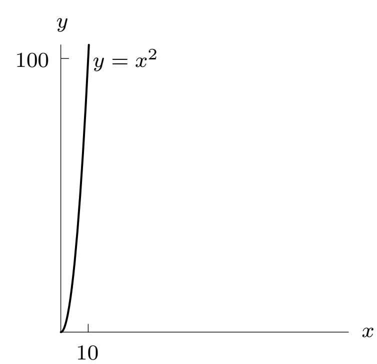


Figure 8.166

(c) The graph in Figure 8.166 is approximately vertical and close to the y axis. Thus, if we measure the arc length up to a certain y-value, the answer is approximately the same as if we had measured the length straight up the y-axis. Hence

$$A(t) \approx y = f(t) = t^2.$$

So

$$A(t) \approx t^2.$$

66. (a) The expression for arc length in terms of a definite integral gives

$$A(t) = \int_0^t \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx = \frac{2\sqrt{t}\sqrt{1+4t} + \text{arcsinh}\,(2\sqrt{t})}{4}.$$

The integral was evaluated using a computer algebra system; different systems may give the answer in different forms. Some may involve $\ln$ instead of arcsinh, which is the inverse function of the hyperbolic sine function.

(b) Figure 8.168 shows that the graphs of $A(t)$ and the graph of $y = t$ look very similar. This suggests that $A(t) \approx t$.

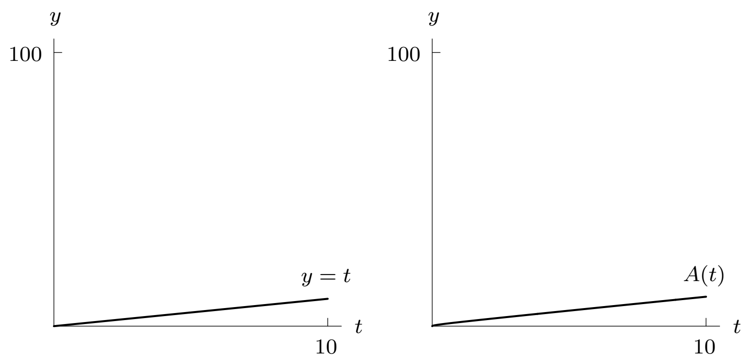


Figure 8.167

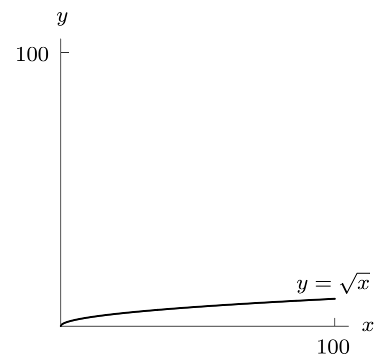


Figure 8.168

(c) The graph in Figure 8.168 is approximately horizontal and close to the x-axis. Thus, if we measure the arc length up to a certain x-value, the answer is approximately the same as if we had measured the length straight along the x-axis. Hence

$$A(t) \approx x = t.$$

So

$$A(t) \approx t.$$

67. (a) Slice the sphere at right angles to the axis of the cylinder. Consider a slice of thickness Δx at distance x from the center of the sphere. The cross-section is an annulus (ring) with internal radius $r_i = a$ and outer radius $r_o = \sqrt{r^2 - x^2}$. Thus

$$\text{Area of annulus } = \pi {r_o}^2 - \pi {r_i}^2 = \pi\left(\sqrt{r^2 - x^2}\right)^2 - \pi a^2 = \pi(r^2 - x^2 - a^2).$$

$$\text{Volume of slice} \approx \pi(r^2 - x^2 - a^2)\Delta x.$$

The lower and upper limits of the integral are where the cylinder meets the sphere, i.e., where $x^2 + a^2 = r^2$, or $x = \pm\sqrt{r^2 - a^2}$. Thus

$$\text{Volume of bead} = \int_{-\sqrt{r^2-a^2}}^{\sqrt{r^2-a^2}} \pi(r^2 - x^2 - a^2)\, dx.$$

