CHAPTER NINE

Solutions for Section 9.1

Exercises

1. The first term is $2^1 + 1 = 3$. The second term is $2^2 + 1 = 5$. The third term is $2^3 + 1 = 9$, the fourth is $2^4 + 1 = 17$, and the fifth is $2^5 + 1 = 33$. The first five terms are $3, 5, 9, 17, 33$.

2. The first term is $1 + (-1)^1 = 1 - 1 = 0$. The second term is $2 + (-1)^2 = 2 + 1 = 3$. The third term is $3 - 1 = 2$ and the fourth is $4 + 1 = 5$. The first five terms are $0, 3, 2, 5, 4$.

3. The first term is $2 \cdot 1/(2 \cdot 1 + 1) = 2/3$. The second term is $2 \cdot 2/(2 \cdot 2 + 1) = 4/5$. The first five terms are

$$2/3, 4/5, 6/7, 8/9, 10/11.$$

4. The first term is $(-1)^1(1/2)^1 = -1/2$. The second term is $(-1)^2(1/2)^2 = 1/4$. The first five terms are

$$-1/2, 1/4, -1/8, 1/16, -1/32.$$

5. The first term is $(-1)^2(1/2)^0 = 1$. The second term is $(-1)^3(1/2)^1 = -1/2$. The first five terms are

$$1, -1/2, 1/4, -1/8, 1/16.$$

6. The first term is $(1 - 1/(1+1))^{(1+1)} = (1/2)^2$. The second term is $(1 - 1/3)^3 = (2/3)^3$. The first five terms are

$$(1/2)^2, (2/3)^3, (3/4)^4, (4/5)^5, (5/6)^6.$$

7. The terms look like powers of 2 so we guess $s_n = 2^n$. This makes the first term $2^1 = 2$ rather than 4. We try instead $s_n = 2^{n+1}$. If we now check, we get the terms $4, 8, 16, 32, 64, \ldots$, which is right.

8. We compare with positive powers of 2, which are $2, 4, 8, 16, 32, \ldots$. Each term is one less, so we take $s_n = 2^n - 1$.

9. We observe that if we subtract 1 from each term of the sequence, we get $1, 4, 9, 16, 25, \ldots$, namely the squares $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$. Thus $s_n = n^2 + 1$.

10. First notice that $s_n = 2n - 1$ is a formula for the general term of the sequence

$$1,\ 3,\ 5,\ 7,\ 9, \ldots.$$

To obtain the alternating signs in the original sequence, we try multiplying by $(-1)^n$. However, checking $(-1)^n(2n-1)$ for $n = 1, 2, 3, \ldots$ gives

$$-1,\ 3,\ -5,\ 7,\ -9, \ldots.$$

To get the correct signs, we multiply by $(-1)^{n+1}$ and take

$$s_n = (-1)^{n+1}(2n - 1).$$

11. The numerator is n. The denominator is then $2n + 1$, so $s_n = n/(2n + 1)$.

12. The denominators are the even numbers, so we try $s_n = 1/(2n)$. To get the signs to alternate, we try multiplying by $(-1)^n$. That gives

$$-1/2, 1/4, -1/6, 1/8, -1/10, \ldots,$$

so we multiply by $(-1)^{n+1}$ instead. Thus $s_n = (-1)^{n+1}/(2n)$.

13. We have $s_2 = s_1 + 2 = 3$ and $s_3 = s_2 + 3 = 6$. Continuing, we get

$$1,\ 3,\ 6,\ 10,\ 15,\ 21.$$

14. We have $s_2 = 2s_1 + 3 = 2 \cdot 1 + 3 = 5$ and $s_3 = 2s_2 + 3 = 2 \cdot 5 + 3 = 13$. Continuing, we get

$$1, 5, 13, 29, 61, 125.$$

15. We have $s_2 = s_1 + 1/2 = 0 + (1/2)^1 = 1/2$ and $s_3 = s_2 + (1/2)^2 = 1/2 + 1/4 = 3/4$. Continuing, we get

$$0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}.$$

16. We have $s_3 = s_2 + 2s_1 = 5 + 2\cdot 1 = 7$ and $s_4 = s_3 + 2s_2 = 7 + 2\cdot 5 = 17$. Continuing, we get

$$1, 5, 7, 17, 31, 65.$$

Problems

17. (a) matches (IV), since the sequence increases toward 1.
(b) matches (III), since the odd terms increase toward 1 and the even terms decrease toward 1.
(c) matches (II), since the sequence decreases toward 0.
(d) matches (I), since the sequence decreases toward 1.

18. (a) matches (II), since the sequence increases toward 2.
(b) matches (III), since the even terms decrease toward 2 and odd terms decrease toward -2.
(c) matches (IV), since the even terms decrease toward 2 and odd terms increase toward 2.
(d) matches (I), since the sequence decreases toward 2.
(e) matches (V), since the even terms decrease toward 2 and odd terms increase toward -2.

19. (a) matches (II), since $\lim_{n\to\infty}(n(n+1)-1) = \infty$.
(b) matches (III), since $\lim_{n\to\infty}(1/(n+1)) = 0$ and $1/(n+1)$ is always positive.
(c) matches (I), since $\lim_{n\to\infty}(1-n^2) = -\infty$.
(d) matches (IV), since $\lim_{n\to\infty}\cos(1/n) = \cos 0 = 1$.
(e) matches (V), since $\sin n$ is bounded above and below by ± 1, so $\lim_{n\to\infty}((\sin n)/n) = 0$ and the sign of $\sin n$ varies as $n \to \infty$.

20. Since $\lim\limits_{n\to\infty} x^n = 0$ if $|x| < 1$ and $|0.2| < 1$, we have $\lim\limits_{n\to\infty}(0.2)^n = 0$, so the sequence converges to 0

21. Since 2^n increases without bound as n increases, the sequence diverges.

22. Since $\lim\limits_{n\to\infty} x^n = 0$ if $|x| < 1$ and $|-0.3| < 1$, we have $\lim\limits_{n\to\infty}(-0.3)^n = 0$, so the sequence converges to 0.

23. Since $\lim\limits_{n\to\infty} x^n = 0$ if $|x| < 1$ and $|e^{-2}| < 1$, we have $\lim\limits_{n\to\infty}(e^{-2n}) = \lim\limits_{n\to\infty}(e^{-2})^n = 0$, so $\lim\limits_{n\to\infty}(3+e^{-2n}) = 3+0 = 3$, so the sequence converges to 3.

24. Since $\lim\limits_{n\to\infty} x^n = 0$ if $|x| < 1$ and $\left|\frac{2}{3}\right| < 1$, we have $\lim\limits_{n\to\infty}\left(\frac{2^n}{3^n}\right) = \lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$, so the sequence converges to 0.

25. We have:

$$\lim_{n\to\infty}\left(\frac{n}{10} + \frac{10}{n}\right) = \lim_{n\to\infty}\frac{n}{10} + \lim_{n\to\infty} 10n.$$

Since $n/10$ gets arbitrarily large and $10/n$ approaches 0 as $n \to \infty$, the sequence diverges.

26. We have:

$$\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0.$$

The terms of the sequence alternate in sign, but they approach 0, so the sequence converges to 0.

27. We have

$$\lim_{n\to\infty}\frac{2n+1}{n} = \lim_{n\to\infty}\left(2 + \frac{1}{n}\right) = 2,$$

so the sequence converges to 2.

28. Since $s_n = \cos(\pi n) = 1$ if n is even and $s_n = \cos(\pi n) = -1$ if n is odd, the values of s_n alternate between 1 and -1, so the limit does not exist. Thus, the sequence diverges.

29. Since $\lim_{n\to\infty} 1/n = 0$ and $-1 \le \sin n \le 1$, the terms approach zero and the sequence converges to 0.

30. As n increases, the term $2n$ is much larger in magnitude than $(-1)^n 5$ and the term $4n$ is much larger in magnitude than $(-1)^n 3$. Thus dividing the numerator and denominator by n and using the fact that $\lim\limits_{n\to\infty} 1/n = 0$, we have

$$\lim_{n\to\infty}\frac{2n+(-1)^n 5}{4n-(-1)^n 3} = \lim_{n\to\infty}\frac{2+(-1)^n 5/n}{4-(-1)^n 3/n} = \frac{1}{2}.$$

Thus, the sequence converges to $1/2$.

31. Since the exponential function 2^n dominates the power function n^3 as $n \to \infty$, the series diverges.

32. The first 6 terms of the sequence for the sampling is

$$\cos 0.5,\ \cos 1.0,\ \cos 1.5,\ \cos 2.0,\ \cos 2.5,\ \cos 3.0$$
$$= 0.878,\ 0.540,\ 0.071,\ -0.416,\ -0.801,\ -0.990.$$

33. The first 6 terms of the sequence for the sampling is

$$(-0.5)^2,\ (0.0)^2,\ (0.5)^2,\ (1.0)^2,\ (1.5)^2,\ (2.0)^2,$$
$$= 0.25,\ 0.00,\ 0.25,\ 1.00,\ 2.25,\ 4.00.$$

34. The first 6 terms of the sequence for the sampling are

$$\frac{\sin 1}{1},\ \frac{\sin 2}{2},\ \frac{\sin 3}{3},\ \frac{\sin 4}{4},\ \frac{\sin 5}{5},\ \frac{\sin 6}{6}$$
$$= 0.841,\ 0.455,\ 0.047,\ -0.189,\ -0.192,\ -0.047.$$

35. The first smoothing gives

$$0,\ 6,\ -6,\ 6,\ -6,\ 6, \ldots$$

The second smoothing gives

$$3,\ 0,\ 2,\ -2,\ 2, \ldots$$

The smoothing process diminishes the peaks and valleys of this alternating sequence.

36. The first smoothing gives

$$0,\ 0,\ 6,\ 6,\ 6,\ 0,\ 0 \ldots$$

The second smoothing gives

$$0,\ 2,\ 4,\ 6,\ 4,\ 2 \ldots$$

The smoothing process spreads out the spike at the fourth term to the neighboring terms.

37. The first smoothing gives

$$1.5,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7 \ldots$$

The second smoothing gives

$$1.75,\ 2.17,\ 3,\ 4,\ 5,\ 6 \ldots$$

Terms which are already the same as their average with their neighbors are not changed.

38. (a) Since month 10 is October, V_{10} is the number of SUVs sold in the US in October 2004.
(b) The difference $V_n - V_{n-1}$ represents the increase in sales between month $(n-1)$ and month n.
(c) The sum $\sum_{i=1}^{12} V_i$ represents the total sales of SUVs in the year 2004 (twelve months). The sum $\sum_{i=1}^{n} V_i$ represents the total sales in the n months starting from January 1, 2004.

39. (a) Since $c_1 = 75.747(1.003)$, $c_2 = 75.747(1.003)^2$, and so on, we have $c_n = 75.747(1.003)^n$.
(b) Since c_n is consumption in year n and c_{n-1} is consumption in year $n-1$, we have

$$c_n - c_{n-1} = 75.747(1.003)^n - 75.747(1.003)^{n-1} = 75.747(1.003)^{n-1}(1.003-1) = 0.227(1.003)^{n-1},$$

and $c_n - c_{n-1}$ represents the increase in consumption in million barrels per day between the year $(n-1)$ and the year n.
(c) The sum represents the total oil consumed in years 1–18, that is, 2003–2020, inclusive.

40. (a) Since you have two parents and four grandparents, $s_1 = 2$ and $s_2 = 4$. In general, $s_n = 2^n$.
(b) Solving $s_n = 6 \cdot 10^9$ gives

$$2^n = 6 \cdot 10^9$$
$$n = \frac{\ln(6 \cdot 10^9)}{\ln 2} = 32.482.$$

Thus, 33 or more generations ago, the number of ancestors is greater than the current population of the world. Since the population of the world 33 generations ago was much smaller than it is now, there must have been overlap among our ancestors.

41. In year 1, the payment is

$$p_1 = 10{,}000 + 0.05(100{,}000) = 15{,}000.$$

The balance in year 2 is $100{,}000 - 10{,}000 = 90{,}000$, so

$$p_2 = 10{,}000 + 0.05(90{,}000) = 14{,}500.$$

The balance in year 3 is 80,000, so

$$p_3 = 10{,}000 + 0.05(80{,}000) = 14{,}000.$$

Thus,

$$\begin{aligned} p_n &= 10{,}000 + 0.05(100{,}000 - (n-1)\cdot 10{,}000) \\ &= 15{,}500 - 500n. \end{aligned}$$

42. (a) (i) Since the number of bacteria doubles every half hour, the number quadruples every hour. Thus

$$\begin{aligned} R_1 &= B_0 \cdot 4 \\ R_2 &= B_0 \cdot 4^2 \\ &\vdots \\ R_n &= B_0 \cdot 4^n. \end{aligned}$$

(ii) Each hour, the number of bacteria is multiplied by a factor a, so

$$F_n = B_0 a^n.$$

The bacteria doubles in number in 10 hours, so

$$F_{10} = 2B_0.$$

Thus,

$$\begin{aligned} B_0 a^{10} &= 2B_0 \\ a &= 2^{1/10}, \end{aligned}$$

so

$$F_n = B_0(2^{1/10})^n = B_0 2^{n/10}.$$

(iii) The ratio is

$$Y_n = \frac{R_n}{F_n} = \frac{B_0 4^n}{B_0 2^{n/10}} = \left(\frac{4}{2^{1/10}}\right)^n = \left(2^{1.9}\right)^n.$$

(b) We want to solve for n making $Y_n = 1{,}000{,}000$:

$$\begin{aligned} \left(2^{1.9}\right)^n &= 1{,}000{,}000 \\ n &= \frac{\ln(1{,}000{,}000)}{\ln(2^{1.9})} = 10.490. \end{aligned}$$

Thus, in about ten and a half hours, there are a million times as many bacteria in the baby formula kept at room temperature.

43. (a) In the first year, $d_1 = 20{,}000(0.12)$, so the car's value at the end of the first year is \$20,000(0.88). In the second year, $d_2 = 20{,}000(0.88)(0.12)$, so the car's value at the end of the second year is \$$20{,}000(0.88)^2$. Similarly, $d_3 = 20{,}000(0.88)^2(0.12)$. In general

$$d_n = 20{,}000(0.88)^{n-1}(0.12).$$

(b) The first year $r_1 = 400$; the second year $r_2 = 400(1.18)$, the third year $r_3 = 400(1.18)^2$. In general, $r_n = 400(1.18)^{n-1}$.

(c) We have

$$\begin{aligned} \text{Total cost} &= d_1 + d_2 + d_3 + r_1 + r_2 + r_3 \\ &= 20{,}000(0.12)(1 + 0.88 + (0.88)^2) + 400(1 + 1.18 + (1.18)^2) \\ &= 7799.52 \text{ dollars.} \end{aligned}$$

(d) A two-year old car has the same pattern of expenses except that the initial price is \$20,000$(0.88)^2$ instead of \$20,000 and that the repair costs start at \$400$(1.18)^2$ instead of \$400. Then

$$\begin{aligned}\text{Total cost } &= 20{,}000(0.88)^2(0.12)(1+0.88+(0.88)^2)+400(1.18)^2(1+1.18+(1.18)^2)\\ &= 6923.05 \text{ dollars.}\end{aligned}$$

Thus, the two-year-old car costs you less and you should buy it.

44. We want to define $\lim\limits_{n\to\infty} s_n = L$ so that s_n is as close to L as we please for all sufficiently large n. Thus, the definition says that for any positive ϵ, there is a value N such that

$$|s_n - L| < \epsilon \quad \text{whenever} \quad n \geq N.$$

45. We use Theorem 9.1, so we must show that s_n is bounded. Since t_n converges, it is bounded so there is a number M, such that $t_n \leq M$ for all n. Therefore $s_n \leq t_n \leq M$ for all n. Since s_n is increasing, $s_1 \leq s_n$ for all n. Thus if we let $K = s_1$, we have $K \leq s_n \leq M$ for all n, so s_n is bounded. Therefore, s_n converges.

46. Each term is 2 more than the previous term, so a recursive definition is $s_n = s_{n-1} + 2$ for $n > 1$ and $s_1 = 1$.

47. Each term is 2 more than the previous term, so a recursive definition is $s_n = s_{n-1} + 2$ for $n > 1$ and $s_1 = 2$. Notice that the even positive integers and odd positive integers have the same recursive definition except for the starting term.

48. Each term is twice the previous term minus one, so a recursive definition is $s_n = 2s_{n-1} - 1$ for $n > 1$ and $s_1 = 3$. We also notice that the differences of consecutive terms are powers of 2, so $s_2 = s_1 + 2$, $s_3 = s_2 + 2^2$, and so on. Thus another recursive definition is $s_n = s_{n-1} + 2^{n-1}$ for $n > 1$ and $s_1 = 3$.

49. The differences between consecutive terms are $4, 9, 16, 25$, so, for example, $s_2 = s_1 + 4$ and $s_3 = s_2 + 9$. Thus, a possible recursive definition is $s_n = s_{n-1} + n^2$ for $n > 1$ and $s_1 = 1$.

50. The differences are $2, 3, 4, 5$, so, for example, $s_2 = s_1 + 2$, $s_3 = s_2 + 3$, and $s_4 = s_3 + 4$. Thus, a recursive definition is $s_n = s_{n-1} + n$ for $n > 1$ and $s_1 = 1$.

51. The numerator and denominator of each term are related to the numerator and denominator of the previous term. The denominator is the previous numerator and the numerator is the sum of the previous numerator and previous denominator. For example,

$$\frac{5}{3} = \frac{2+3}{3} \text{ and } \frac{8}{5} = \frac{3+5}{5}.$$

If we simplify, we get

$$\frac{5}{3} = \frac{2}{3} + 1, \text{ and } \frac{8}{5} = \frac{3}{5} + 1.$$

In words, we turn the previous term upside down and add 1. Thus, a recursive definition is $s_n = \dfrac{1}{s_{n-1}} + 1$ for $n > 1$ and $s_1 = 1$.

52. For $n > 1$, if $s_n = 3n - 2$, then $s_{n-1} = 3(n-1) - 2 = 3n - 5$, so

$$s_n - s_{n-1} = (3n-2) - (3n-5) = 3,$$

giving

$$s_n = s_{n-1} + 3.$$

In addition, $s_1 = 3 \cdot 1 - 2 = 1$.