(b) Using a computer algebra system to evaluate the integral, we have

$$\text{Volume of bead} = \frac{4\pi}{3}\left(r^2 - a^2\right)^{3/2}.$$

CHECK YOUR UNDERSTANDING

1. True. Since $y = \pm\sqrt{9-x^2}$ represent the top and bottom halves of the sphere, slicing disks perpendicular to the x-axis gives

$$\text{Volume of slice } \approx \pi y^2 \Delta x = \pi(9-x^2)\Delta x$$
$$\text{Volume} = \int_{-3}^{3} \pi(9-x^2)\,dx.$$

2. False. Evaluating does not give the volume of a cone $\pi r^2 h/3$:

$$\int_0^h \pi(r-y)\,dy = \pi\left(ry - \frac{y^2}{2}\right)\bigg|_0^h = \pi\left(rh - \frac{h^2}{2}\right).$$

Alternatively, you can show by slicing that the integral representing this volume is $\int_0^h \pi r^2(1-y/h)^2\,dy$.

3. False. Using the table of integrals (VI-28 and VI-30) or a trigonometric substitution gives

$$\int_0^r \pi\sqrt{r^2-y^2}\,dy = \frac{\pi}{2}\left(y\sqrt{r^2-y^2} + r^2 \arcsin\left(\frac{y}{r}\right)\right)\bigg|_0^r = \frac{\pi r^2}{2}(\arcsin 1 - \arcsin 0) = \frac{\pi^2 r^2}{4}.$$

The volume of a hemisphere is $2\pi r^3/3$.

Alternatively, you can show by slicing that the integral representing this volume is $\int_0^r \pi(r^2-y^2)\,dy$.

4. True. Horizontal slicing gives rectangular slabs of length l, thickness Δy, and width $w = 2\sqrt{r^2-y^2}$. So the volume of one slab is $2l\sqrt{r^2-y^2}\Delta y$, and the integral is $\int_{-r}^{r} 2l\sqrt{r^2-y^2}\,dy$.
5. False. Volume is always positive, like area.
6. False. The population density needs to be approximately constant on each ring. This is only true if the population density is a function of r, the distance from the center of the city.
7. False. Since the density varies with y, the region must be sliced perpendicular to the y-axis, along the lines of constant y.
8. False. Although the density is greater near the center, the area of the suburbs is much larger than the area of the inner city, and population is determined by both area and density. In fact, the population of the inner city:

$$\int_0^1 (10-3r)2\pi r dr = 2\pi(5r^2 - r^3)\bigg|_0^1 = 8\pi$$

is less than the population of the suburbs:

$$\int_1^2 (10-3r)2\pi r dr = 2\pi(5r^2 - r^3)\bigg|_1^2 = 16\pi.$$

9. True. One way to look at it is that the center of mass should not change if you change the units by which you measure the masses. If you double the masses, that is no different than using as a new unit of mass half the old unit. Alternatively, let the masses be m_1, m_2, and m_3 located at x_1, x_2, and x_3. Then the center of mass is given by:

$$\bar{x} = \frac{x_1 m_1 + x_2 m_2 + x_3 m_3}{m_1 + m_2 + m_3}.$$

Doubling the masses does not change the center of mass, since it doubles both the numerator and the denominator.

10. False. The center of mass of a circular ring (for example, a coin with a hole in it) is at the center.
11. True. The density of particles hitting the target is approximately constant on concentric rings.
12. False. If the density were constant this would be true, but suppose that all the mass on the left half is concentrated at $x = 0$ and all the mass on the right side is concentrated at $x = 3$. In order for the rod to balance at $x = 2$, the weight on the left side must be half the weight on the right side.
13. False. Work is the product of force and distance moved, so the work done in either case is 200 ft-lb.
14. True. Displacement in the same direction as the force gives positive work; displacement in the opposite direction as the force gives negative work.

15. False. Since the water pressure increases with depth, the force on the lower half of the new dam is greater than the force on the upper half of the new dam, which is the same as the force on the old dam. Thus the force on the new dam is more than double the force on the old dam.

16. True. Since pressure increases with depth and we want the pressure to be approximately constant on each strip, we use horizontal strips.

17. False. The pressure is positive and when integrated gives a positive force.

18. True. Although work is expressed in an integral, the average value is also expressed in an integral. We have:

$$\text{Average value of the force} = \frac{1}{4-1}\int_1^4 F(x)dx.$$

Thus if we multiply the average force by 3, we get $\int_1^4 F(x)dx$, which is the work done.