53. For $n > 1$, if $s_n = n(n+1)/2$, then $s_{n-1} = (n-1)(n-1+1)/2 = n(n-1)/2$. Since

$$s_n = \frac{1}{2}(n^2+n) = \frac{n^2}{2} + \frac{n}{2} \quad \text{and} \quad s_{n-1} = \frac{1}{2}(n^2-n) = \frac{n^2}{2} - \frac{n}{2},$$

we have

$$s_n - s_{n-1} = \frac{n}{2} + \frac{n}{2} = n,$$

so

$$s_n = s_{n-1} + n.$$

In addition, $s_1 = 1(2)/2 = 1$.

54. For $n > 1$, if $s_n = 2n^2 - n$, then $s_{n-1} = 2(n-1)^2 - (n-1) = 2n^2 - 5n + 3$, so

$$s_n - s_{n-1} = (2n^2 - n) - (2n^2 - 5n + 3) = 4n - 3,$$

giving

$$s_n = s_{n-1} + 4n - 3.$$

In addition, $s_1 = 2 \cdot 1^2 - 1 = 1$.

55. (a) The bottom row contains k cans, the next one contains $(k-1)$ cans, then $(k-2)$ and so on. Thus, there are k rows. Since the top row contains 1 can, the second contains 2 cans, etc, we have $a_n = n$.

(b) Since the n^{th} row contains n cans, $a_n = n$,

$$T_n = T_{n-1} + a_n$$

gives

$$T_n = T_{n-1} + n, \quad \text{for } n > 1.$$

In addition, $T_1 = 1$.

(c) If $T_n = \frac{1}{2}n(n+1)$, then $T_{n-1} = \frac{1}{2}(n-1)n$, so

$$T_n - T_{n-1} = \frac{1}{2}n(n+1) - \frac{1}{2}n(n-1) = \frac{n}{2}(n+1-(n-1)) = n.$$

In addition, $T_1 = \frac{1}{2} \cdot 1(2) = 1$.

56. (a) The first 12 terms are

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144.$$

(b) The sequence of ratios is

$$1,\ 2,\ \frac{3}{2},\ \frac{5}{3},\ \frac{8}{5},\ \frac{13}{8},\ \frac{21}{13},\ \frac{34}{21},\ \frac{55}{34},\ \frac{89}{55}\ldots.$$

To three decimal places, the first ten ratios are

$$1,\ 2,\ 1.500,\ 1.667,\ 1.600,\ 1.625,\ 1.615,\ 1.619,\ 1.618,\ 1.618.$$

It appears that the sequence of ratios is converging to $r = 1.618$. We find $(1.618)^2 = 2.618 = 1.618 + 1$ so r seems to satisfy $r^2 = r + 1$. Alternatively, by the quadratic formula, the positive root of $x^2 - x - 1 = 0$ is $(1 + \sqrt{5})/2 = 1.618$.

(c) If we multiply both sides of the equation $r^2 = r + 1$ by Ar^{n-2}, we obtain

$$Ar^n = Ar^{n-1} + Ar^{n-2}.$$

Thus, if $s_n = Ar^n$, then $s_{n-1} = Ar^{n-1}$ and $s_{n-2} = Ar^{n-2}$, so the sequence satisfies $s_n = s_{n-1} + s_{n-2}$.

57. The sequence seems to converge. By the 25^{th} term it stabilizes to four decimal places at $L = 0.7391$.

58. The sequence oscillates up and down, but by the 20^{th} term it stabilizes to 4 decimal places at $L = 0.5671$.

Solutions for Section 9.2

Exercises

1. Yes, $a = 2$, ratio $= 1/2$.

2. Yes, $a = 1$, ratio $= -1/2$.

3. No. Ratio between successive terms is not constant: $\dfrac{1/3}{1/2} = 0.66\ldots$, while $\dfrac{1/4}{1/3} = 0.75$.

4. Yes, $a = 5$, ratio $= -2$.

5. No. Ratio between successive terms is not constant: $\dfrac{2x^2}{x} = 2x$, while $\dfrac{3x^3}{2x^2} = \dfrac{3}{2}x$.

6. No. Ratio between successive terms is not constant: $\dfrac{6z^2}{3z} = 2z$, while $\dfrac{9z^3}{6z^2} = \dfrac{3}{2}z$.

7. Yes, $a = 1$, ratio $= 2z$.

8. Yes, $a = y^2$, ratio $= y$.

9. Yes, $a = 1$, ratio $= -x$.

10. Yes, $a = 1$, ratio $= -y^2$.

11. Sum $= \dfrac{1}{1-2z}$, $|z| < 1/2$

12. Sum $= \dfrac{y^2}{1-y}$, $|y| < 1$

13. Sum $= \dfrac{1}{1-(-x)} = \dfrac{1}{1+x}$, $|x| < 1$

14. Sum $= \dfrac{1}{1-(-y^2)} = \dfrac{1}{1+y^2}$, $|y| < 1$.

15. The series has 26 terms. The first term is $a = 2$ and the constant ratio is $x = 0.1$, so

$$\text{Sum } = \frac{a(1-x^{26})}{(1-x)} = \frac{2(1-(0.1)^{26})}{0.9} = 2.222.$$

16. The series has 10 terms. The first term is $a = 0.2$ and the constant ratio is $x = 0.1$, so

$$\text{Sum } = \frac{0.2(1-x^{10})}{(1-x)} = \frac{0.2(1-(0.1)^{10})}{0.9} = 0.222.$$

17. The series has 9 terms. The first term is $a = 0.00002$ and the constant ratio is $x = 0.1$, so

$$\text{Sum } = \frac{0.00002(1-x^{9})}{(1-x)} = \frac{0.00002(1-(0.1)^{9})}{0.9} = 0.0000222.$$

18. $-2+1-\dfrac{1}{2}+\dfrac{1}{4}-\dfrac{1}{8}+\dfrac{1}{16}-\cdots = \displaystyle\sum_{n=0}^{\infty}(-2)\left(-\frac{1}{2}\right)^n$, a geometric series.

Let $a = -2$ and $x = -\frac{1}{2}$. Then

$\displaystyle\sum_{n=0}^{\infty}(-2)\left(-\frac{1}{2}\right)^n = \frac{a}{1-x} = \frac{-2}{1-(-\frac{1}{2})} = -\frac{4}{3}.$

19. $3+\dfrac{3}{2}+\dfrac{3}{4}+\dfrac{3}{8}\cdots+\dfrac{3}{2^{10}} = 3\left(1+\dfrac{1}{2}+\cdots+\dfrac{1}{2^{10}}\right) = \dfrac{3\left(1-\frac{1}{2^{11}}\right)}{1-\frac{1}{2}} = \dfrac{3\left(2^{11}-1\right)}{2^{10}}$

20. Using the formula for the sum of an infinite geometric series,

$$\sum_{n=4}^{\infty}\left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^4+\left(\frac{1}{3}\right)^5+\cdots = \left(\frac{1}{3}\right)^4\left(1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\cdots\right) = \frac{(\frac{1}{3})^4}{1-\frac{1}{3}} = \frac{1}{54}$$

21. Using the formula for the sum of a finite geometric series,

$$\sum_{n=4}^{20}\left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^4+\left(\frac{1}{3}\right)^5+\cdots+\left(\frac{1}{3}\right)^{20} = \left(\frac{1}{3}\right)^4\left(1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\cdots\left(\frac{1}{3}\right)^{16}\right) = \frac{(1/3)^4(1-(1/3)^{17})}{1-(1/3)} = \frac{3^{17}-1}{2\cdot 3^{20}}.$$

Problems

22. Yes. If the original series is finite, then

$$\text{Original series } = a + ax + ax^2 + \cdots + ax^{n-1},$$

then the new series obtained by multiplying termwise by c is

$$\text{New series } = ca + cax + cax^2 + \cdots + cax^{n-1},$$

which is also a geometric series: its first term is ca and the constant ratio of successive terms is still x. The argument works the same way for an infinite geometric series.

23. Yes. If the original finite geometric series is

$$\text{Original series } = a + ax + ax^2 + \cdots + ax^{n-1},$$

then the new series obtained by taking reciprocals termwise is

$$\text{New series } = \frac{1}{a} + \frac{1}{ax} + \frac{1}{ax^2} + \cdots + \frac{1}{ax^{n-1}},$$

which is also a geometric series: its first term is $1/a$ and the constant ratio of successive terms is $1/x$. The argument works the same way for an infinite geometric series.

24. Since the amount of ampicillin excreted during the time interval between tablets is 250 mg, we have

$$\text{Amount of ampicillin excreted} = \text{Original quantity} - \text{Final quantity}$$
$$250 = Q - (0.04)Q.$$

Solving for Q gives, as before,

$$Q = \frac{250}{1-0.04} \approx 260.42.$$

25. (a) The amount of atenolol in the blood is given by $Q(t) = Q_0e^{-kt}$, where $Q_0 = Q(0)$ and k is a constant. Since the half-life is 6.3 hours,

$$\frac{1}{2} = e^{-6.3k}, \quad k = -\frac{1}{6.3}\ln\frac{1}{2} \approx 0.11.$$

After 24 hours

$$Q = Q_0e^{-k(24)} \approx Q_0e^{-0.11(24)} \approx Q_0(0.07).$$

Thus, the percentage of the atenolol that remains after 24 hours $\approx 7\%$.

(b)

$$\begin{aligned}
Q_0 &= 50\\
Q_1 &= 50 + 50(0.07)\\
Q_2 &= 50 + 50(0.07) + 50(0.07)^2\\
Q_3 &= 50 + 50(0.07) + 50(0.07)^2 + 50(0.07)^3\\
&\vdots\\
Q_n &= 50 + 50(0.07) + 50(0.07)^2 + \cdots + 50(0.07)^n = \frac{50(1-(0.07)^{n+1})}{1-0.07}
\end{aligned}$$

(c)

$$\begin{aligned}
P_1 &= 50(0.07)\\
P_2 &= 50(0.07) + 50(0.07)^2\\
P_3 &= 50(0.07) + 50(0.07)^2 + 50(0.07)^3\\
P_4 &= 50(0.07) + 50(0.07)^2 + 50(0.07)^3 + 50(0.07)^4\\
&\vdots\\
P_n &= 50(0.07) + 50(0.07)^2 + 50(0.07)^3 + \cdots + 50(0.07)^n\\
&= 50(0.07)\left(1 + (0.07) + (0.07)^2 + \cdots + (0.07)^{n-1}\right) = \frac{0.07(50)(1-(0.07)^n)}{1-0.07}
\end{aligned}$$

26. (a)

$$\begin{aligned}
P_1 &= 0\\
P_2 &= 250(0.04)\\
P_3 &= 250(0.04) + 250(0.04)^2\\
P_4 &= 250(0.04) + 250(0.04)^2 + 250(0.04)^3\\
&\vdots\\
P_n &= 250(0.04) + 250(0.04)^2 + 250(0.04)^3 + \cdots + 250(0.04)^{n-1}
\end{aligned}$$

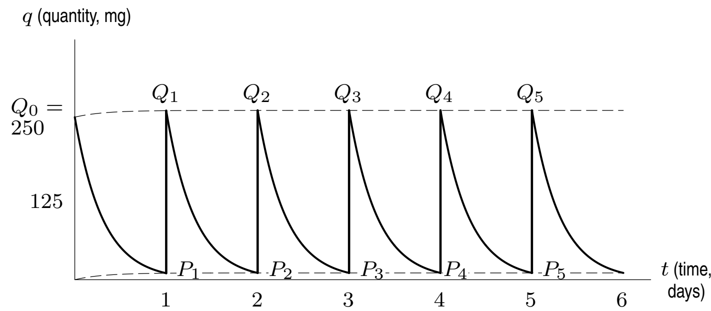
(b) $P_n = 250(0.04)\left(1 + (0.04) + (0.04)^2 + (0.04)^3 + \cdots + (0.04)^{n-2}\right) = 250\dfrac{0.04(1-(0.04)^{n-1})}{1-0.04}$

(c)

$$\begin{aligned} P &= \lim_{n\to\infty} P_n \\ &= \lim_{n\to\infty} 250\frac{0.04(1-(0.04)^{n-1})}{1-0.04} \\ &= \frac{(250)(0.04)}{0.96} = 0.04Q \approx 10.42 \end{aligned}$$

Thus, $\lim_{n\to\infty} P_n = 10.42$ and $\lim_{n\to\infty} Q_n = 260.42$. We would expect these limits to differ because one is right before taking a tablet, one is right after. We would expect the difference between them to be 250 mg, the amount of ampicillin in one tablet.

27.



28. (a) Let h_n be the height of the n^{th} bounce after the ball hits the floor for the n^{th} time. Then from Figure 9.1,

$$\begin{aligned} h_0 &= \text{height before first bounce } = 10 \text{ feet}, \\ h_1 &= \text{height after first bounce } = 10\left(\frac{3}{4}\right) \text{ feet}, \\ h_2 &= \text{height after second bounce } = 10\left(\frac{3}{4}\right)^2 \text{ feet}. \end{aligned}$$

Generalizing gives

$$h_n = 10\left(\frac{3}{4}\right)^n.$$

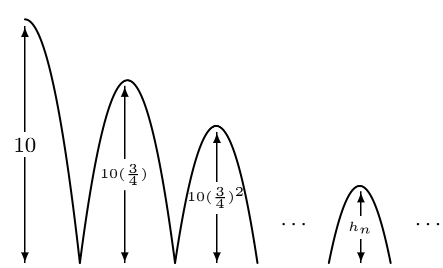


Figure 9.1

(b) When the ball hits the floor for the first time, the total distance it has traveled is just $D_1 = 10$ feet. (Notice that this is the same as $h_0 = 10$.) Then the ball bounces back to a height of $h_1 = 10\left(\frac{3}{4}\right)$, comes down and hits the floor for the second time. See Figure 9.1. The total distance it has traveled is

$$D_2 = h_0 + 2h_1 = 10 + 2\cdot 10\left(\frac{3}{4}\right) = 25 \text{ feet}.$$

Then the ball bounces back to a height of $h_2 = 10\left(\frac{3}{4}\right)^2$, comes down and hits the floor for the third time. It has traveled

$$D_3 = h_0 + 2h_1 + 2h_2 = 10 + 2\cdot 10\left(\frac{3}{4}\right) + 2\cdot 10\left(\frac{3}{4}\right)^2 = 25 + 2\cdot 10\left(\frac{3}{4}\right)^2 = 36.25 \text{ feet.}$$

Similarly,

$$\begin{aligned} D_4 &= h_0 + 2h_1 + 2h_2 + 2h_3 \\ &= 10 + 2\cdot 10\left(\frac{3}{4}\right) + 2\cdot 10\left(\frac{3}{4}\right)^2 + 2\cdot 10\left(\frac{3}{4}\right)^3 \\ &= 36.25 + 2\cdot 10\left(\frac{3}{4}\right)^3 \\ &\approx 44.69 \text{ feet.} \end{aligned}$$

(c) When the ball hits the floor for the n^{th} time, its last bounce was of height h_{n-1}. Thus, by the method used in part (b), we get

$$\begin{aligned} D_n &= h_0 + 2h_1 + 2h_2 + 2h_3 + \cdots + 2h_{n-1} \\ &= 10 + \underbrace{2\cdot 10\left(\frac{3}{4}\right) + 2\cdot 10\left(\frac{3}{4}\right)^2 + 2\cdot 10\left(\frac{3}{4}\right)^3 + \cdots + 2\cdot 10\left(\frac{3}{4}\right)^{n-1}}_{\text{finite geometric series}} \\ &= 10 + 2\cdot 10\cdot\left(\frac{3}{4}\right)\left(1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^{n-2}\right) \\ &= 10 + 15\left(\frac{1-\left(\frac{3}{4}\right)^{n-1}}{1-\left(\frac{3}{4}\right)}\right) \\ &= 10 + 60\left(1-\left(\frac{3}{4}\right)^{n-1}\right). \end{aligned}$$

29. (a) The acceleration of gravity is 32 ft/sec^2 so acceleration $= 32$ and velocity $v = 32t + C$. Since the ball is dropped, its initial velocity is 0 so $v = 32t$. Thus the position is $s = 16t^2 + C$. Calling the initial position $s = 0$, we have $s = 6t$. The distance traveled is h so $h = 16t$. Solving for t we get $t = \frac{1}{4}\sqrt{h}$.

(b) The first drop from 10 feet takes $\frac{1}{4}\sqrt{10}$ seconds. The first full bounce (to $10\cdot(\frac{3}{4})$ feet) takes $\frac{1}{4}\sqrt{10\cdot(\frac{3}{4})}$ seconds to rise, therefore the same time to come down. Thus, the full bounce, up and down, takes $2(\frac{1}{4})\sqrt{10\cdot(\frac{3}{4})}$ seconds. The next full bounce takes $2(\frac{1}{4})10\cdot(\frac{3}{4})^2 = 2(\frac{1}{4})\sqrt{10}\left(\sqrt{\frac{3}{4}}\right)^2$ seconds. The n^{th} bounce takes $2(\frac{1}{4})\sqrt{10}\left(\sqrt{\frac{3}{4}}\right)^n$ seconds. Therefore the

$$\begin{aligned} &\text{Total amount of time} \\ &= \frac{1}{4}\sqrt{10} + \underbrace{\frac{2}{4}\sqrt{10}\sqrt{\frac{3}{4}} + \frac{2}{4}\sqrt{10}\left(\sqrt{\frac{3}{4}}\right)^2 + \frac{2}{4}\sqrt{10}\left(\sqrt{\frac{3}{4}}\right)^3}_{\text{Geometric series with } a = \frac{2}{4}\sqrt{10}\sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{10}\sqrt{\frac{3}{4}} \text{ and } x = \sqrt{\frac{3}{4}}} + \cdots \\ &= \frac{1}{4}\sqrt{10} + \frac{1}{2}\sqrt{10}\sqrt{\frac{3}{4}}\left(\frac{1}{1-\sqrt{3/4}}\right) \text{ seconds.} \end{aligned}$$

30.

$$\begin{aligned} \text{Total present value, in dollars} &= 1000 + 1000e^{-0.04} + 1000e^{-0.04(2)} + 1000e^{-0.04(3)} + \cdots \\ &= 1000 + 1000(e^{-0.04}) + 1000(e^{-0.04})^2 + 1000(e^{-0.04})^3 + \cdots \end{aligned}$$

This is an infinite geometric series with $a = 1000$ and $x = e^{(-0.04)}$, and sum

$$\text{Total present value, in dollars} = \frac{1000}{1-e^{-0.04}} = 25{,}503.$$

31. The amount of additional income generated directly by people spending their extra money is $\$100(0.8) = \$80$ million. This additional money in turn is spent, generating another $(\$100(0.8))\,(0.8) = \$100(0.8)^2$ million. This continues indefinitely, resulting in

$$\text{Total additional income} = 100(0.8) + 100(0.8)^2 + 100(0.8)^3 + \cdots = \frac{100(0.8)}{1-0.8} = \$400 \text{ million}$$

32. The total of the spending and respending of the additional income is given by the series: Total additional income $= 100(0.9) + 100(0.9)^2 + 100(0.9)^3 + \cdots = \frac{100(0.9)}{1-0.9} = \$900$ million.
Notice the large effect of changing the assumption about the fraction of money spent has: the additional spending more than doubles.

Solutions for Section 9.3

Exercises

1. We use the integral test with $f(x) = 1/x^3$ to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\int_1^\infty \frac{1}{x^3}dx$ converges or diverges:

$$\int_1^\infty \frac{1}{x^3}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}dx = \lim_{b\to\infty} \frac{-1}{2x^2}\bigg|_1^b = \lim_{b\to\infty}\left(\frac{-1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2}.$$

Since the integral $\int_1^\infty \frac{1}{x^3}dx$ converges, we conclude from the integral test that the series $\sum_{n=1}^\infty \frac{1}{n^3}$ converges.

2. We use the integral test with $f(x) = x/(x^2+1)$ to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\int_1^\infty \frac{x}{x^2+1}dx$ converges or diverges:

$$\int_1^\infty \frac{x}{x^2+1}dx = \lim_{b\to\infty}\int_1^b \frac{x}{x^2+1}dx = \lim_{b\to\infty} \frac{1}{2}\ln(x^2+1)\bigg|_1^b = \lim_{b\to\infty}\left(\frac{1}{2}\ln(b^2+1)-\frac{1}{2}\ln 2\right) = \infty.$$

Since the integral $\int_1^\infty \frac{x}{x^2+1}dx$ diverges, we conclude from the integral test that the series $\sum_{n=1}^\infty \frac{n}{n^2+1}$ diverges.

3. We use the integral test with $f(x) = 1/e^x$ to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\int_1^\infty \frac{1}{e^x}dx$ converges or diverges:

$$\int_1^\infty \frac{1}{e^x}dx = \lim_{b\to\infty}\int_1^b e^{-x}dx = \lim_{b\to\infty} -e^{-x}\bigg|_1^b = \lim_{b\to\infty}\left(-e^{-b}+e^{-1}\right) = e^{-1}.$$

Since the integral $\int_1^\infty \frac{1}{e^x}dx$ converges, we conclude from the integral test that the series $\sum_{n=1}^\infty \frac{1}{e^n}$ converges. We can also observe that this is a geometric series with ratio $x = 1/e < 1$, and hence it converges.

4. We use the integral test with $f(x) = 1/(x(\ln x)^2)$ to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ converges or diverges:

$$\int_2^\infty \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty}\int_2^b \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty} \frac{-1}{\ln x}\bigg|_2^b = \lim_{b\to\infty}\left(\frac{-1}{\ln b}+\frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Since the integral $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ converges, we conclude from the integral test that the series $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ converges.

5. The improper integral $\int_1^\infty x^{-3}\,dx$ converges to $\frac{1}{2}$, since

$$\int_1^b x^{-3}\,dx = \left.\frac{x^{-2}}{-2}\right|_1^b = \frac{b^{-2}}{-2} - \frac{1^{-2}}{-2} = \frac{1}{-2b^2} + \frac{1}{2}$$

and

$$\lim_{b\to\infty}\left(\frac{1}{-2b^2} + \frac{1}{2}\right) = \frac{1}{2}.$$

The terms of the series $\sum_{n=2}^{\infty} n^{-3}$ form a right hand sum for the improper integral; each term represents the area of a rectangle of width 1 fitting completely under the graph of the function x^{-3}. (See Figure 9.2.) Thus the sequence of partial sums is bounded above by $1/2$. Since the partial sums are increasing (every new term added is positive) the series is guaranteed to converge to some number less than or equal to $1/2$ by Theorem 9.1.

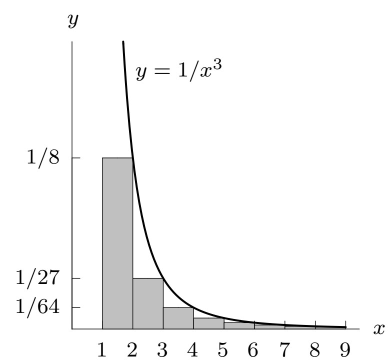


Figure 9.2

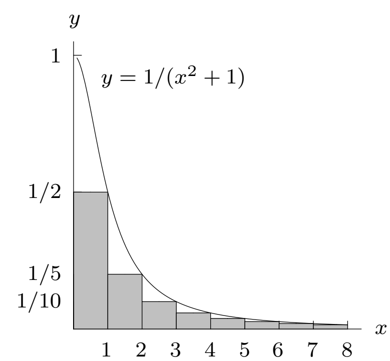


Figure 9.3

6. The improper integral $\int_0^\infty \frac{1}{x^2+1}\,dx$ converges to $\frac{\pi}{2}$, since

$$\int_0^b \frac{1}{x^2+1}\,dx = \arctan x|_0^b = \arctan b - \arctan 0 = \arctan b,$$

and $\lim_{b\to\infty} \arctan b = \frac{\pi}{2}$. The terms of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ form a right hand sum for the improper integral; each term represents the area of a rectangle of width 1 fitting completely under the graph of the function $\frac{1}{x^2+1}$. (See Figure 9.3.) Thus the sequence of partial sums is bounded above by $\frac{\pi}{2}$. Since the partial sums are increasing (every new term added is positive), the series is guaranteed to converge to some number less than or equal to $\pi/2$ by Theorem 9.1.

7. The integral test requires that $f(x) = x^2$, which is not decreasing.

8. The integral test requires that $f(x) = (-1)^x/x$. However $(-1)^x$ is not defined for all x.

9. The integral test requires that $f(x) = e^{-x}\sin x$, which is not positive, nor is it decreasing.

Problems

10. Using the integral test, we compare the series with

$$\int_0^\infty \frac{3}{x+2}\,dx = \lim_{b\to\infty}\int_0^b \frac{3}{x+2}\,dx = 3\ln|x+2|\Big|_0^b.$$

Since $\ln(b+2)$ is unbounded as $b\to\infty$, the integral diverges and therefore so does the series.

11. We use the integral test and calculate the corresponding improper integral, $\int_1^\infty 3/(2x-1)^2 dx$:

$$\int_1^\infty \frac{3\,dx}{(2x-1)^2} = \lim_{b\to\infty}\int_1^b \frac{3\,dx}{(2x-1)^2} = \lim_{b\to\infty}\left.\frac{-3/2}{(2x-1)}\right|_1^b = \lim_{b\to\infty}\left(\frac{-3/2}{(2b-1)}+\frac{3}{2}\right) = \frac{3}{2}.$$

Since the integral converges, the series $\displaystyle\sum_{n=1}^{\infty}\frac{3}{(2n-1)^2}$ converges.

12. We use the integral test and calculate the corresponding improper integral, $\int_0^\infty 2/\sqrt{2+x}\,dx$:

$$\int_0^\infty \frac{2}{\sqrt{2+x}}\,dx = \lim_{b\to\infty}\int_0^b \frac{2\,dx}{\sqrt{2+x}} = \lim_{b\to\infty} 4(2+x)^{1/2}\Big|_0^b = \lim_{b\to\infty} 4\left((2+b)^{1/2}-2^{1/2}\right).$$

Since the limit does not exist, the integral diverges, so the series $\displaystyle\sum_{n=1}^{\infty}\frac{2}{\sqrt{2+n}}$ diverges.

13. Writing $a_n = n/(n+1)$, we have $\lim_{n\to\infty} a_n = 1$ so the series diverges by Property 3 of Theorem 9.2.

14. We use the integral test and calculate the corresponding improper integral, $\int_1^\infty 4/(2x+1)^3 dx$. Using the substitution $w = 2x+1$, we have

$$\int_1^\infty \frac{4\,dx}{(2x+1)^3} = \lim_{b\to\infty}\int_1^b \frac{4\,dx}{(2x+1)^3} = \lim_{b\to\infty} -\left.\frac{1}{(2x+1)^2}\right|_1^b = \lim_{b\to\infty}\left(-\frac{1}{(2b+1)^2}+\frac{1}{9}\right) = \frac{1}{9}.$$

Since the integral converges, the series $\displaystyle\sum_{n=1}^{\infty}\frac{4}{(2n+1)^3}$ converges.

15. Using the integral test, we compare the series with

$$\int_0^\infty \frac{3}{x^2+4}\,dx = \lim_{b\to\infty}\int_0^b \frac{3}{x^2+4}\,dx = \frac{3}{2}\lim_{b\to\infty}\arctan\left(\frac{x}{2}\right)\Big|_0^b = \frac{3}{2}\lim_{b\to\infty}\arctan\left(\frac{b}{2}\right) = \frac{3\pi}{4},$$

by integral table V-24. Since the integral converges so does the series.

16. The series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$ is a convergent geometric series, but $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is the divergent harmonic series.

If $\displaystyle\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^n+\frac{1}{n}\right)$ converged, then $\displaystyle\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^n+\frac{1}{n}\right)-\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n = \sum_{n=1}^{\infty}\frac{1}{n}$ would converge by Theorem 9.2.

Therefore $\displaystyle\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^n+\frac{1}{n}\right)$ diverges.

17. The series can be written as

$$\sum_{n=1}^{\infty}\frac{n+2^n}{n2^n} = \sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{n}\right).$$

If this series converges, then $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{n}\right)-\sum_{n=1}^{\infty}\frac{1}{2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ would converge by Theorem 9.2. Since this is the harmonic series, which diverges, then the series $\displaystyle\sum_{n=1}^{\infty}\frac{n+2^n}{n}$ diverges.

18. Let $a_n = (\ln n)/n$ and $f(x) = (\ln x)/x$. We use the integral test and consider the improper integral

$$\int_c^\infty \frac{\ln x}{x}dx.$$

Since

$$\int_c^R \frac{\ln x}{x}dx = \frac{1}{2}(\ln x)^2\Big|_c^R = \frac{1}{2}\left((\ln R)^2-(\ln c)^2\right),$$

and $\ln R$ grows without bound as $R\to\infty$, the integral diverges. Therefore, the integral test tells us that the series, $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$, also diverges.

19. We use the integral test and calculate the corresponding improper integral, $\int_3^\infty (x+1)/(x^2+2x+2)\,dx$:

$$\int_3^\infty \frac{x+1}{x^2+2x+2}\,dx = \lim_{b\to\infty}\int_3^b \frac{x+1}{x^2+2x+2}\,dx = \lim_{b\to\infty}\frac{1}{2}\ln|x^2+2x+2|\Big|_3^b = \lim_{b\to\infty}\frac{1}{2}(\ln(b^2+2b+2)-\ln 17).$$

Since the limit does not exist (it is ∞), the integral diverges, so the series $\displaystyle\sum_{n=3}^{\infty}\frac{n+1}{n^2+2n+2}$ diverges.

20. Using $\ln(2^n) = n\ln 2$, we see that

$$\sum \frac{1}{\ln(2^n)} = \sum \frac{1}{(\ln 2)n}.$$

The series on the right is the harmonic series multiplied by $1/\ln 2$. Since the harmonic series diverges, $\sum_{n=1}^\infty 1/\ln(2^n)$ diverges.

21. Using $\ln(2^n) = n\ln 2$, we see that

$$\sum_{n=1}^{\infty}\frac{1}{(\ln(2^n))^2} = \sum_{n=1}^{\infty}\frac{1}{(\ln 2)^2 n^2}.$$

Since $\sum 1/n^2$ converges, $\sum 1/((\ln(2))^2 n^2)$ converges by property 1 of Theorem 9.2.

22. (a) With $a_n = \ln((n+1)/n)$ we have

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n \\ &= \ln(2/1) + \ln(3/2) + \ln(4/3) + \cdots + \ln(n/(n-1)) + \ln((n+1)/n) \\ &= \ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n}{n-1}\cdot\frac{n+1}{n}\right) = \ln(n+1). \end{aligned}$$

(b) Since the limit of the partial sums, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\ln(n+1)$, does not exist, the series diverges.

23. (a) Using $r = e^{\ln r}$ and $n = e^{\ln n}$ we have $r^{\ln n} = e^{(\ln r)(\ln n)} = n^{\ln r}$.

(b) By part (a) we have $r^{\ln n} = n^{\ln r} = 1/n^{-\ln r}$. Since the p-series $\sum 1/n^p$ converges if and only if $p > 1$, the series $\sum_{n=1}^\infty 1/n^{-\ln r}$ converges if and only if $-\ln r > 1$, which is equivalent to $\ln r < -1$ or $r < 1/e$. Thus $\sum_{n=1}^\infty r^{\ln n}$ converges if $0 < r < 1/e$ and diverges if $r \geq 1/e$.

24. (a) A common denominator is $k(k+1)$ so

$$\frac{1}{k} - \frac{1}{k+1} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{k+1-k}{k(k+1)} = \frac{1}{k(k+1)}.$$

(b) Using the result of part (a), the partial sum can be written as

$$S_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}.$$

All of the intermediate terms cancel out, leaving only the first and last terms. Thus $S_{10} = 1 - \dfrac{1}{11}$ and $S_n = 1 - \dfrac{1}{n+1}$.

(c) The limit of S_n as $n\to\infty$ is $\displaystyle\lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$. Thus the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ converges to 1.

25. (a) The partial sum

$$S_4 = \ln\left(\frac{1\cdot 3}{2\cdot 2}\right) + \ln\left(\frac{2\cdot 4}{3\cdot 3}\right) + \ln\left(\frac{3\cdot 5}{4\cdot 4}\right).$$

Using the property $\ln(A) + \ln(B) = \ln(AB)$, we get

$$S_4 = \ln\left(\frac{1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5}{2\cdot 2\cdot 3\cdot 3\cdot 4\cdot 4}\right).$$

The intermediate factors cancel out, leaving only $\ln\left(\dfrac{1\cdot 5}{2\cdot 4}\right)$, so $S_4 = \ln\left(\dfrac{5}{8}\right)$.

(b) For the partial sum S_n, similar steps yield

$$S_n = \ln\left(\frac{1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5\cdots(n-1)(n+1)}{2\cdot 2\cdot 3\cdot 3\cdot 4\cdot 4\cdots n\cdot n}\right).$$

As before, most of the factors cancel, leaving $S_n = \ln\left(\dfrac{n+1}{2n}\right)$.

(c) The limit of $S_n = \ln\left(\frac{n+1}{2n}\right)$ as $n \to \infty$ is $\lim_{n\to\infty} \ln\left(\frac{n+1}{2n}\right) = \ln\left(\frac{1}{2}\right)$. Thus the series $\sum_{k=2}^{\infty} \ln\left(\frac{(k-1)(k+1)}{k^2}\right)$ converges to $\ln\left(\frac{1}{2}\right)$.

26. Let S_n be the n^{th} partial sum for $\sum a_n$ and let T_n be the n^{th} partial sum for $\sum b_n$. Then the n^{th} partial sums for $\sum(a_n+b_n)$, $\sum(a_n-b_n)$, and $\sum ka_n$ are S_n+T_n, S_n-T_n, and kS_n, respectively. To show that these series converge, we have to show that the limits of their partial sums exist. By the properties of limits,

$$\lim_{n\to\infty}(S_n+T_n) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} T_n$$
$$\lim_{n\to\infty}(S_n-T_n) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} T_n$$
$$\lim_{n\to\infty} kS_n = k\lim_{n\to\infty} S_n.$$

This proves that the limits of the partial sums exist, so the series converge.

27. Let S_n be the n-th partial sum for $\sum a_n$ and let T_n be the n-th partial sum for $\sum b_n$. Suppose that $S_N = T_N + k$. Since $a_n = b_n$ for $n \geq N$, we have $S_n = T_n + k$ for $n \geq N$. Hence if S_n converges to a limit, so does T_n, and vice versa. Thus, $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

28. We have $a_n = S_n - S_{n-1}$. If $\sum a_n$ converges, then $S = \lim_{n\to\infty} S_n$ exists. Hence $\lim_{n\to\infty} S_{n-1}$ exists and is equal to S also. Thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

29. From Property 1 in Theorem 9.2, we know that if $\sum a_n$ converges, then so does $\sum ka_n$.

Now suppose that $\sum a_n$ diverges and $\sum ka_n$ converges for $k \neq 0$. Thus using Property 1 and replacing $\sum a_n$ by $\sum ka_n$, we know that the following series converges:

$$\sum \frac{1}{k}(ka_n) = \sum a_n.$$

Thus, we have arrived at a contradiction, which means our original assumption, that $\sum_{n=1}^{\infty} ka_n$ converged, must be wrong.

30. A typical partial sum of the series $\sum_{n=1}^{\infty}(a_{n+1}-a_n)$, say S_5, shows what happens in the general case:

$$S_5 = (a_2-a_1)+(a_3-a_2)+(a_4-a_3)+(a_5-a_4)+(a_6-a_5) = a_6 - a_1$$

as all of the intermediate terms cancel out. The same thing will happen in the general partial sum: $S_n = a_{n+1} - a_1$.

Now the series $\sum_{n=1}^{\infty}(a_{n+1}-a_n)$ converges if the sequence of partial sums S_n has a limit as $n \to \infty$. Since we're assuming that the original series $\sum_{n=1}^{\infty} a_n$ converges, we know that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = 0$ by property 3 of Theorem 9.2. Thus

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(a_{n+1}-a_1) = 0 - a_1 = -a_1.$$

Since the sequence of partial sums converges (to $-a_1$), the series $\sum_{n=1}^{\infty}(a_{n+1}-a_n)$ converges (also to $-a_1$).