19. True. For an income stream $P(t)$ from $t = 0$ to $t = M$, we have:

$$\text{Present value} = \int_0^M P(t)e^{-rt}dt,$$

and

$$\text{Future value} = \int_0^M P(t)e^{r(M-t)}dt.$$

Since M and r are constant, we can factor out e^{rM} from the integral for the future value to get:

$$\text{Future value} = e^{rM}(\text{Present value}).$$

Since $r > 0$ and $M > 0$, this means $e^{rM} > 1$ so the future value is greater than the present value.

20. False. Since $p(x) < 0$ for $x < 0$, it cannot be a probability density function.

21. False. It is true that $p(x) \geq 0$ for all x, but we also need $\int_{-\infty}^{\infty} p(x)dx = 1$. Since $p(x) = 0$ for $x \leq 0$, we need only check the integral from 0 to ∞. We have

$$\int_0^\infty xe^{-x^2}dx = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Big|_0^b = \frac{1}{2}.$$

22. False. The volume also depends on how far away the region is from the axis of revolution. For example, let R be the rectangle $0 \leq x \leq 8, 0 \leq y \leq 1$ and let S be the rectangle $0 \leq x \leq 3, 0 \leq y \leq 2$. Then rectangle R has area greater than rectangle S. However, when you revolve R about the x-axis you get a cylinder, lying on its side, of radius 1 and length 8, which has volume 8π. When you revolve S about the x-axis, you get a cylinder of radius 2 and length 3, which has volume 12π. Thus the second volume is larger, even though the region revolved has smaller area.

23. False. Suppose that the graph of f starts at the point $(0, 100)$ and then goes down to $(1, 0)$ and from there on goes along the x-axis. For example, if $f(x) = 100(x-1)^2$ on the interval $[0, 1]$ and $f(x) = 0$ on the interval $[1, 10]$, then f is differentiable on the interval $[0, 10]$. The arc length of the graph of f on the interval $[0, 1]$ is at least 100, while the arc length on the interval $[1, 10]$ is 9.

24. True. Since f is concave up, f' is an increasing function, so $f'(x) \geq f'(0) = 3/4$ on the interval $[0, 4]$. Thus $\sqrt{1 + (f'(x))^2} \geq \sqrt{1 + 9/16} = 5/4$. Then we have:

$$\text{Arc length} = \int_0^4 \sqrt{1+(f'(x))^2}dx \geq \int_0^4 \frac{5}{4}dx = 5.$$

25. False. Since f is concave down, this means that $f'(x)$ is decreasing, so $f'(x) \leq f'(0) = 3/4$ on the interval $[0, 4]$. However, it could be that $f'(x)$ becomes negative so that $(f'(x))^2$ becomes large, making the integral for the arc length large also. For example, $f(x) = (3/4)x - x^2$ is concave down and $f'(0) = 3/4$, but $f(0) = 0$ and $f(4) = -13$, so the graph of f on the interval $[0, 4]$ has arc length at least 13.

26. False. Note that p is the density function for the population, not the cumulative density function. Thus $p(10) = 1/2$ means that the probability of x lying in a small interval of length Δx around $x = 10$ is about $(1/2)\Delta x$.

27. True. This follows directly from the definition of the cumulative density function.

28. True. The interval from $x = 9.98$ to $x = 10.04$ has length 0.06. Assuming that the value of $p(x)$ is near $1/2$ for $9.98 < x < 10.04$, the fraction of the population in that interval is $\int_{9.98}^{10.04} p(x)dx \approx (1/2)(0.06) = 0.03$.

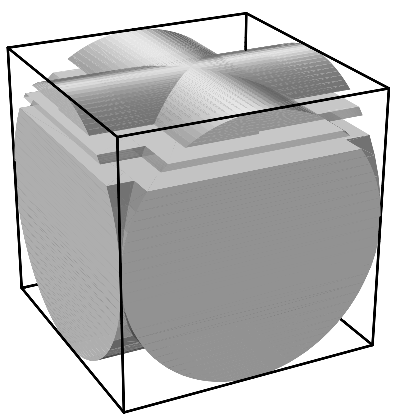
29. False. Note that p is the density function for the population, not the cumulative density function. Thus $p(10) = p(20)$ means that x values near 10 are as likely as x values near 20.