31. If $a_n = 1$ for all n, then $\sum a_n$ diverges but $\sum(a_{n+1}-a_n) = \sum 0$ converges. If $a_n = n$ for all n, then $\sum a_n$ diverges, and $\sum a_{n+1} - a_n = \sum 1$ diverges

32. (a) Let N an integer with $N \geq c$. Consider the series $\sum_{i=N+1}^{\infty} a_i$. The partial sums of this series are increasing because all the terms in the series are positive. We show the partial sums are bounded using the right-hand sum in Figure 9.4. We see that for each positive integer k

$$f(N+1) + f(N+2) + \cdots + f(N+k) \leq \int_N^{N+k} f(x)\,dx.$$

Since $f(n) = a_n$ for all n, and $c \le N$, we have

$$a_{N+1} + a_{N+2} + \cdots + a_{N+k} \le \int_c^{N+k} f(x)\,dx.$$

Since $f(x)$ is a positive function, $\int_c^{N+k} f(x)\,dx \le \int_c^b f(x)\,dx$ for all $b \ge N+k$. Since f is positive and $\int_c^\infty f(x)\,dx$ is convergent, $\int_c^{N+k} f(x)\,dx < \int_c^\infty f(x)\,dx$, so we have

$$a_{N+1} + a_{N+2} + \cdots + a_{N+k} \le \int_c^\infty f(x)\,dx \quad \text{for all } k.$$

Thus, the partial sums of the series $\sum_{i=N+1}^{\infty} a_i$ are bounded and increasing, so this series converges by Theorem 9.1.

Now use Theorem 9.2, property 2, to conclude that $\sum_{i=1}^{\infty} a_i$ converges.

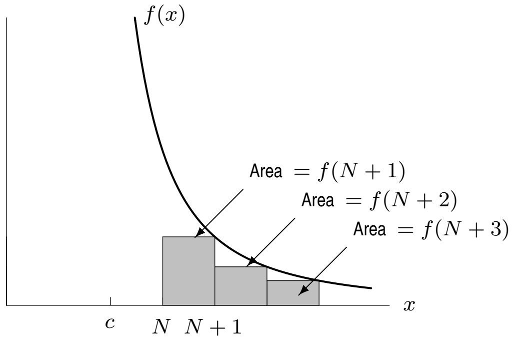


Figure 9.4

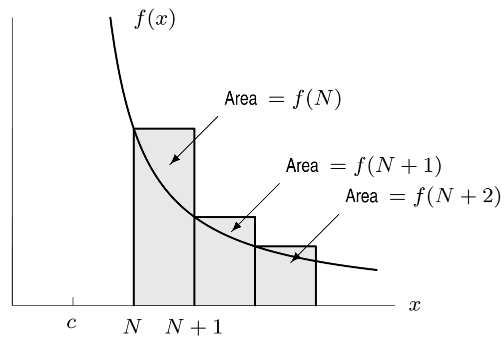


Figure 9.5

(b) We now suppose $\int_c^\infty f(x)\,dx$ diverges. In Figure 9.5 we see that for each positive integer k

$$\int_N^{N+k+1} f(x)\,dx \le f(N) + f(N+1) + \cdots + f(N+k).$$

Since $f(n) = a_n$ for all n, we have

$$\int_N^{N+k+1} f(x)\,dx \le a_N + a_{N+1} + \cdots + a_{N+k}.$$

Since $f(x)$ is defined for all $x \ge c$, if $\int_c^\infty f(x)\,dx$ is divergent, then $\int_N^\infty f(x)\,dx$ is divergent. So as $k \to \infty$, the the integral $\int_N^{N+k+1} f(x)\,dx$ diverges, so the partial sums of the series $\sum_{i=N}^{\infty} a_i$ diverge. Thus, the series $\sum_{i=1}^{\infty} a_i$ diverges.

More precisely, suppose the series converged. Then the partial sums would be bounded. (The partial sums would be less than the sum of the series, since all the terms in the series are positive.) But that would imply that the integral converged, by Theorem 9.1 on Convergence of Monotone Bounded Sequences. This contradicts the assumption that $\int_N^\infty f(x)\,dx$ is divergent.

33. (a) Show that the sum of each group of fractions is more than $1/2$.
(b) Explain why this shows that the harmonic series does not converge.

(a) Notice that

$$\begin{aligned}
&\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2} \\
&\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2} \\
&\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > \frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16} = \frac{8}{16} = \frac{1}{2}.
\end{aligned}$$

In the same way, we can see that the sum of the fractions in each grouping is greater than $1/2$.

(b) Since the sum of the first n groups is greater than $n/2$, it follows that the partial sums of the harmonic series are not bounded. Thus, the harmonic series diverges.

34. (a) Since for $x > 0$,

$$\int \frac{1}{x\ln x}dx = \ln(\ln x) + C$$

we have

$$\int_2^\infty \frac{1}{x\ln x}dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\ln x}dx = \lim_{b\to\infty}(\ln(\ln b) - \ln(\ln 2)) = \infty.$$

The series diverges by the integral test.

(b) The terms in each group are decreasing so we can bound each group as follows:

$$\frac{1}{3\ln 3} + \frac{1}{4\ln 4} > \frac{1}{4\ln 4} + \frac{1}{4\ln 4} = \frac{1}{2\ln 4}$$

and

$$\frac{1}{5\ln 5} + \frac{1}{6\ln 6} + \frac{1}{7\ln 7} + \frac{1}{8\ln 8} > 4\frac{1}{8\ln 8} = \frac{1}{2\ln 8}.$$

Similarly, the group whose final term is $1/(2^n\ln(2^n))$ is greater than $1/(2\ln(2^n)) = 1/(2(\ln 2)n)$. Thus

$$\sum_{n=2}^{2^N}\frac{1}{n\ln n} > \sum_{n=1}^{N}\frac{1}{2(\ln 2)n}.$$

The series on the right is the harmonic series multiplied by the constant $1/(2\ln 2)$. Since the harmonic series diverges, $\sum 1/(n\ln n)$ diverges.

35. (a) The left-hand sum approximation to $\displaystyle\int_1^n \frac{1}{x}\,dx$ in Figure 9.6 shows that

$$\begin{aligned}\int_1^n \frac{dx}{x} &< 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\\ \ln n - \ln 1 &< 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\\ 0 &< 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n.\end{aligned}$$

Thus, $0 < a_n$.

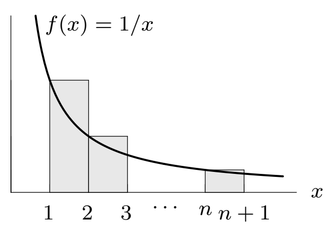


Figure 9.6

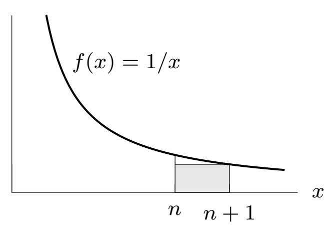


Figure 9.7

(b) We calculate

$$\begin{aligned}a_n - a_{n+1} &= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \ln n - \left(\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right) - \ln(n+1)\right)\\ &= \ln(n+1) - \ln n - \frac{1}{n+1}.\end{aligned}$$

But using the right sum with one rectangle in Figure 9.7, we see that

$$\begin{aligned}\int_n^{n+1}\frac{dx}{x} &> \frac{1}{n+1}\\ \ln(n+1) - \ln n &> \frac{1}{n+1}.\end{aligned}$$

Thus

$$a_n - a_{n+1} = \ln(n+1) - \ln n - \frac{1}{n+1} > 0.$$
$$a_n > a_{n+1}.$$

(c) Since a_n is a decreasing sequence bounded below by 0, Theorem 9.1 ensures that $\lim_{n\to\infty} a_n$ exists.
(d) The sequence converges slowly, but a calculator or computer gives $a_{200} = 0.5797$. For comparison, $a_{100} = 0.5822$, $a_{500} = 0.5782$. Thus, $\gamma = 0.58$. More extensive calculations show that $\gamma = 0.577216$.

36. (a) A calculator or computer gives

$$\sum_1^{20} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{20^2} = 1.596.$$

(b) Since $\sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, the answer to part (a) gives

$$\frac{\pi^2}{6} \approx 1.596$$
$$\pi \approx \sqrt{6 \cdot 1.596} = 3.09$$

(c) A calculator or computer gives

$$\sum_1^{100} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{100^2} = 1.635,$$

so

$$\frac{\pi^2}{6} \approx 1.635$$
$$\pi \approx \sqrt{6 \cdot 1.635} = 3.13.$$

(d) The error in approximating $\pi^2/6$ by $\sum_1^{20} 1/n^2$ is the tail of the series $\sum_{21}^{\infty} 1/n^2$. From Figure 9.8, we see that

$$\sum_{21}^{\infty} \frac{1}{n^2} < \int_{20}^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{20}^{\infty} = \frac{1}{20} = 0.05.$$

A similar argument leads to a bound for the error in approximating $\pi^2/6$ by $\sum_1^{100} 1/n^2$ as

$$\sum_{101}^{\infty} \frac{1}{n^2} < \int_{100}^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{100}^{\infty} = \frac{1}{100} = 0.01.$$

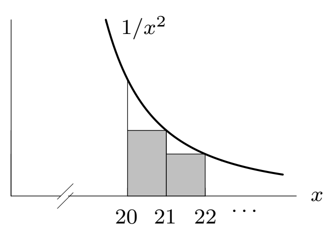


Figure 9.8

37. (a) We have $e > 1 + 1 + 1/2 + 1/6 + 1/24 = 65/24 = 2.708$.
(b) We have

$$\frac{1}{n!} = \frac{1}{1\cdot 2\cdot 3\cdot 4\cdots n} \le \frac{1}{1\cdot 2\cdot 2\cdot 2\cdots 2} = \frac{1}{2^{n-1}}.$$

(c) The inequality in part (b) can be used to replace the given series with a geometric series that we can sum.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} < 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{1 - 1/2} = 3.$$

38. (a) The right-hand sum for $\int_0^N x^N dx$ with $\Delta x = 1$ is the sum $1^5 \cdot 1 + 2^5 \cdot 1 + 3^5 \cdot 1 + \cdots + N^5 \cdot 1 = S_N$. This sum is greater than the integral because the integrand x^5 is increasing on the interval $0 < x < N$. Since $\int_0^N x^5 dx = N^6/6$, we have $S_N > N^6/6$.

(b) The left-hand sum for $\int_1^{N+1} x^N dx$ with $\Delta x = 1$ is the sum $1^5 \cdot 1 + 2^5 \cdot 1 + 3^5 \cdot 1 + \cdots + N^5 \cdot 1 = S_N$. This sum is less than the integral because the integrand x^5 is increasing on the interval $1 < x < N + 1$. Since $\int_1^{N+1} x^5 dx = ((N+1)^6 - 1)/6$, we have $S_N < ((N+1)^6 - 1)/6$.

(c) By parts (a) and (b) we have

$$\frac{N^6/6}{N^6/6} = 1 < \frac{S_N}{N^6/6} < \frac{((N+1)^6 - 1)/6}{N^6/6} = (1 + \frac{1}{N})^6 - \frac{1}{N^6}.$$

Since both $\lim_{N\to\infty} 1 = 1$ and $\lim_{N\to\infty}((1 + \frac{1}{N})^6 - \frac{1}{N^6}) = 1$, we conclude that the limit in the middle also equals 1, $\lim_{N\to\infty} S_N/(N^6/6) = 1$.

Solutions for Section 9.4

Exercises

1. Let $a_n = 1/(n-3)$, for $n \geq 4$. Since $n - 3 < n$, we have $1/(n-3) > 1/n$, so

$$a_n > \frac{1}{n}.$$

The harmonic series $\sum_{n=4}^{\infty} \frac{1}{n}$ diverges, so the comparison test tells us that the series $\sum_{n=4}^{\infty} \frac{1}{n-3}$ also diverges.

2. Let $a_n = 1/(n^2+2)$. Since $n^2 + 2 > n^2$, we have $1/(n^2+2) < 1/n^2$, so

$$0 < a_n < \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ also converges.

3. Let $a_n = e^{-n}/n^2$. Since $e^{-n} < 1$, for $n \geq 1$,we have $\frac{e^{-n}}{n^2} < \frac{1}{n^2}$, so

$$0 < a_n < \frac{1}{n^2}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$ also converges.

4. Let $a_n = 1/(3^n+1)$. Since $3^n + 1 > 3^n$, we have $1/(3^n+1) < 1/3^n = \left(\frac{1}{3}\right)^n$, so

$$0 < a_n < \left(\frac{1}{3}\right)^n.$$

Thus we can compare the series $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ with the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$. This geometric series converges since $|1/3| < 1$, so the comparison test tells us that $\sum_{n=1}^{\infty} \frac{1}{3^n+1}$ also converges.

5. Let $a_n = 1/(n^4+e^n)$. Since $n^4 + e^n > n^4$, we have

$$\frac{1}{n^4+e^n} < \frac{1}{n^4},$$

so

$$0 < a_n < \frac{1}{n^4}.$$

Since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges, the comparison test tells us that the series $\sum_{n=1}^{\infty} \frac{1}{n^4+e^n}$ also converges.

6. Since $\ln n \leq n$ for $n \geq 2$, we have $1/\ln n \geq 1/n$, so the series diverges by comparison with the harmonic series, $\sum 1/n$.

7. Let $a_n = n^2/(n^4+1)$. Since $n^4+1 > n^4$, we have $\dfrac{1}{n^4+1} < \dfrac{1}{n^4}$, so

$$a_n = \frac{n^2}{n^4+1} < \frac{n^2}{n^4} = \frac{1}{n^2},$$

therefore

$$0 < a_n < \frac{1}{n^2}.$$

Since the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the comparison test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$ converges also.

8. We know that $|\sin n| < 1$, so

$$\left|\frac{n \sin n}{n^3+1}\right| \leq \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Since the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, comparison gives that $\displaystyle\sum_{n=1}^{\infty} \left|\frac{n \sin n}{n^3+1}\right|$ converges. Thus, by Theorem 9.6, $\displaystyle\sum_{n=1}^{\infty} \frac{n \sin n}{n^3+1}$ converges.

9. Let $a_n = (2^n+1)/(n2^n-1)$. Since $n2^n - 1 < n2^n + n = n(2^n+1)$, we have

$$\frac{2^n+1}{n2^n-1} > \frac{2^n+1}{n(2^n+1)} = \frac{1}{n}.$$

Therefore, we can compare the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$ with the divergent harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. The comparison test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$ also diverges.

10. Since $a_n = 1/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = 1/(2n+2)!$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{1}{(2n+2)!}}{\dfrac{1}{(2n)!}} = \frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{(2n+2)(2n+1)},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

Since $L = 0$, the ratio test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ converges.

11. Since $a_n = (n!)^2/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = ((n+1)!)^2/(2n+2)!$. Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{((n+1)!)^2}{(2n+2)!}}{\dfrac{(n!)^2}{(2n)!}} = \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2}.$$

However, since $(n+1)! = (n+1)n!$ and $(2n+2)! = (2n+2)(2n+1)(2n)!$, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2(n!)^2(2n)!}{(2n+2)(2n+1)(2n)!(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n+1}{4n+2},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{4}.$$

Since $L < 1$, the ratio test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges.

12. Since $a_n = (2n)!/(n!(n+1)!)$, replacing n by $n+1$ gives $a_{n+1} = (2n+2)!/((n+1)!(n+2)!)$. Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{(2n+2)!}{(n+1)!(n+2)!}}{\dfrac{(2n)!}{n!(n+1)!}} = \frac{(2n+2)!}{(n+1)!(n+2)!} \cdot \frac{n!(n+1)!}{(2n)!}.$$

However, since $(n+2)! = (n+2)(n+1)n!$ and $(2n+2)! = (2n+2)(2n+1)(2n)!$, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(2n+2)(2n+1)}{(n+2)(n+1)} = \frac{2(2n+1)}{n+2},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 4.$$

Since $L > 1$, the ratio test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!}$ diverges.

13. Since $a_n = 1/(r^n n!)$, replacing n by $n+1$ gives $a_{n+1} = 1/(r^{n+1}(n+1)!)$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{1}{r^{n+1}(n+1)!}}{\dfrac{1}{r^n n!}} = \frac{r^n n!}{r^{n+1}(n+1)!} = \frac{1}{r(n+1)},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{r} \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since $L = 0$, the ratio test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{r^n n!}$ converges for all $r > 0$.

14. Since $a_n = 1/(ne^n)$, replacing n by $n+1$ gives $a_{n+1} = 1/(n+1)e^{n+1}$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{1}{(n+1)e^{n+1}}}{\dfrac{1}{ne^n}} = \frac{ne^n}{(n+1)e^{n+1}} = \left(\frac{n}{n+1}\right)\frac{1}{e}.$$

Therefore

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{e} < 1.$$

Since $L < 1$, the ratio test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{ne^n}$ converges.

15. Since $a_n = 2^n/(n^3+1)$, replacing n by $n+1$ gives $a_{n+1} = 2^{n+1}/((n+1)^3+1)$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{2^{n+1}}{(n+1)^3+1}}{\dfrac{2^n}{n^3+1}} = \frac{2^{n+1}}{(n+1)^3+1} \cdot \frac{n^3+1}{2^n} = 2\frac{n^3+1}{(n+1)^3+1},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 2.$$

Since $L > 1$ the ratio test tells us that the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n^3+1}$ diverges.

16. Even though the first term is negative, the terms alternate in sign, so it is an alternating series.

17. Since $\cos(n\pi) = (-1)^n$, this is an alternating series.

18. Since $(-1)^n \cos(n\pi) = (-1)^{2n} = 1$, this is not an alternating series.

19. Since $a_n = \cos n$ is not always positive, this is not an alternating series.

20. Let $a_n = 1/\sqrt{n}$. Then replacing n by $n+1$ we have $a_{n+1} = 1/\sqrt{n+1}$. Since $\sqrt{n+1} > \sqrt{n}$, we have $\dfrac{1}{\sqrt{n+1}} < \dfrac{1}{\sqrt{n}}$, hence $a_{n+1} < a_n$. In addition, $\lim_{n\to\infty} a_n = 0$ so $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test.