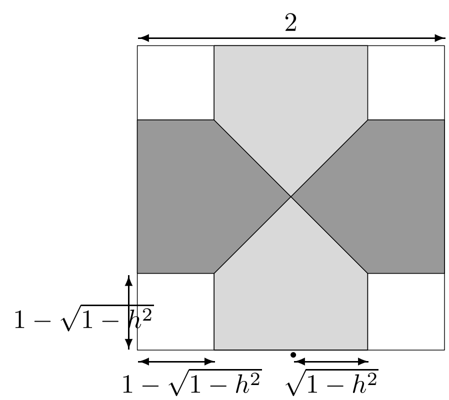
30. True. By the definition of the cumulative distribution function, $P(20) - P(10) = 0$ is the fraction of the population having x values between 10 and 20.

PROJECTS FOR CHAPTER EIGHT

1. Let us make coordinate axes with the origin at the center of the box. The x and y axes will lie along the central axes of the cylinders, and the (height) axis will extend vertically to the top of the box. If one slices the cylinders horizontally, one gets a cross. The cross is what you get if you cut out four corner squares from a square of side length 2. If h is the height of the cross above (or below) the xy plane, the equation of a cylinder is $h^2 + y^2 = 1$ (or $h^2 + x^2 = 1$). Thus the "armpits" of the cross occur where $y^2 - 1 = -h^2 = x^2 - 1$ for some fixed height h—that is, out $\sqrt{1-h^2}$ units from the center, or $1 - \sqrt{1-h^2}$ units away from the edge. Each corner square has area $(1 - \sqrt{1-h^2})^2 = 2 - h^2 - 2\sqrt{1-h^2}$. The whole big square has area 4. Therefore, the area of the cross is

$$4 - 4(2 - h^2 - 2\sqrt{1-h^2}) = -4 + 4h^2 + 8\sqrt{1-h^2}.$$





We integrate this from $h = -1$ to $h = 1$, and obtain the volume, V:

$$\begin{aligned} V &= \int_{-1}^{1} -4 + 4h^2 + 8\sqrt{1-h^2}\, dh \\ &= \left[-4h + \frac{4h^3}{3} + 8 \cdot \frac{1}{2}\left(h\sqrt{1-h^2} + \arcsin h \right) \right]\Bigg|_{-1}^{1} \\ &= -8 + \frac{8}{3} + 4\pi = 4\pi - \frac{16}{3} \approx 7.23. \end{aligned}$$

This is a reasonable answer, as the volume of the cube is 8, and the volume of one cylinder alone is $2\pi \approx 6.28$.

2. (a) Let y represent height, and let x represent horizontal distance from the lowest point of the cable. Then the stretched cable is a parabola of the form $y = kx^2$ passing through the point $(1280/2, 143) = (640, 143)$. Therefore, $143 = k(640)^2$ so $k \approx 3.491 \times 10^{-4}$. To find the arc length of the parabola, we take twice the arc length of the part to the right of the lowest point. Since $dy/dx = 2kx$,

$$\text{Arc Length} = 2\int_0^{640} \sqrt{1 + (2kx)^2}\, dx = 2\int_0^{640} \sqrt{1 + 4k^2x^2}\, dx.$$

The easiest way to find this integral is to substitute the value of k and find the integral's value numerically, giving

$$\text{Arc Length} \approx 1321.4 \text{ meters}.$$

Alternatively, we can make the substitution $w = 2kx$:

$$\begin{aligned}
\text{Arc Length} &= \frac{2}{2k}\int_0^{1280k} \sqrt{1+w^2}\,dw \\
&= \frac{1}{k}\int_0^{1280k} \sqrt{1+w^2}\,dw \\
&= \frac{1}{2k}\left(w\sqrt{1+w^2}\Big|_0^{1280k}\right) + \frac{1}{2k}\left(\int_0^{1280k} \frac{1}{\sqrt{1+w^2}}\,dw\right) \\
&\quad \text{[Using the integral table, Formula VI-29, or substitute } w=\tan\theta] \\
&= \frac{1}{2k}\left(1280k\sqrt{1+(1280k)^2}\right) + \frac{1}{2k}\left(\ln\left|x+\sqrt{1+x^2}\right|\Big|_0^{1280k}\right) \\
&= \frac{1}{2k}\left(1280k\sqrt{1+(1280k)^2}\right) + \frac{1}{2k}\left(\ln\left|1280k+\sqrt{1+(1280k)^2}\right|\right) \\
&\approx 1321.4 \text{ meters.}
\end{aligned}$$

(b) Adding 0.05% to the length of the cable gives a cable length of $(1321.4)(1.0005) = 1322.1$. We now want to calculate the new shape of the parabola; that is, we want to find a new k so that the arc length is 1322.1. Since

$$\text{Arc Length} = 2\int_0^{640} \sqrt{1+4k^2x^2}\,dx$$

we can find k numerically by trial and error. Trying values close to our original value of k, we find $k \approx 3.52 \times 10^{-4}$. To find the sag for this new k, we find the height $y = kx^2$ for which the cable hangs from the towers. This is

$$y = k(640)^2 \approx 144.2.$$

Thus the cable sag is 144.2 meters, over a meter more than on a cold winter day. Notice, though, that although the length increases by 0.05%, the sag increases by more:$144.2/143 \approx 1.0084$, an increase of 0.84%.

3. (a) Revolving the semi-circle $y = \sqrt{r^2 - x^2}$ around the x-axis yields the sphere of radius r. See Figure 8.169. Differentiating yields:

$$\frac{dy}{dx} = \frac{-1}{\sqrt{r^2-x^2}} \cdot x = -\frac{x}{y}.$$

Thus, substituting $-x/y$ for $f'(x)$, we get

$$\begin{aligned}
\text{Surface area} &= 2\pi\int_{-r}^{r} y\sqrt{1+\frac{x^2}{y^2}}\,dx = 2\pi\int_{-r}^{r}\sqrt{x^2+y^2}\,dx \\
&= 2\pi r\int_{-r}^{r} dx = 4\pi r^2.
\end{aligned}$$

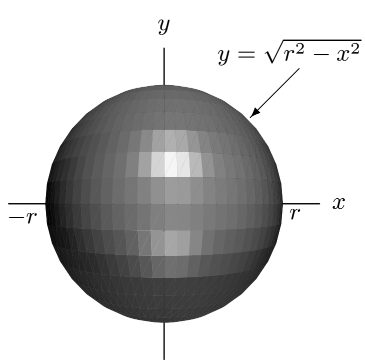


Figure 8.169

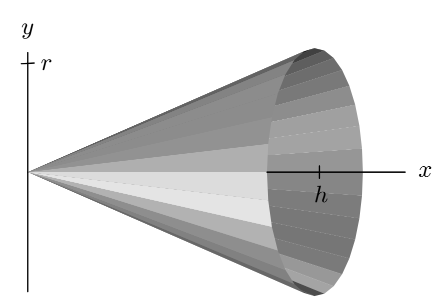


Figure 8.170

(b) Revolving the line $y = rx/h$ around the x-axis yields the cone. The base of the cone is a circle with area πr^2. See Figure 8.170. The area of the rest of the cone is

$$\begin{aligned}\text{Surface area} &= 2\pi \int_0^h y\sqrt{1+\frac{r^2}{h^2}}\,dx = 2\pi\sqrt{1+\frac{r^2}{h^2}}\left(\frac{r}{h}\int_0^h x\,dx\right)\\ &= 2\pi\frac{r}{h}\frac{h^2}{2}\sqrt{1+\frac{r^2}{h^2}} = \pi r\sqrt{r^2+h^2}\end{aligned}$$

Adding the area of the base, we get

$$\text{Total surface area of cone } = \pi r^2 + \pi r\sqrt{r^2+h^2}.$$

(c) We find the volume of $y = 1/x$ revolved about the x-axis as x runs from 1 to ∞. See Figure 8.171.