21. Let $a_n = 1/(2n+1)$. Then replacing n by $n+1$ gives $a_{n+1} = 1/(2n+3)$. Since $2n+3 > 2n+1$, we have

$$0 < a_{n+1} = \frac{1}{2n+3} < \frac{1}{2n+1} = a_n.$$

We also have $\lim_{n\to\infty} a_n = 0$. Therefore, the alternating series test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$ converges.

22. Let $a_n = 1/(n^2+2n+1) = 1/(n+1)^2$. Then replacing n by $n+1$ gives $a_{n+1} = 1/(n+2)^2$. Since $n+2 > n+1$, we have

$$\frac{1}{(n+2)^2} < \frac{1}{(n+1)^2}$$

so

$$0 < a_{n+1} < a_n.$$

We also have $\lim_{n\to\infty} a_n = 0$. Therefore, the alternating series test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+2n+1}$ converges.

23. Let $a_n = 1/e^n$. Then replacing n by $n+1$ we have $a_{n+1} = 1/e^{n+1}$. Since $e^{n+1} > e^n$, we have $\dfrac{1}{e^{n+1}} < \dfrac{1}{e^n}$, hence $a_{n+1} < a_n$. In addition, $\lim_{n\to\infty} a_n = 0$ so $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$ converges by the alternating series test. We can also observe that the series is geometric with ratio $x = -1/e$ can hence converges since $|x| < 1$.

24. We have

$$\frac{a_n}{b_n} = \frac{(5n+1)/(3n^2)}{1/n} = \frac{5n+1}{3n},$$

so

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5n+1}{3n} = \frac{5}{3} = c \neq 0.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series, the original series diverges.

25. We have

$$\frac{a_n}{b_n} = \frac{((1+n)/(3n))^n}{(1/3)^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n,$$

so

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e = c \neq 0.$$

Since $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a convergent geometric series, the original series converges.

26. The n^{th} term $a_n = 1/(n^4-7)$ behaves like $1/n^4$ for large n, so we take $b_n = 1/n^4$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(n^4-7)}{1/n^4} = \lim_{n\to\infty} \frac{n^4}{n^4-7} = 1.$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^4$ converges because $p = 4 > 1$. Therefore $\sum 1/(n^4-7)$ also converges.

27. The n^{th} term $a_n = (n+1)/(n^2+2)$ behaves like $n/n^2 = 1/n$ for large n, so we take $b_n = 1/n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n+1)/(n^2+2)}{1/n} = \lim_{n\to\infty} \frac{n^2+n}{n^2+2} = 1.$$

The limit comparison test applies with $c = 1$. Since the harmonic series $\sum 1/n$ diverges, the series $\sum(n+1)/(n^2+2)$ also diverges.

28. The n^{th} term $a_n = (n^3 - 2n^2 + n + 1)/(n^4 - 2)$ behaves like $n^3/n^4 = 1/n$ for large n, so we take $b_n = 1/n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n^3 - 2n^2 + n + 1)/(n^4 - 2)}{1/n} = \lim_{n\to\infty} \frac{n^4 - 2n^3 + n^2 + n}{n^4 - 2} = 1.$$

The limit comparison test applies with $c = 1$. The harmonic series $\sum 1/n$ diverges. Thus $\sum \left(n^3 - 2n^2 + n + 1\right) / \left(n^4 - 2\right)$ also diverges.

29. The n^{th} term $a_n = 2^n/(3^n - 1)$ behaves like $2^n/3^n$ for large n, so we take $b_n = 2^n/3^n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n/(3^n - 1)}{2^n/3^n} = \lim_{n\to\infty} \frac{3^n}{3^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 3^{-n}} = 1.$$

The limit comparison test applies with $c = 1$. The geometric series $\sum 2^n/3^n = \sum (2/3)^n$ converges. Therefore $\sum 2^n/(3^n - 1)$ also converges.

30. The n^{th} term $a_n = 1/(2\sqrt{n} + \sqrt{n+2})$ behaves like $1/(3\sqrt{n})$ for large n, so we take $b_n = 1/(3\sqrt{n})$. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2\sqrt{n} + \sqrt{n+2})}{1/(3\sqrt{n})} &= \lim_{n\to\infty} \frac{3\sqrt{n}}{2\sqrt{n} + \sqrt{n+2}} \\
&= \lim_{n\to\infty} \frac{3\sqrt{n}}{\sqrt{n}\left(2 + \sqrt{1 + 2/n}\right)} \\
&= \lim_{n\to\infty} \frac{3}{2 + \sqrt{1 + 2/n}} = \frac{3}{2 + \sqrt{1+0}} \\
&= 1.
\end{aligned}$$

The limit comparison test applies with $c = 1$. The series $\sum 1/(3\sqrt{n})$ diverges because it is a multiple of a p-series with $p = 1/2 < 1$. Therefore $\sum 1/(2\sqrt{n} + \sqrt{n+2})$ also diverges.

31. The n^{th} term,

$$a_n = \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{4n^2 - 2n},$$

behaves like $1/(4n^2)$ for large n, so we take $b_n = 1/(4n^2)$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(4n^2 - 2n)}{1/(4n^2)} = \lim_{n\to\infty} \frac{4n^2}{4n^2 - 2n} = \lim_{n\to\infty} \frac{1}{1 - 1/(2n)} = 1.$$

The limit comparison test applies with $c = 1$. The series $\sum 1/(4n^2)$ converges because it is a multiple of a p-series with $p = 2 > 1$. Therefore $\sum \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$ also converges.

Problems

32. The comparison test requires that $a_n = (-1)^n/n^2$ be positive. It is not.

33. The comparison test requires that $a_n = \sin n$ be positive for all n. It is not.

34. With $a_n = (-1)^n$, we have $|a_{n+1}/a_n| = 1$, and $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, so the test gives no information.

35. With $a_n = \sin n$, we have $|a_{n+1}/a_n| = |\sin(n+1)/\sin n|$, which does not have a limit as $n \to \infty$, so the test does not apply.

36. The sequence $a_n = n$ does not satisfy either $a_{n+1} < a_n$ or $\lim_{n\to\infty} a_n = 0$.

37. The alternating series test requires $a_n = \sin n$ be positive, which it is not. This is not an alternating series.

38. The alternating series test requires $a_n = 2 - 1/n$ which is positive and satisfies $a_{n+1} < a_n$ but $\lim_{n\to\infty} a_n = 2 \neq 0$.

39. The partial sums are $S_1 = 1$, $S_2 = -1$, $S_3 = 2$, $S_{10} = -5$, $S_{11} = 6$, $S_{100} = -50$, $S_{101} = 51$, $S_{1000} = -500$, $S_{1001} = 501$, which appear to be oscillating further and further from 0. This series does not converge.

40. The partial sums look like: $S_1 = 1$, $S_2 = 0.9$, $S_3 = 0.91$, $S_4 = 0.909$, $S_5 = 0.9091$, $S_6 = 0.90909$. The series appears to be converging to $0.909090\ldots$ or $10/11$.

Since $a_n = 10^{-k}$ is positive and decreasing and $\lim\limits_{n\to\infty} 10^{-n} = 0$, the alternating series test confirms the convergence of the series.

41. The partial sums look like: $S_1 = 1$, $S_2 = 0$, $S_3 = 0.5$, $S_4 = 0.3333$, $S_5 = 0.375$, $S_{10} = 0.3679$, $S_{20} = 0.3679$, and higher partial sums agree with these first 4 decimal places. The series appears to be converging to about 0.3679.

Since $a_n = 1/n!$ is positive and decreasing and $\lim_{n\to\infty} 1/n! = 0$, the alternating series test confirms the convergence of this series.

42. We use the ratio test and calculate

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(0.1)^{n+1}/(n+1)!}{(0.1)^n/n!} = \lim_{n\to\infty} \frac{0.1}{n+1} = 0.$$

Since the limit is less than 1, the series converges.

43. We use the ratio test and calculate

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n!/(n+1)^2}{(n-1)!/n^2} = \lim_{n\to\infty} \left(\frac{n!}{(n-1)!} \cdot \frac{n^2}{(n+1)^2} \right) = \lim_{n\to\infty} \left(n \cdot \frac{n^2}{(n+1)^2} \right).$$

Since the limit does not exist (it is ∞), the series diverges.

44. The first few terms of the series may be written

$$1 + e^{-1} + e^{-2} + e^{-3} + \cdots;$$

this is a geometric series with $a = 1$ and $x = e^{-1} = 1/e$. Since $|x| < 1$, the geometric series converges to $S = \dfrac{1}{1-x} = \dfrac{1}{1-e^{-1}} = \dfrac{e}{e-1}$.

45. The first few terms of the series may be written

$$e + e^2 + e^3 + \cdots = e + e \cdot e + e \cdot e^2 + \cdots;$$

this is a geometric series with $a = e$ and $x = e$. Since $|x| > 1$, this geometric series diverges.

46. Let $a_n = 1/\sqrt{3n-1}$. Then replacing n by $n+1$ gives $a_{n+1} = 1/\sqrt{3(n+1)-1}$. Since

$$\sqrt{3(n+1)-1} > \sqrt{3n-1},$$

we have

$$a_{n+1} < a_n.$$

In addition, $\lim_{n\to\infty} a_n = 0$ so the alternating series test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{3n-1}}$ converges.

47. Since the exponential, 2^n, grows faster than the power, n^2, the terms are growing in size. Thus, $\displaystyle\lim_{n\to\infty} a_n \neq 0$. We conclude that this series diverges.

48. Since $0 \le |\sin n| \le 1$ for all n, we may be able to compare with $1/n^2$. We have $0 \le |\sin n/n^2| \le 1/n^2$ for all n. So $\sum |\sin n/n^2|$ converges by comparison with the convergent series $\sum(1/n^2)$. Therefore $\sum(\sin n/n^2)$ also converges, since absolute convergence implies convergence by Theorem 9.6.

49. Note that $\cos(n\pi)/n = (-1)^n/n$, so this is an alternating series. Therefore, since $1/(n+1) < 1/n$ and $\lim_{n\to\infty} 1/n = 0$, we see that $\sum(\cos(n\pi)/n)$ converges by the alternating series test.

50. As $n \to \infty$, we see that

$$\frac{n+2}{n^2-1} \to \frac{n}{n^2} = \frac{1}{n}.$$

Since $\sum(1/n)$ diverges, we expect our series to have the same behavior.

More precisely, for all $n \ge 2$, we have

$$0 \le \frac{1}{n} = \frac{n}{n^2} \le \frac{n+2}{n^2-1},$$

so $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-1}$ diverges by comparison with the divergent series $\displaystyle\sum \frac{1}{n}$.

51. Since

$$\frac{3}{\ln n^2} = \frac{3}{2\ln n},$$

our series behaves like the series $\sum 1/\ln n$. More precisely, for all $n \geq 2$, we have

$$0 \leq \frac{1}{n} \leq \frac{1}{\ln n} \leq \frac{3}{2\ln n} = \frac{3}{\ln n^2},$$

so $\displaystyle\sum_{n=2}^{\infty} \frac{3}{\ln n^2}$ diverges by comparison with the divergent series $\displaystyle\sum \frac{1}{n}$.

52. Let $a_n = 1/\sqrt{n^2(n+2)}$. Since $n^2(n+2) = n^3 + 2n^2 > n^3$, we have

$$0 < a_n < \frac{1}{n^{3/2}}.$$

Since the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, the comparison test tells us that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2(n+2)}}$$

also converges.

53. Let $a_n = n(n+1)/\sqrt{n^3+2n^2}$. Since $n^3 + 2n^2 = n^2(n+2)$, we have

$$a_n = \frac{n(n+1)}{n\sqrt{n+2}} = \frac{n+1}{\sqrt{n+2}}$$

so a_n grows without bound as $n \to \infty$, therefore the series $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{\sqrt{n^3+2n^2}}$ diverges.

54. The n^{th} partial sum of the series is given by

$$S_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n},$$

so the absolute value of the first term omitted is $1/(n+1)$. By Theorem 9.9, we know that the value, S, of the sum differs from S_n by less than $1/(n+1)$. Thus, we want to choose n large enough so that $1/(n+1) \leq 0.01$. Solving this inequality for n yields $n \geq 99$, so we take 99 or more terms in our partial sum.

55. The n^{th} partial sum of the series is given by

$$S_n = 1 - \frac{2}{3} + \frac{4}{9} - \cdots + (-1)^n \left(\frac{2}{3}\right)^n,$$

so the absolute value of the first term omitted is $(2/3)^{n+1}$. By Theorem 9.9, we know that the value, S, of the sum differs from S_n by less than $(2/3)^{n+1}$. Thus, we want to choose n large enough so that $(2/3)^{n+1} \leq 0.01$. Solving this inequality for n yields $n \geq 10.358$, so we take 11 or more terms in our partial sum.

56. The n^{th} partial sum of the series is given by

$$S_n = \frac{1}{2} - \frac{1}{24} + \frac{1}{720} - \cdots + \frac{(-1)^{n-1}}{(2n)!},$$

so the absolute value of the first term omitted is $1/(2n+2)!$. By Theorem 9.9, we know that the value, S, of the sum differs from S_n by less than $1/(2n+2)!$. Thus, we want to choose n large enough so that $1/(2n+2)! \leq 0.01$. Substituting $n = 2$ into the expression $1/(2n+2)!$ yields $1/720$ which is less than 0.01. We therefore take 2 or more terms in our partial sum.

57. Both $\displaystyle\sum \frac{(-1)^n}{2^n} = \sum \left(\frac{-1}{2}\right)^n$ and $\displaystyle\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$ are convergent geometric series. Thus $\displaystyle\sum \frac{(-1)^n}{2^n}$ is absolutely convergent.

58. The series $\sum \frac{(-1)^n}{2n}$ converges by the alternating series test. However $\sum \frac{1}{2n}$ diverges because it is a multiple of the harmonic series. Thus $\sum \frac{(-1)^n}{2n}$ is conditionally convergent.

59. Since
$$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)=1,$$
the n^{th} term $a_n=(-1)^n\left(1+\frac{1}{n^2}\right)$ does not tend to zero as $n\to\infty$. Thus, the series $\sum(-1)^n\left(1+\frac{1}{n^2}\right)$ is divergent.

60. The series $\sum \frac{(-1)^n}{n^4+7}$ converges by the alternating series test. Moreover, the series $\sum \frac{1}{n^4+7}$ converges by comparison with the convergent p-series $\sum \frac{1}{n^4}$. Thus $\sum \frac{(-1)^n}{n^4+7}$ is absolutely convergent.

61. Since $0\le c_n\le 2^{-n}$ for all n, and since $\sum 2^{-n}$ is a convergent geometric series, $\sum c_n$ converges by the Comparison Test. Similarly, since $2^n\le a_n$, and since $\sum 2^n$ is a divergent geometric series, $\sum a_n$ diverges by the Comparison Test. We do not have enough information to determine whether or not $\sum b_n$ and $\sum d_n$ converge.

62. (a) The sum $\sum a_n\cdot b_n=\sum 1/n^5$, which converges, as a p-series with $p=5$, or by the integral test:
$$\int_1^\infty \frac{1}{x^5}\,dx=\lim_{b\to\infty}\left.\frac{x^{-4}}{(-4)}\right|_1^b=\lim_{b\to\infty}\frac{b^{-4}}{(-4)}+\frac{1}{4}=\frac{1}{4}.$$
Since this improper integral converges, $\sum a_n\cdot b_n$ also converges.

(b) This is an alternating series that satisfies the conditions of the alternating series test: the terms are decreasing and have limit 0, so $\sum(-1)^n/\sqrt{n}$ converges.

(c) We have $a_nb_n=1/n$, so $\sum a_nb_n$ is the harmonic series, which diverges.

63. Since $\lim_{n\to\infty} a_n/b_n=0$, for large enough n we have $|a_n/b_n|<1/2$ and thus $0\le|a_n|<b_n/2<b_n$. By the comparison test applied to $\sum|a_n|$ and $\sum b_n$, the series $\sum|a_n|$ converges. The series $\sum a_n$ converges absolutely and thus it converges.

64. Since $\lim_{n\to\infty} a_n/b_n=\infty$, for large enough n we have $a_n/b_n>1$ and thus $a_n>b_n$. By the comparison test applied to $\sum a_n$ and $\sum b_n$, the series $\sum a_n$ diverges.

65. Each term in $\sum b_n$ is greater than or equal to a_1 times a term in the harmonic series:
$$\begin{aligned}
b_1 &= a_1\cdot 1\\
b_2 &= \frac{a_1+a_2}{2}>a_1\cdot\frac{1}{2}\\
b_3 &= \frac{a_1+a_2+a_3}{3}>a_1\cdot\frac{1}{3}\\
&\vdots\\
b_n &= \frac{a_1+a_2+\cdots+a_n}{n}>a_1\cdot\frac{1}{n}
\end{aligned}$$
Adding these inequalities gives
$$\sum b_n>a_1\sum\frac{1}{n}.$$
Since the harmonic series $\sum 1/n$ diverges, a_1 times the harmonic series also diverges. Then, by the comparison test, the series $\sum b_n$ diverges.

66. Suppose we let $c_n=(-1)^na_n$. (We have just given the terms of the series $\sum(-1)^na_n$ a new name.) Then
$$|c_n|=|(-1)^na_n|=|a_n|.$$
Thus $\sum|c_n|$ converges, and by Theorem 9.6,
$$\sum c_n=\sum(-1)^na_n\quad\text{converges.}$$

67. (a) Since

$$\begin{aligned} |a_n| &= a_n && \text{if } a_n \geq 0 \\ |a_n| &= -a_n && \text{if } a_n < 0, \end{aligned}$$

we have

$$\begin{aligned} a_n + |a_n| &= 2|a_n| && \text{if } a_n \geq 0 \\ a_n + |a_n| &= 0 && \text{if } a_n < 0. \end{aligned}$$

Thus, for all n,

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

(b) If $\sum |a_n|$ converges, then $\sum 2|a_n|$ is convergent, so, by comparison, $\sum (a_n + |a_n|)$ is convergent. Then

$$\sum ((a_n + |a_n|) - |a_n|) = \sum a_n$$

is convergent, as it is the difference of two convergent series.

68. The limit

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1,$$

so the series converges.

69. The limit

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{5n+1}{3n^2} = 0 < 1,$$

so the series converges.