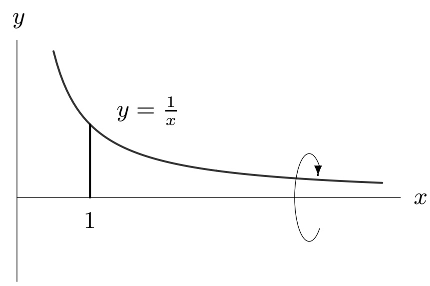


Figure 8.171

$$\text{Volume } = \int_1^\infty \pi y^2\,dx = \pi\int_1^\infty \frac{1}{x^2}\,dx = \pi\lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \pi\lim_{b\to\infty}\left.\frac{-1}{x}\right|_1^b = \pi$$

Thus, the volume of this solid is finite and equal to π.

(d) Now we show the surface area of this solid is unbounded. We have

$$\text{Surface area } = 2\pi\int_1^\infty y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

We cannot easily compute the antiderivative of $\frac{1}{x}\sqrt{1+\frac{1}{x^4}}$, so we bound the integral from below by noticing that

$$\sqrt{1+\frac{1}{x^4}} \geq 1.$$

Thus we see that

$$\text{Surface area } \geq 2\pi\int_1^\infty \frac{1}{x}\,dx = 2\pi\lim_{b\to\infty}\int_1^b\frac{dx}{x} = 2\pi\lim_{b\to\infty}\ln x\Big|_1^b.$$

Since $\ln x$ goes to infinity as x goes to infinity, the surface area is unbounded.

Alternatively, we can try calculating

$$2\pi\int_1^b \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

for larger and larger values of b. We would see that the integral seems to diverge.

(e) For a solid generated by the revolution of a curve $y = f(x)$ for $a \le x \le b$,

$$\text{Volume} = \int_a^b \pi y^2\,dx$$

and

$$\text{Surface area} = \int_a^b 2\pi y\sqrt{1+(f'(x))^2}\,dx.$$

The volume and the surface area will be equal if

$$f(x) = 2\sqrt{1+(f'(x))^2}.$$

We find a function $y = f(x)$ which satisfies this relation:

$$\begin{aligned}
y &= 2\sqrt{1+\left(\frac{dy}{dx}\right)^2}\\
\frac{y^2}{4} &= 1+\left(\frac{dy}{dx}\right)^2\\
\frac{dy}{dx} &= \sqrt{\frac{y^2}{4}-1}\\
\frac{dy}{\sqrt{y^2-4}} &= \frac{1}{2}\,dx\\
\int \frac{dy}{\sqrt{y^2-4}} &= \int \frac{1}{2}\,dx\\
\ln|y+\sqrt{y^2-4}| &= \frac{x}{2}+C\\
y+\sqrt{y^2-4} &= Ae^{x/2}
\end{aligned}$$

Notice in the third line we have used the fact that $dy/dx \ge 0$. Any function, $y = f(x)$, which satisfies this relationship has the required property.

4. (a) We want to find a such that $\int_0^\infty p(v)\,dv = \lim_{r\to\infty} a\int_0^r v^2e^{-mv^2/2kT}\,dv = 1$. Therefore,

$$\frac{1}{a} = \lim_{r\to\infty}\int_0^r v^2e^{-mv^2/2kT}\,dv.$$

To evaluate the integral, use integration by parts with the substitutions $u = v$ and $w' = ve^{-mv^2/2kT}$:

$$\begin{aligned}
\int_0^r \underbrace{v}_{u}\underbrace{ve^{-mv^2/2kT}}_{w'}\,dv &= \underbrace{v}_{u}\underbrace{\frac{e^{-mv^2/2kT}}{-m/kT}}_{w}\Bigg|_0^r - \int_0^r \underbrace{1}_{u'}\underbrace{\frac{e^{-mv^2/2kT}}{-m/kT}}_{w}\,dv\\
&= -\frac{kTr}{m}e^{-mr^2/2kT} + \frac{kT}{m}\int_0^r e^{-mv^2/2kT}\,dv.
\end{aligned}$$

From the normal distribution we know that $\displaystyle\int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx = \frac{1}{2}$, so

$$\int_0^\infty e^{-x^2/2}\,dx = \frac{\sqrt{2\pi}}{2}.$$

Therefore in the above integral, make the substitution $x = \sqrt{\frac{m}{kT}}v$, so that $dx = \sqrt{\frac{m}{kT}}\,dv$, or $dv = \sqrt{\frac{kT}{m}}\,dx$. Then

$$\frac{kT}{m}\int_0^r e^{-mv^2/2kT}\,dv = \left(\frac{kT}{m}\right)^{3/2}\int_0^{\sqrt{\frac{m}{kT}}r} e^{-x^2/2}\,dx.$$