Solutions for Section 9.5

Exercises

1. Yes.

2. No, because it contains negative powers of x.

3. No, each term is a power of a different quantity.

4. Yes. It's a polynomial, or a series with all coefficients beyond the 7th being zero.

5. The general term can be written as $\dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!} x^n$ for $n \geq 1$. Other answers are possible.

6. The general term can be written as $\dfrac{p(p-1)(p-2)\cdots(p-n+1)}{n!} x^n$ for $n \geq 1$. Other answers are possible.

7. The general term can be written as $\dfrac{(-1)^k (x-1)^{2k}}{(2k)!}$ for $k \geq 0$. Other answers are possible.

8. The general term can be written as $\dfrac{(-1)^{k+1}(x-1)^{2k+1}}{(2(k-1))!}$ for $k \geq 1$ or as $\dfrac{(-1)^k (x-1)^{2k+3}}{(2k)!}$ for $k \geq 0$. Other answers are possible.

9. The general term can be written as $\dfrac{(x-a)^n}{2^{n-1} \cdot n!}$ for $n \geq 1$. Other answers are possible.

10. The general term can be written as $\dfrac{(k+1)(x+5)^{2k+1}}{(k-1)!}$ for $k \geq 1$ or as $\dfrac{(k+2)(x+5)^{2k+3}}{k!}$ for $k \geq 0$. Other answers are possible.

11. This series may be written as

$$1 + 5x + 25x^2 + \cdots$$

so $C_n = 5^n$. Using the ratio test, with $a_n = 5^n x^n$, we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = |x| \lim_{n\to\infty} \frac{5^{n+1}}{5^n} = 5|x|.$$

Thus the radius of convergence is $R = 1/5$.

12. Since $C_n = n^3$, replacing n by $n+1$ gives $C_{n+1} = (n+1)^3$. Using the ratio test, with $a_n = n^3x^n$, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{(n+1)^3}{n^3} = |x|\left(\frac{n+1}{n}\right)^3.$$

We have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x|.$$

Thus the radius of convergence is $R = 1$.

13. Since $C_n = (n+1)/(2^n+n)$, replacing n by $n+1$ gives $C_{n+1} = (n+2)/(2^{n+1}+n+1)$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{(n+2)/(2^{n+1}+n+1)}{(n+1)/(2^n+n)} = |x|\frac{n+2}{2^{n+1}+n+1}\cdot\frac{2^n+n}{n+1} = |x|\frac{n+2}{n+1}\cdot\frac{2^n+n}{2^{n+1}+n+1}.$$

Since

$$\lim_{n\to\infty}\frac{n+2}{n+1} = 1$$

and

$$\lim_{n\to\infty}\left(\frac{2^n+n}{2^{n+1}+n+1}\right) = \frac{1}{2}\lim_{n\to\infty}\left(\frac{2^n+n}{2^n+(n+1)/2}\right) = \frac{1}{2},$$

because 2^n dominates n as $n\to\infty$, we have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \frac{1}{2}|x|.$$

Thus the radius of convergence is $R = 2$.

14. Since $C_n = 2^n/n$, replacing n by $n+1$ gives $C_{n+1} = 2^{n+1}/(n+1)$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x-1|\frac{|C_{n+1}|}{|C_n|} = |x-1|\frac{2^{n+1}/(n+1)}{2^n/n} = |x-1|\frac{2^{n+1}}{(n+1)}\cdot\frac{n}{2^n} = 2|x-1|\left(\frac{n}{n+1}\right),$$

so

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 2|x-1|.$$

Thus the radius of convergence is $R = \frac{1}{2}$.

15. To find R, we consider the following limit, where the coefficient of the n^{th} term is given by $C_n = n^2$.

$$\begin{aligned}\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\left|\frac{(n+1)^2x^{n+1}}{n^2x^n}\right| &= \lim_{n\to\infty}|x|\frac{n^2+2n+1}{n^2}\\ &= |x|\lim_{n\to\infty}\left(\frac{1+(2/n)+(1/n^2)}{1}\right) = |x|.\end{aligned}$$

Thus, the radius of convergence is $R = 1$.

16. The coefficient of the n^{th} term is $C_n = (-1)^{n+1}/n^2$. Now consider the ratio

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n^2x^{n+1}}{(n+1)^2x^n}\right| \to |x| \quad \text{as} \quad n\to\infty.$$

Thus, the radius of convergence is $R = 1$.

17. Here the coefficient of the n^{th} term is $C_n = (2^n/n!)$. Now we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2^{n+1}/(n+1)!)x^{n+1}}{(2^n/n!)x^n}\right| = \frac{2|x|}{n+1} \to 0 \text{ as } n\to\infty.$$

Thus, the radius of convergence is $R = \infty$, and the series converges for all x.

18. Here the coefficient of the n^{th} term is $C_n = n/(2n+1)$. Now we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((n+1)/(2n+3))x^{n+1}}{(n/(2n+1))x^n}\right| = \frac{(n+1)(2n+1)}{n(2n+3)}|x| \to |x| \text{ as } n\to\infty.$$

Thus, by the ratio test, the radius of convergence is $R = 1$.

19. Here $C_n = (2n)!/(n!)^2$. We have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2(n+1))!/((n+1)!)^2 x^{n+1}}{(2n)!/(n!)^2 x^n}\right| = \frac{(2(n+1))!}{(2n)!} \cdot \frac{(n!)^2}{((n+1)!)^2}|x|$$
$$= \frac{(2n+2)(2n+1)|x|}{(n+1)^2} \to 4|x| \text{ as } n \to \infty.$$

Thus, the radius of convergence is $R = 1/4$.

20. Here the coefficient of the n^{th} term is $C_n = (2n+1)/n$. Applying the ratio test, we consider:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((2n+3)/(n+1))x^{n+1}}{((2n+1)/n)x^n}\right| = |x|\frac{2n+3}{2n+1} \cdot \frac{n}{n+1} \to |x| \text{ as } n \to \infty.$$

Thus, the radius of convergence is $R = 1$.

21. We write the series as

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots,$$

so

$$a_n = (-1)^{n-1}\frac{x^{2n-1}}{2n-1}.$$

Replacing n by $n+1$, we have

$$a_{n+1} = (-1)^{n+1-1}\frac{x^{2(n+1)-1}}{2(n+1)-1} = (-1)^n\frac{x^{2n+1}}{2n+1}.$$

Thus

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{(-1)^n x^{2n+1}}{2n+1}\right| \cdot \left|\frac{2n-1}{(-1)^{n-1}x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2,$$

so

$$L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{2n-1}{2n+1}x^2 = x^2.$$

By the ratio test, this series converges if $L < 1$, that is, if $x^2 < 1$, so $R = 1$.

22. (a) The general term of the series is x^n/n if n is odd and $-x^n/n$ if n is even, so $C_n = (-1)^{n-1}/n$, and we can use the ratio test. We have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x|\lim_{n\to\infty}\frac{|(-1)^n/(n+1)|}{|(-1)^{n-1}/n|} = |x|\lim_{n\to\infty}\frac{n}{n+1} = |x|.$$

Therefore the radius of convergence is $R = 1$. This tells us that the power series converges for $|x| < 1$ and does not converge for $|x| > 1$. Notice that the radius of convergence does not tell us what happens at the endpoints, $x = \pm 1$.

(b) The endpoints of the interval of convergence are $x = \pm 1$. At $x = 1$, we have the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$

This is an alternating series with $a_n = 1/n$, so by the alternating series test, it converges. At $x = -1$, we have the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots$$

This is the negative of the harmonic series, so it does not converge. Therefore the right endpoint is included, and the left endpoint is not included in the interval of convergence, which is $-1 < x \le 1$.

23. Let $C_n = 2^n/n$. Then replacing n by $n+1$ gives $C_{n+1} = 2^{n+1}/(n+1)$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{2^{n+1}/(n+1)}{2^n/n} = |x|\frac{2^{n+1}}{n+1} \cdot \frac{n}{2^n} = 2|x|\left(\frac{n}{n+1}\right).$$

Thus

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 2|x|.$$

The radius of convergence is $R = 1/2$.

For $x = 1/2$ the series becomes the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.

For $x = -1/2$ the series becomes the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges. See Example 8 on page 460.

Problems

24. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{x^n}\right| = \frac{|x|}{3}.$$

Since $|x|/3 < 1$ when $|x| < 3$, the radius of convergence is 3 and the series converges for $-3 < x < 3$.

We check the endpoints:

$$x = 3: \quad \sum_{n=0}^{\infty} \frac{x^n}{3^n} = \sum_{n=0}^{\infty} \frac{3^n}{3^n} = \sum_{n=0}^{\infty} 1^n \quad \text{which diverges.}$$

$$x = -3: \quad \sum_{n=0}^{\infty} \frac{x^n}{3^n} = \sum_{n=0}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n \quad \text{which diverges.}$$

The series diverges at both the endpoints, so the interval of convergence is $-3 < x < 3$.

25. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| = \frac{n}{n+1} \cdot |x-3|.$$

Since $n/(n+1) \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-3|.$$

The series converges for $|x-3| < 1$. The radius of convergence is 1 and the series converges for $2 < x < 4$.

We check the endpoints. For $x = 2$, we have

$$\sum_{n=2}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=2}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}.$$

This is the alternating harmonic series and converges. For $x = 4$, we have

$$\sum_{n=2}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=2}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=2}^{\infty} \frac{1}{n}.$$

This is the harmonic series and diverges. The series converges at $x = 2$ and diverges at $x = 4$. Therefore, the interval of convergence is $2 \le x < 4$.

26. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2 x^{2(n+1)}}{2^{2(n+1)}} \cdot \frac{2^{2n}}{n^2 x^{2n}}\right| = \left(\frac{n+1}{n}\right)^2 \cdot \frac{x^2}{4}.$$

Since $(n+1)/n \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{x^2}{4}.$$

We have $x^2/4 < 1$ when $|x| < 2$. The radius of convergence is 2 and the series converges for $-2 < x < 2$.

We check the endpoints. For $x = -2$, we have

$$\sum_{n=1}^{\infty} \frac{n^2 x^{2n}}{2^{2n}} = \sum_{n=1}^{\infty} \frac{n^2(-2)^{2n}}{2^{2n}} = \sum_{n=1}^{\infty} n^2,$$

which diverges. Similarly, for $x = 2$, we have

$$\sum_{n=1}^{\infty} \frac{n^2 x^{2n}}{2^{2n}} = \sum_{n=1}^{\infty} \frac{n^2 2^{2n}}{2^{2n}} = \sum_{n=1}^{\infty} n^2,$$

which diverges. The series diverges at both endpoints, so the interval of convergence is $-2 < x < 2$.

27. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+1)^2} \cdot \frac{2^n n^2}{(-1)^n(x-5)^n}\right| = \left(\frac{n}{n+1}\right)^2 \cdot \frac{|x-5|}{2}.$$

Since $n/(n+1) \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-5|}{2}.$$

We have $|x-5|/2 < 1$ when $|x-5| < 2$. The radius of convergence is 2 and the series converges for $3 < x < 7$.

We check the endpoints. For $x = 3$, we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n(x-5)^n}{2^n n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n(3-5)^n}{2^n n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n(-2)^n}{2^n n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

This is a p-series with $p = 2$ and it converges. For $x = 7$, we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n(x-5)^n}{2^n n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n(7-5)^n}{2^n n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{2^n n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

Since $\sum \frac{1}{n^2}$ converges, the alternating series $\sum \frac{(-1)^n}{n^2}$ also converges. The series converges at both its endpoints, so the interval of convergence is $3 \le x \le 7$.

28. The coefficient of the n^{th} term of the binomial power series is given by

$$C_n = \frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!}.$$

To apply the ratio test, consider

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= |x|\left|\frac{p(p-1)(p-2)\cdots(p-(n-1))(p-n)/(n+1)!}{p(p-1)(p-2)\cdots(p-(n-1))/n!}\right| \\ &= |x|\left|\frac{p-n}{n+1}\right| = |x|\left|\frac{p}{n+1} - \frac{n}{n+1}\right| \to |x| \text{ as } n \to \infty.\end{aligned}$$

Thus, the radius of convergence is $R = 1$.

29. The k^{th} coefficient in the series $\sum kC_kx^k$ is $D_k = k\cdot C_k$. We are given that the series $\sum C_kx^k$ has radius of convergence R by the ratio test, so

$$|x|\lim_{k\to\infty}\frac{|C_{k+1}|}{|C_k|} = \frac{|x|}{R}.$$

Thus, applying the ratio test to the new series, we have

$$\lim_{k\to\infty}\left|\frac{D_{k+1}x^{k+1}}{D_kx^k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)C_{k+1}}{kC_k}\right||x| = \frac{|x|}{R}.$$

Hence the new series has radius of convergence R.

30. The radius of convergence, R, is between 5 and 7.

31. The series is centered at $x = -7$. Since the series converges at $x = 0$, which is a distance of 7 from $x = -7$, the radius of convergence, R, is at least 7. Since the series diverges at $x = -17$, which is a distance of 10 from $x = -7$, the radius of convergence is no more than 10. That is, $7 \le R \le 10$.

32. The radius of convergence of the series, R, is at least 4 but no larger than 7.

(a) False. Since $10 > R$ the series diverges.
(b) True. Since $3 < R$ the series converges.
(c) False. Since $1 < R$ the series converges.
(d) Not possible to determine since the radius of convergence may be more or less than 6.

33. The series is centered at $x = 3$. Since the series converges at $x = 7$, which is a distance of 4 from $x = 3$, we know $R \ge 4$. Since the series diverges at $x = 10$, which is a distance of 7 from $x = 3$, we know $R \le 7$. That is, $4 \le R \le 7$.

Since $x = 11$ is a distance of 8 from $x = 3$, the series diverges at $x = 11$.

Since $x = 5$ is a distance of 2 from $x = 3$, the series converges there.

Since $x = 0$ is a distance of 3 from $x = 3$, the series converges at $x = 3$.

34. (a) We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2}\cdot\frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right|$$
$$= \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2}\cdot\frac{2^{2n}(n!)^2}{x^{2n}}$$
$$= \frac{x^2}{4(n+1)^2}.$$

For a fixed value of x, we have

$$\frac{x^2}{4(n+1)^2} \to 0 \quad \text{as} \quad n \to \infty.$$

The series converges for all x, so the domain of $J(x)$ is all real numbers.

(b) Since

$$J(x) = 1 - \frac{x^2}{4} + \cdots,$$

we have $J(0) = 1$.

(c) We have

$$S_0(x) = 1$$
$$S_1(x) = 1 - \frac{x^2}{4}$$
$$S_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$
$$S_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$
$$S_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}.$$

(d) The value of $J(1)$ can be approximated using partial sums. Substituting $x = 1$ into the partial sum polynomials, we have

$$S_0(1) = 1$$
$$S_1(1) = 0.75$$
$$S_2(1) = 0.765625$$
$$S_3(1) = 0.765191$$
$$S_4(1) = 0.765198.$$

We estimate that $J(1) \approx 0.765$. Theorem 9.9 can be used to bound the error.

(e) We see from the series that $J(x)$ is an even function, so $J(-1) = J(1)$. Thus, $J(-1) \approx 0.765$.

35. (a) We have

$$f(x) = 1 + x + \frac{x^2}{2} + \cdots,$$

so

$$f(0) = 1 + 0 + 0 + \cdots = 1.$$

(b) To find the domain of f, we find the interval of convergence.

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|x^{n+1}/(n+1)!|}{|x^n/n!|} = \lim_{n\to\infty}\left(\frac{|x|^{n+1}n!}{|x|^n(n+1)!}\right) = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Thus the series converges for all x, so the domain of f is all real numbers.

(c) Differentiating term-by-term gives

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$
$$= 0 + 1 + 2\frac{x}{2!} + 3\frac{x^2}{3!} + 4\frac{x^3}{4!} + \cdots$$
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Thus, the series for f and f' are the same, so

$$f(x) = f'(x).$$

(d) We guess $f(x) = e^x$.

36. (a) Since only odd powers are involved in the series for $g(x)$,

$$g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

we see that $g(x)$ is odd. Substituting $x = 0$ gives $g(0) = 0$.

(b) Differentiating term-by-term gives

$$\begin{aligned} g'(x) &= 1 - 3\frac{x^2}{3!} + 5\frac{x^4}{5!} - 7\frac{x^6}{7!} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \end{aligned}$$

$$\begin{aligned} g''(x) &= 0 - 2\frac{x}{2!} + 4\frac{x^3}{4!} - 6\frac{x^5}{6!} + \cdots \\ &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots. \end{aligned}$$

So we see $g''(x) = -g(x)$.

(c) We guess $g(x) = \sin x$ since then $g'(x) = \cos x$ and $g''(x) = -\sin x = g(x)$. We check $g(0) = 0 = \sin 0$ and $g'(0) = 1 = \cos 0$.

37. (a) We have

$$\begin{aligned} (p(x))^2 &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^2 \\ &= 1 - 2\cdot\frac{x^2}{2} + \left(-\frac{x^2}{2!}\right)^2 + 2\frac{x^4}{4!} - 2\frac{x^6}{6!} - 2\frac{x^2}{2!}\cdot\frac{x^4}{4!}\cdots \\ &= 1 - x^2 + \left(\frac{1}{4} + \frac{1}{12}\right)x^4 - x^6\left(\frac{1}{3\cdot 5\cdot 4!} + \frac{1}{4!}\right)\cdots \\ &= 1 - x^2 + \frac{x^4}{3} - \frac{2}{45}x^6\cdots. \end{aligned}$$

$$\begin{aligned} (q(x))^2 &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^2 = x^2\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)^2 \\ &= x^2\left(1 - 2\frac{x^2}{3!} + \left(-\frac{x^2}{3!}\right)^2 + 2\frac{x^4}{5!}\cdots\right) \\ &= x^2\left(1 - \frac{x^2}{3} + x^4\left(\frac{1}{(3!)^2} + \frac{1}{5\cdot 4\cdot 3}\right)\cdots\right) \\ &= x^2\left(1 - \frac{x^2}{3} + \frac{2}{45}x^4\cdots\right) \\ &= x^2 - \frac{x^4}{3} + \frac{2}{45}x^6\cdots. \end{aligned}$$

Thus, up to terms in x^6, we have

$$(p(x))^2 + (q(x))^2 = 1.$$

(b) The result of part (a) suggests that $p(x)$ and $q(x)$ could be the sine and cosine. Since $p(x)$ is even and $q(x)$ is odd, we guess that $p(x) = \cos x$ and $q(x) = \sin x$.

Solutions for Chapter 9 Review

Exercises

1. As n increases, the term $4n$ is much larger than 3 and $7n$ is much larger than 5. Thus dividing the numerator and denominator by n and using the fact that $\lim_{n\to\infty} 1/n = 0$, we have

$$\lim_{n\to\infty} \frac{3+4n}{5+7n} = \lim_{n\to\infty} \frac{(3/n)+4}{(5/n)+7} = \frac{4}{7}.$$

Thus, the sequence converges to $4/7$.

2. We have:

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = 1.$$

The terms of the sequence do not approach 0, so the sequence diverges.

3. The first eight terms of the sequence are:

$$\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0.$$

The sequence then repeats this pattern, so it diverges.

4. Since $1/n$ approaches zero and $\ln n$ becomes arbitrarily large as $n \to \infty$, the sequence diverges.

5. If $b = 1$, then the sum is 6. If $b \neq 1$, we use the formula for the sum of a finite geometric series. This is a six-term geometric series ($n = 6$) with initial term $a = b^5$ and constant ratio $x = b$:

$$\text{Sum } = \frac{a(1-x^n)}{1-x} = \frac{b^5(1-b^6)}{1-b}.$$

6. This is a geometric series with $k-2$ terms in it, so $n = k-2$. The initial term is $a = (0.5)^3 = 0.125$ and the constant ratio is $x = 0.5$. Using the formula for the sum of a finite geometric series, we get

$$\text{Sum } = \frac{a(1-x^n)}{1-x} = \frac{0.125(1-(0.5)^{k-2})}{1-0.5} = 0.25(1-(0.5)^{k-2}).$$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n+5}{4^n} = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n + \sum_{n=0}^{\infty}\frac{5}{4^n},$ a sum of two geometric series.

$$\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = \frac{1}{1-\frac{3}{4}} = 4$$

$$\sum_{n=0}^{\infty}\frac{5}{4^n} = \frac{5}{1-\frac{1}{4}} = \frac{20}{3}$$

so $\displaystyle\sum_{n=0}^{\infty}\frac{3^n+5}{4^n} = 4 + \frac{20}{3} = \frac{32}{3}.$

8. We use the integral test to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\displaystyle\int_1^{\infty}\frac{1}{(x+2)^2}\,dx$ converges or diverges:

$$\begin{aligned}
\int_1^{\infty}\frac{1}{(x+2)^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{(x+2)^2}\,dx \\
&= \lim_{b\to\infty}\int_3^b \frac{1}{w^2}\,dw \qquad \text{(Substitute } w = x+2\text{)} \\
&= \lim_{b\to\infty} -\frac{1}{w}\Big|_3^b \\
&= \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{3}\right) = \frac{1}{3}.
\end{aligned}$$

Since the integral $\displaystyle\int_1^{\infty}\frac{1}{(x+2)^2}\,dx$ converges, we conclude from the integral test that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+2)^2}$ converges.

9. We use the integral test to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\int_1^\infty \frac{3x^2+2x}{x^3+x^2+1}dx$ converges or diverges. The integral can be calculated using the substitution $w = x^3+x^2+1$, $dw = (3x^2+2x)\,dx$.

$$\begin{aligned}\int_1^\infty \frac{3x^2+2x}{x^3+x^2+1}dx &= \lim_{b\to\infty}\int_1^b \frac{3x^2+2x}{x^3+x^2+1}dx \\ &= \lim_{b\to\infty} \ln|x^3+x^2+1|\Big|_1^b \\ &= \lim_{b\to\infty}\left(\ln|b^3+b^2+1| - \ln 3\right) = \infty.\end{aligned}$$

Since the integral $\int_1^\infty \frac{3x^2+2x}{x^3+x^2+1}dx$ diverges, we conclude from the integral test that the series $\sum_{n=1}^{\infty}\frac{3n^2+2n}{n^3+n^2+1}$ diverges.

10. We use the integral test to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\int_0^\infty xe^{-x^2}dx$ converges or diverges:

$$\int_0^\infty xe^{-x^2}dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}dx = \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^b = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2}+\frac{1}{2}\right) = \frac{1}{2}.$$

Since the integral $\int_0^\infty xe^{-x^2}dx$ converges, we conclude from the integral test that the series $\sum_{n=0}^{\infty} ne^{-n^2}$ converges.

11. We use the integral test to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\int_2^\infty \frac{2}{x^2-1}\,dx$ converges or diverges:

$$\begin{aligned}\int_2^\infty \frac{2}{x^2-1}\,dx &= \lim_{b\to\infty}\int_2^b \frac{2}{x^2-1}\,dx \\ &= \lim_{b\to\infty}\left(\int_2^b\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx\right) \qquad \text{(Using partial fractions)} \\ &= \lim_{b\to\infty}\left(\ln|x-1| - \ln|x+1|\Big|_2^b\right) \\ &= \lim_{b\to\infty}\left(\ln\left|\frac{x-1}{x+1}\right|\Big|_2^b\right) \\ &= \lim_{b\to\infty}\left(\ln\left|\frac{b-1}{b+1}\right| - \ln\left(\frac{1}{3}\right)\right) = \ln 1 - \ln\frac{1}{3} = \ln 3.\end{aligned}$$

Since the integral $\int_2^\infty \frac{2}{x^2-1}\,dx$ converges, we conclude that the series $\sum_{n=2}^{\infty}\frac{2}{n^2-1}$ converges.

12. Let $a_n = n^2/(3n^2+4)$. Since $3n^2+4 > 3n^2$, we have $\frac{n^2}{3n^2+4} < \frac{1}{3}$, so

$$0 < a_n < \left(\frac{1}{3}\right)^n.$$

The geometric series $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ converges, so the comparison test tells us that the series $\sum_{n=1}^{\infty}\left(\frac{n^2}{3n^2+4}\right)^n$ also converges.

13. Let $a_n = 1/(n\sin^2 n)$. Since $0 < \sin^2 n < 1$, for any positive integer n, we have $n\sin^2 n < n$, so $\dfrac{1}{n\sin^2 n} > \dfrac{1}{n}$, thus

$$a_n > \frac{1}{n}.$$

The harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so the comparison test tells us that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sin^2 n}$ also diverges.

14. The n^{th} term $a_n = \sqrt{n-1}/(n^2+3)$ behaves like $\sqrt{n}/n^2 = 1/n^{3/2}$ for large n, so we take $b_n = 1/n^{3/2}$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{n-1}/(n^2+3)}{1/n^{3/2}} = \lim_{n\to\infty}\frac{n^{3/2}\sqrt{n-1}}{n^2+3} = \lim_{n\to\infty}\frac{n^2\sqrt{1-1/n}}{n^2(1+3/n^2)} = 1.$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^{3/2}$ converges because $p = 3/2 > 1$. Therefore $\sum\sqrt{n-1}/(n^2+3)$ also converges.

15. The n^{th} term $a_n = (n^3-2n^2+n+1)/(n^5-2)$ behaves like $n^3/n^5 = 1/n^2$ for large n, so we take $b_n = 1/n^2$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(n^3-2n^2+n+1)/(n^5-2)}{1/n^2} = \lim_{n\to\infty}\frac{n^5-2n^4+n^3+n^2}{n^5-2} = 1.$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^2$ converges because $p = 2 > 1$. Therefore the series $\sum\left(n^3-2n^2+n+1\right)/\left(n^5-2\right)$ also converges.

16. The n^{th} term is $a_n = \sin(1/n^2)$. When n is large, $1/n^2$ is near zero, so $\sin(1/n^2)$ is near $1/n^2$. We see that $\sin(1/n^2)$ behaves like $1/n^2$ for large n, so we take $b_n = 1/n^2$. We have

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\sin(1/n^2)}{1/n^2}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\\ &= 1.\end{aligned}$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^2$ converges because $p = 2 > 1$. Therefore $\sum\sin(1/n^2)$ also converges.

17. The n^{th} term $a_n = 1/(\sqrt{n^3-1})$ behaves like $1/\sqrt{n^3} = 1/n^{3/2}$ for large n, so we take $b_n = 1/n^{3/2}$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/\sqrt{n^3-1}}{1/n^{3/2}} = \lim_{n\to\infty}\frac{n^{3/2}}{\sqrt{n^3-1}} = \lim_{n\to\infty}\frac{n^{3/2}}{n^{3/2}\sqrt{1-1/n^3}} = \lim_{n\to\infty}\frac{1}{\sqrt{1-1/n^3}} = \frac{1}{\sqrt{1-0}} = 1.$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^{3/2}$ converges because $p = 3/2 > 1$. Therefore $\sum 1/\sqrt{n^3-1}$ also converges.

18. Since $a_n = 1/(2^n n!)$, replacing n by $n+1$ gives $a_{n+1} = 1/(2^{n+1}(n+1)!)$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{1}{2^{n+1}(n+1)!}}{\dfrac{1}{2^n n!}} = \frac{2^n n!}{2^{n+1}(n+1)!} = \frac{1}{2(n+1)},$$

so

$$L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{1}{2n+2} = 0.$$

Since $L < 1$, the ratio test tells us that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n n!}$ converges.

19. Since $a_n = n!(n+1)!/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = (n+1)!(n+2)!/(2n+2)!$. Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\dfrac{(n+1)!(n+2)!}{(2n+2)!}}{\dfrac{n!(n+1)!}{(2n)!}} = \frac{(n+1)!(n+2)!}{(2n+2)!}\cdot\frac{(2n)!}{n!(n+1)!}.$$

However, since $(n+2)! = (n+2)(n+1)n!$ and $(2n+2)! = (2n+2)(2n+1)(2n)!$, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+2)(n+1)}{(2n+2)(2n+1)} = \frac{n+2}{2(2n+1)},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{4}.$$

Since $L < 1$, the ratio test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(2n)!}$ converges.

20. Let $a_n = 1/(n^2+1)$. Then replacing n by $n+1$ gives $a_{n+1} = 1/((n+1)^2+1)$. Since $(n+1)^2+1 > n^2+1$, we have

$$0 < \frac{1}{(n+1)^2+1} < \frac{1}{n^2+1},$$

so

$$0 < a_{n+1} < a_n.$$

We also have $\lim_{n\to\infty} a_n = 0$, therefore, the alternating series test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ converges.

21. Let $a_n = 1/\sqrt{n^2+1}$. Then replacing n by $n+1$ we have $a_{n+1} = 1/\sqrt{(n+1)^2+1}$. Since $\sqrt{(n+1)^2+1} > \sqrt{n^2+1}$, we have

$$\frac{1}{\sqrt{(n+1)^2+1}} < \frac{1}{\sqrt{n^2+1}},$$

so

$$0 < a_{n+1} < a_n.$$

In addition, $\lim_{n\to\infty} a_n = 0$ so $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ converges by the alternating series test.

22. Since $f(x) = 1/(x+1)$ is continuous, positive and decreasing, we apply the integral test, and we obtain

$$\int_1^{\infty} \frac{1}{x+1} dx = \lim_{b\to\infty} \int_1^b \frac{1}{1+x}\, dx = \lim_{b\to\infty} (\ln(b+1) - \ln 2) = \infty.$$

Since this improper integral diverges, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$ also diverges. We can also observe the series is the harmonic series, with the first term missing, and hence diverges by Property 2 of Theorem 9.2.

23. This is a p-series with $p > 1$, so it converges.

24. We use the integral test to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\displaystyle\int_3^{\infty} \frac{2}{\sqrt{x-2}}\, dx$ converges or diverges:

$$\begin{aligned}
\int_3^{\infty} \frac{2}{\sqrt{x-2}}\, dx &= \lim_{b\to\infty} \int_3^b \frac{2}{\sqrt{x-2}}\, dx \\
&= \lim_{b\to\infty} \int_1^b \frac{2}{\sqrt{w}}\, dw \qquad \text{(Substitute } w = x-2.\text{)} \\
&= \lim_{b\to\infty} 4\sqrt{w}\Big|_1^b = \infty.
\end{aligned}$$

Since the limit does not exist, the integral $\displaystyle\int_3^{\infty} \frac{2}{\sqrt{x-2}}\, dx$ diverges, and we conclude from the integral test that the series $\displaystyle\sum_{n=3}^{\infty} \frac{2}{\sqrt{n-2}}$ diverges. The limit comparison test with $b_n = 1/\sqrt{n}$ can also be used.

25. This is an alternating series. Let $a_n = 1/(\sqrt{n}+1)$. Then $\lim_{n\to\infty} a_n = 0$. Now replace n by $n+1$ to give $a_{n+1} = 1/(\sqrt{n+1}+1)$. Since $\sqrt{n+1}+1 > \sqrt{n}+1$, we have $\dfrac{1}{\sqrt{n+1}+1} < \dfrac{1}{\sqrt{n}+1}$, so

$$0 < a_{n+1} = \frac{1}{\sqrt{n+1}+1} < \frac{1}{\sqrt{n}+1} = a_n.$$

Therefore, the alternating series test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}+1}$ converges.

26. Writing $a_n = n^2/(n^2+1)$, we have $\lim_{n\to\infty} a_n = 1$ so the series diverges by Property 3 of Theorem 9.2.

27. We use the integral test to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\displaystyle\int_1^{\infty} \frac{x^2}{x^3+1} dx$ converges or diverges:

$$\int_1^{\infty} \frac{x^2}{x^3+1}\, dx = \lim_{b\to\infty} \int_1^{b} \frac{x^2}{x^3+1}\, dx = \lim_{b\to\infty} \frac{1}{3} \ln|x^3+1| \Big|_1^b = \lim_{b\to\infty} \left(\frac{1}{3}\ln(b^3+1) - \frac{1}{3}\ln 2\right).$$

Since the limit does not exist, the integral $\displaystyle\int_1^{\infty} \frac{x^2}{x^3+1} dx$ diverges an so we conclude from the integral test that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ diverges. The limit comparison test with $b_n = 1/n$ can also be used.

28. We use the ratio test. Since $a_n = 3^n/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = 3^{n+1}/(2n+2)!$. Thus

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}/(2n+2)!}{3^n/(2n)!} = \frac{3^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{3^n}.$$

Since $(2n+2)! = (2n+2)(2n+1)(2n)!$, we have

$$\frac{a_{n+1}}{a_n} = \frac{3}{(2n+2)(2n+1)},$$

so

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0.$$

Since $L < 1$, the ratio test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$ converges.

29. We use the ratio test. Since $a_n = (2n)!/(n!)^2$, replacing n by $n+1$ gives $a_{n+1} = (2n+2)!/((n+1)!)^2$. Thus

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(2n+2)!}{((n+1)!)^2}}{\dfrac{(2n)!}{(n!)^2}} = \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{(2n)!}.$$

Since $(2n+2)! = (2n+2)(2n+1)(2n)!$ and $(n+1)! = (n+1)n!$, we have

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)},$$

therefore

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4.$$

As $L > 1$ the ratio test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ diverges.

30. The series can be written as

$$\sum_{n=1}^{\infty} \frac{n^2+2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as a p-series with $p > 1$, we see $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2^n}{n^2 2^n}$ converges by Theorem 9.2.

31. Let $a_n = 2^{-n}\dfrac{(n+1)}{(n+2)} = \left(\dfrac{n+1}{n+2}\right)\left(\dfrac{1}{2^n}\right)$. Since $\dfrac{(n+1)}{(n+2)} < 1$ and $\dfrac{1}{2^n} = \left(\dfrac{1}{2}\right)^n$, we have

$$0 < a_n < \left(\frac{1}{2}\right)^n,$$

so that we can compare the series $\sum_{n=1}^{\infty} 2^{-n}\dfrac{(n+1)}{(n+2)}$ with the convergent geometric series $\sum_{n=1}^{\infty}\left(\dfrac{1}{2}\right)^n$. The comparison test tells us that

$$\sum_{n=1}^{\infty} 2^{-n}\frac{(n+1)}{(n+2)}$$

also converges.

32. We have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(2n+3)!}\cdot\frac{(2n+1)!}{2^n} = \lim_{n\to\infty}\frac{2}{(2n+3)(2n+2)} = 0,$$

so the series converges by the ratio test, since $L < 1$.

33. Since there is an n in the numerator and a $\sqrt{n}$ in the denominator, the terms in this series are increasing in magnitude. We have

$$\lim_{n\to\infty}\left|\frac{n+1}{\sqrt{n}}(-1)^n\right| = \lim_{n\to\infty}\frac{n+1}{\sqrt{n}} = \infty,$$

so $\lim_{n\to\infty}(-1)^n(n+1)/\sqrt{n}$ does not approach zero. Therefore, the series diverges by Property 3 of Theorem 9.2.

34. The series can be written as

$$\sum_{n=0}^{\infty}\frac{2+3^n}{5^n} = \sum_{n=0}^{\infty}\left(\frac{2}{5^n}+\frac{3^n}{5^n}\right) = \sum_{n=0}^{\infty}\left(2\left(\frac{1}{5}\right)^n + \left(\frac{3}{5}\right)^n\right).$$

The series $\sum_{n=0}^{\infty}\left(\dfrac{1}{5}\right)^n$ is a geometric series which converges because $|\frac{1}{5}| < 1$. Likewise, the geometric series $\sum_{n=0}^{\infty}\left(\dfrac{3}{5}\right)^n$ converges because $|\frac{3}{5}| < 1$. Since both series converge, Property 1 of Theorem 9.2 tells us that the series $\sum_{n=0}^{\infty}\dfrac{2+3^n}{5^n}$ also converges.

35. Writing $a_n = 1/(2+\sin n)$, we have $\lim_{n\to\infty} a_n$ does not exist, so the series diverges by Property 3 of Theorem 9.2.