Substituting this into Equation 4a we get

$$\frac{1}{a} = \lim_{r\to\infty}\left(-\frac{kTr}{m}e^{-mr^2/2kT} + \left(\frac{kT}{m}\right)^{3/2}\int_0^{\sqrt{\frac{m}{kT}}r} e^{-x^2/2}\,dx\right) = 0 + \left(\frac{kT}{m}\right)^{3/2}\cdot\frac{\sqrt{2\pi}}{2}.$$

Therefore, $a = \frac{2}{\sqrt{2\pi}}(\frac{m}{kT})^{3/2}$. Substituting the values for k, T, and m gives $a \approx 3.4\times 10^{-8}$.

(b) To find the median, we wish to find the speed x such that

$$\int_0^x p(v)\,dv = \int_0^x av^2e^{-\frac{mv^2}{2kT}}\,dv = \frac{1}{2},$$

where $a = \frac{2}{\sqrt{2\pi}}(\frac{m}{kT})^{3/2}$. Using a calculator, by trial and error we get $x\approx 441$ m/sec.
To find the mean, we find

$$\int_0^\infty vp(v)\,dv = \int_0^\infty av^3e^{-\frac{mv^2}{2kT}}\,dv.$$

This integral can be done by substitution. Let $u = v^2$, so $du = 2vdv$. Then

$$\begin{aligned}\int_0^\infty av^3e^{-\frac{mv^2}{2kT}}\,dv &= \frac{a}{2}\int_{v=0}^{v=\infty} v^2e^{-\frac{mv^2}{2kT}}2v\,dv\\ &= \frac{a}{2}\int_{u=0}^{u=\infty} ue^{-\frac{mu}{2kT}}\,du\\ &= \lim_{r\to\infty}\frac{a}{2}\int_0^r ue^{-\frac{mu}{2kT}}\,du.\end{aligned}$$

Now, using the integral table, we have

$$\begin{aligned}\int_0^\infty av^3e^{-\frac{mv^2}{2kT}}\,dv &= \lim_{r\to\infty}\frac{a}{2}\left[-\frac{2kT}{m}ue^{-\frac{mu}{2kT}} - \left(-\frac{2kT}{m}\right)^2e^{-\frac{mu}{2kT}}\right]\Bigg|_0^r\\ &= \frac{a}{2}\left(-\frac{2kT}{m}\right)^2\\ &\approx 457.7 \text{ m/sec.}\end{aligned}$$

The maximum for $p(v)$ will be at a point where $p'(v) = 0$.

$$\begin{aligned}p'(v) &= a(2v)e^{-\frac{mv^2}{2kT}} + av^2\left(-\frac{2mv}{2kT}\right)e^{-\frac{mv^2}{2kT}}\\ &= ae^{-\frac{mv^2}{2kT}}\left(2v - v^3\frac{m}{kT}\right).\end{aligned}$$

Thus $p'(v) = 0$ at $v = 0$ and at $v = \sqrt{\dfrac{2kT}{m}} \approx 405$. It's obvious that $p(0) = 0$, and that $p\to 0$ as $v\to\infty$. So $v = 405$ gives us a maximum: $p(405)\approx 0.002$.

(c) The mean, as we found in part (b), is $\dfrac{a}{2}\dfrac{4k^2T^2}{m^2} = \dfrac{4}{\sqrt{2\pi}}\dfrac{k^{1/2}T^{1/2}}{m^{1/2}}$. It is clear, then, that as T increases so does the mean. We found in part (b) that $p(v)$ reached its maximum at $v = \sqrt{\dfrac{2kT}{m}}$. Thus

$$\begin{aligned}\text{The maximum value of } p(v) &= \frac{2}{\sqrt{2\pi}}\left(\frac{m}{kT}\right)^{3/2}\frac{2kT}{m}e^{-1}\\ &= \frac{4}{e\sqrt{2\pi}}\frac{m^{1/2}}{kT^{1/2}}.\end{aligned}$$

Thus as T increases, the maximum value decreases.