36. We use the integral test to determine whether this series converges or diverges. To do so we determine whether the corresponding improper integral $\displaystyle\int_3^{\infty}\frac{1}{(2x-5)^3}\,dx$ converges or diverges:

$$\begin{aligned}\int_3^{\infty}\frac{1}{(2x-5)^3}\,dx &= \frac{1}{2}\lim_{b\to\infty}\int_1^b\frac{1}{w^3}\,dw \qquad \text{(Substitute } w = 2x-5\text{)}\\ &= -\frac{1}{2}\lim_{b\to\infty}\left.\frac{1}{2w^2}\right|_1^b\\ &= -\frac{1}{2}\lim_{b\to\infty}\left(\frac{1}{2b^2}-\frac{1}{2}\right) = \frac{1}{4}.\end{aligned}$$

Since the integral $\displaystyle\int_3^{\infty}\frac{1}{(2x-5)^3}\,dx$ converges, we conclude from the integral test that the series $\displaystyle\sum_{n=3}^{\infty}\frac{1}{(2n-5)^3}$ converges. The limit comparison test, with $b_n = 1/n^3$ can also be used.

37. The n^{th} term $a_n = 1/(n^3-3)$ behaves like $1/n^3$ for large n, so we take $b_n = 1/n^3$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/(n^3-3)}{1/n^3} = \lim_{n\to\infty}\frac{n^3}{n^3-3} = 1.$$

The limit comparison test applies with $c = 1$. The p-series $\sum 1/n^3$ converges because $p = 3 > 1$. Therefore $\sum 1/(n^3-$ 3 also converges.

38. Note that
$$\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^3}=1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots$$
is an alternating series with the absolute values of the terms decreasing to 0. Thus, the series converges by the alternating series test.

39. Since $\ln(1+1/k)=\ln((k+1)/k)=\ln(k+1)-\ln k$, the n^{th} partial sum of this series is
$$\begin{aligned}S_n&=\sum_{k=1}^{n}\ln\left(1+\frac{1}{k}\right)\\&=\sum_{k=1}^{n}\ln(k+1)-\sum_{k=1}^{n}\ln k\\&=(\ln 2+\ln 3+\cdots+\ln(n+1))-(\ln 1+\ln 2+\cdots+\ln n)\\&=\ln(n+1)-\ln 1\\&=\ln(n+1).\end{aligned}$$

Thus, the partial sums, S_n, grow without bound as $n\to\infty$, so the series diverges by the definition.

40. The ratio test gives
$$L=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)/2^{n+1}}{n/2^n}=\lim_{n\to\infty}\frac{n+1}{2n}=\frac{1}{2},$$
so the series converges since $L<1$.

41. Since $\ln n$ grows much more slowly than n, we suspect that $(\ln n)^2<n$ for large n. This can be confirmed with L'Hopital's rule.
$$\lim_{n\to\infty}\frac{(\ln n)^2}{n}=\lim_{n\to\infty}\frac{2(\ln n)/n}{1}=\lim_{n\to\infty}\frac{2(\ln n)}{n}=0.$$
Therefore, for large n, we have $(\ln n)^2/n<1$, and hence for large n,
$$\frac{1}{n}<\frac{1}{(\ln n)^2}.$$
Thus $\sum_{n=2}^{\infty}1/(\ln n)^2$ diverges by comparison with the divergent harmonic series $\sum 1/n$.

42. Since $C_n=n$, replacing n by $n+1$ gives $C_{n+1}=n+1$. Using the ratio test with $a_n=nx^n$, we have
$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=|x|\lim_{n\to\infty}\frac{|C_{n+1}|}{|C_n|}=|x|\lim_{n\to\infty}\frac{n+1}{n}=|x|.$$
Thus the radius of convergence is $R=1$.

43. Let $C_n=\dfrac{(2n)!}{(n!)^2}$. Then replacing n by $n+1$, we have $C_{n+1}=\dfrac{(2n+2)!}{((n+1)!)^2}$. Thus, with $a_n=(2n)!x^n/(n!)^2$, we have
$$\frac{|a_{n+1}|}{|a_n|}=|x|\frac{|C_{n+1}|}{|C_n|}=|x|\frac{(2n+2)!/((n+1)!)^2}{(2n)!/(n!)^2}=|x|\frac{(2n+2)!}{(2n)!}\cdot\frac{(n!)^2}{((n+1)!)^2}.$$
Since $(2n+2)!=(2n+2)(2n+1)(2n)!$ and $(n+1)!=(n+1)n!$ we have
$$\frac{|C_{n+1}|}{|C_n|}=\frac{(2n+2)(2n+1)}{(n+1)(n+1)},$$
so
$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=|x|\lim_{n\to\infty}\frac{|C_{n+1}|}{|C_n|}=|x|\lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)}=|x|\lim_{n\to\infty}\frac{4n+2}{n+1}=4|x|,$$
so the radius of convergence of this series is $R=1/4$.

44. Let $C_n = 2^n + n^2$. Then replacing n by $n+1$ gives $C_{n+1} = 2^{n+1} + (n+1)^2$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{2^{n+1} + (n+1)^2}{2^n + n^2} = 2|x|\left(\frac{2^n + \frac{1}{2}(n+1)^2}{2^n + n^2}\right).$$

Since 2^n dominates n^2 as $n \to \infty$, we have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 2|x|.$$

Thus the radius of convergence is $R = \frac{1}{2}$.

45. Let $C_n = 1/(n!+1)$. Then replacing n by $n+1$ gives $C_{n+1} = 1/((n+1)!+1)$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{1/((n+1)!+1)}{1/(n!+1)} = |x|\frac{n!+1}{(n+1)!+1}.$$

Since $n!$ and $(n+1)!$ dominate the constant term 1 as $n \to \infty$ and $(n+1)! = (n+1)\cdot n!$ we have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 0.$$

Thus the radius of convergence is $R = \infty$.

Problems

46. The series $\sum \frac{(-1)^n}{n^{1/2}}$ converges by the alternating series test. However $\sum \frac{1}{n^{1/2}}$ diverges because it is a p-series with $p = 1/2 \le 1$. Thus $\sum \frac{(-1)^n}{n^{1/2}}$ is conditionally convergent.

47. Since

$$\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$$

the series $\sum (-1)^n \frac{n}{n+1}$ does not converge. It is a divergent series.

48. The series can be written as

$$\sum_{n=1}^{\infty}\frac{n^r + r^n}{n^r r^n} = \sum_{n=1}^{\infty}\frac{1}{r^n} + \sum_{n=1}^{\infty}\frac{1}{n^r}.$$

If $0 < r < 1$, both series diverge, but if $r > 1$ both series converge.

If $r = 1$ the given series becomes $\sum_{n=1}^{\infty}\frac{n+1}{n}$ which diverges.

By Theorem 9.2 the given series converges if $r > 1$.

49. We use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{3^{n+1}(n+1)^2}\cdot\frac{3^n n^2}{x^n}\right| = \left(\frac{n}{n+1}\right)^2\cdot\frac{|x|}{3}.$$

Since $n/(n+1) \to 1$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{3}.$$

We have $|x|/3 < 1$ when $|x| < 3$. The radius of convergence is 3 and the series converges for $-3 < x < 3$.

We check the endpoints. For $x = -3$, we have

$$\sum_{n=1}^{\infty}\frac{x^n}{3^n n^2} = \sum_{n=1}^{\infty}\frac{(-3)^n}{3^n n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

We know $\sum \frac{1}{n^2}$ is a p-series with $p = 2$ so it converges. Therefore the alternating series $\sum \frac{(-1)^n}{n^2}$ also converges. For $x = 3$, we have

$$\sum_{n=1}^{\infty}\frac{x^n}{3^n n^2} = \sum_{n=1}^{\infty}\frac{3^n}{3^n n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

This is a p-series with $p = 2$ and it converges. The series converges at both its endpoints and the interval of convergence is $-3 \le x \le 3$.

50. We use the ratio test:
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(x-2)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{(-1)^n(x-2)^n}\right| = \frac{|x-2|}{5}.$$
Since $|x-2|/5 < 1$ when $|x-2| < 5$, the radius of convergence is 5 and the series converges for $-3 < x < 7$.

We check the endpoints:
$$x = -3: \quad \sum_{n=0}^{\infty} \frac{(-1)^n(x-2)^n}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n(-3-2)^n}{5^n} = \sum_{n=0}^{\infty} 1 \quad \text{which diverges.}$$
$$x = 7: \quad \sum_{n=0}^{\infty} \frac{(-1)^n(x-2)^n}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n(7-2)^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n \quad \text{which diverges.}$$
The series diverges at both the endpoints, so the interval of convergence is $-3 < x < 7$.

51. We use the ratio test:
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}x^{n+1}}{n+1} \cdot \frac{n}{(-1)^n x^n}\right| = \frac{n}{n+1} \cdot |x|.$$
Since $n/(n+1) \to 1$ as $n \to \infty$, we have
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x|.$$
The series converges for $|x| < 1$. The radius of convergence is 1 and the series converges for $-1 < x < 1$.

We check the endpoints. For $x = -1$, we have
$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}.$$
This is the harmonic series and diverges. For $x = 1$, we have
$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n(1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$
This is the alternating harmonic series and converges. The series diverges at $x = -1$ and converges at $x = 1$. Therefore, interval of convergence is $-1 < x \le 1$.

52. The series converges for $|x-2| = 2$ and diverges for $|x-2| = 4$, thus the radius of convergence of the series, R, is at least 2 but no larger than 4.

(a) False. If $x = 7$ then $|x-2| = 5$, so the series diverges.
(b) False. If $x = 1$ then $|x-2| = 1$, so the series converges.
(c) True. If $x = 0.5$ then $|x-2| = 1.5$, so the series converges.
(d) If $x = 5$ then $|x-2| = 3$ and it is not possible to determine whether or not the series converges at this point.
(e) False. If $x = -3$ then $|x-2| = 5$, so the series diverges.

53. (a) Using an argument similar to Example 5 in Section 9.5, we take
$$a_n = (-1)^n \frac{t^{2n}}{(2n)!},$$
so, replacing n by $n+1$,
$$a_{n+1} = (-1)^{n+1} \frac{t^{2(n+1)}}{(2(n+1))!} = (-1)^{n+1} \frac{t^{2n+2}}{(2n+2)!}.$$
Thus,
$$\frac{|a_{n+1}|}{|a_n|} = \frac{|(-1)^{n+1}t^{2n+2}/(2n+2)!|}{|(-1)^n t^{2n}/(2n)!|} = \frac{t^2}{(2n+2)(2n+1)},$$
so
$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{t^2}{(2n+2)(2n+1)} = 0.$$
The radius of convergence is therefore ∞, so the series converges for all t. Therefore, the domain of h is all real numbers.

(b) Since h involves only even powers,

$$h(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots,$$

h is an even function.

(c) Differentiating term-by-term, we have

$$\begin{aligned} h'(t) &= 0 - 2\frac{t}{2!} + 4\frac{t^3}{4!} - 6\frac{t^6}{6!} + \cdots \\ &= -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \cdots. \end{aligned}$$

$$\begin{aligned} h''(t) &= -1 + 3\frac{t^2}{3!} - 5\frac{t^4}{5!} + \cdots \\ &= -1 + \frac{t^2}{2!} - \frac{t^4}{4!} + \cdots. \end{aligned}$$

So we see $h''(t) = -h(t)$.

54. (a) It is easier to work with the value of the car first and then find the yearly losses. The value of the car goes down by 10% a year. Thus, the value at the end of the first years is $v_1 = 30{,}000(0.9)$. The value at the end of the second year is $v_2 = 30{,}000(0.9)^2$. The value at the end of n years is $v_n = 30{,}000(0.9)^n$. Thus, the losses in the first four years are

$$\begin{aligned} l_1 &= 30{,}000(0.1) \\ l_2 &= v_1 - v_2 = 30{,}000(0.9) - 30{,}000(0.9)^2 = 30{,}000(0.9)(0.1) \\ l_3 &= v_2 - v_3 = 30{,}000(0.9)^2 - 30{,}000(0.9)^3 = 30{,}000(0.9)^2(0.1) \\ l_4 &= v_3 - v_4 = 30{,}000(0.9)^3 - 30{,}000(0.9)^4 = 30{,}000(0.9)^3(0.1). \end{aligned}$$

Thus,

$$l_n = v_{n-1} - v_n = 30{,}000(0.9)^{n-1}(0.1) = 3000(0.9)^{n-1}.$$

(b) In the first year, $m_1 = 500$; in the second year, $m_2 = 500(1.2)$; in the third year, $m_3 = 500(1.2)^2$. Thus

$$m_n = 500(1.2)^{n-1}.$$

(c) We want to find n such that $m_n \geq l_n$, so

$$500(1.2)^{n-1} \geq 3000(0.9)^{n-1}.$$

We solve

$$\begin{aligned} 500(1.2)^{n-1} &= 3000(0.9)^{n-1} \\ \frac{(1.2)^{n-1}}{(0.9)^{n-1}} &= \frac{3000}{500} \\ \left(\frac{1.2}{0.9}\right)^{n-1} &= 6 \\ (n-1)\ln\left(\frac{1.2}{0.9}\right) &= \ln 6 \\ n - 1 &= \frac{\ln 6}{\ln(1.2/0.9)} \\ n &= 6.228 + 1 = 7.228. \end{aligned}$$

So, maintenance first exceeds losses in year 8. In year 7,

$$l_7 = 3000(0.9)^6 = \$1594, \quad m_7 = 500(1.2)^6 = \$1493.$$

In year 8,

$$l_8 = 3000(0.9)^7 = \$1435, \quad m_8 = 500(1.2)^7 = \$1792.$$

55.

$$\begin{aligned}\text{Present value of first coupon} &= \frac{50}{1.06}\\ \text{Present value of second coupon} &= \frac{50}{(1.06)^2},\ \text{etc.}\end{aligned}$$

$$\begin{aligned}\text{Total present value} &= \underbrace{\frac{50}{1.06}+\frac{50}{(1.06)^2}+\cdots+\frac{50}{(1.06)^{10}}}_{\text{coupons}}+\underbrace{\frac{1000}{(1.06)^{10}}}_{\text{principal}}\\ &= \frac{50}{1.06}\left(1+\frac{1}{1.06}+\cdots+\frac{1}{(1.06)^9}\right)+\frac{1000}{(1.06)^{10}}\\ &= \frac{50}{1.06}\left(\frac{1-\left(\frac{1}{1.06}\right)^{10}}{1-\frac{1}{1.06}}\right)+\frac{1000}{(1.06)^{10}}\\ &= 368.004+558.395\\ &= \$926.40\end{aligned}$$

56.

$$\begin{aligned}\text{Present value of first coupon} &= \frac{50}{1.04}\\ \text{Present value of second coupon} &= \frac{50}{(1.04)^2},\ \text{etc.}\end{aligned}$$

$$\begin{aligned}\text{Total present value} &= \underbrace{\frac{50}{1.04}+\frac{50}{(1.04)^2}+\cdots+\frac{50}{(1.04)^{10}}}_{\text{coupons}}+\underbrace{\frac{1000}{(1.04)^{10}}}_{\text{principal}}\\ &= \frac{50}{1.04}\left(1+\frac{1}{1.04}+\cdots+\frac{1}{(1.04)^9}\right)+\frac{1000}{(1.04)^{10}}\\ &= \frac{50}{1.04}\left(\frac{1-\left(\frac{1}{1.04}\right)^{10}}{1-\frac{1}{1.04}}\right)+\frac{1000}{(1.04)^{10}}\\ &= 405.545+675.564\\ &= \$1081.11\end{aligned}$$

57. (a)

$$\begin{aligned}\text{Present value of first coupon} &= \frac{50}{1.05}\\ \text{Present value of second coupon} &= \frac{50}{(1.05)^2},\ \text{etc.}\end{aligned}$$

$$\begin{aligned}\text{Total present value} &= \underbrace{\frac{50}{1.05}+\frac{50}{(1.05)^2}+\cdots+\frac{50}{(1.05)^{10}}}_{\text{coupons}}+\underbrace{\frac{1000}{(1.05)^{10}}}_{\text{principal}}\\ &= \frac{50}{1.05}\left(1+\frac{1}{1.05}+\cdots+\frac{1}{(1.05)^9}\right)+\frac{1000}{(1.05)^{10}}\\ &= \frac{50}{1.05}\left(\frac{1-\left(\frac{1}{1.05}\right)^{10}}{1-\frac{1}{1.05}}\right)+\frac{1000}{(1.05)^{10}}\\ &= 386.087+613.913\\ &= \$1000\end{aligned}$$

(b) When the interest rate is 5%, the present value equals the principal.

(c) When the interest rate is more than 5%, the present value is smaller than it is when interest is 5% and must therefore be less than the principal. Since the bond will sell for around its present value, it will sell for less than the principal; hence the description *trading at discount*.

(d) When the interest rate is less than 5%, the present value is more than the principal. Hence the bond will be selling for more than the principal, and is described as *trading at a premium*.

58. The amount of cephalexin in the body is given by $Q(t) = Q_0 e^{-kt}$, where $Q_0 = Q(0)$ and k is a constant. Since the half-life is 0.9 hours,

$$\frac{1}{2} = e^{-0.9k}, \quad k = -\frac{1}{0.9}\ln\frac{1}{2} \approx 0.8.$$

(a) After 6 hours

$$Q = Q_0 e^{-k(6)} \approx Q_0 e^{-0.8(6)} = Q_0(0.01).$$

Thus, the percentage of the cephalexin that remains after 6 hours $\approx 1\%$.

(b)

$$\begin{aligned} Q_1 &= 250 \\ Q_2 &= 250 + 250(0.01) \\ Q_3 &= 250 + 250(0.01) + 250(0.01)^2 \\ Q_4 &= 250 + 250(0.01) + 250(0.01)^2 + 250(0.01)^3 \end{aligned}$$

(c)

$$\begin{aligned} Q_3 &= \frac{250(1-(0.01)^3)}{1-0.01} \\ &\approx 252.5 \\ Q_4 &= \frac{250(1-(0.01)^4)}{1-0.01} \\ &\approx 252.5 \end{aligned}$$

Thus, by the time a patient has taken three cephalexin tablets, the quantity of drug in the body has leveled off to 252.5 mg.

(d) Looking at the answers to part (b) shows that

$$\begin{aligned} Q_n &= 250 + 250(0.01) + 250(0.01)^2 + \cdots + 250(0.01)^{n-1} \\ &= \frac{250(1-(0.01)^n)}{1-0.01}. \end{aligned}$$

(e) In the long run, $n \to \infty$. So,

$$Q = \lim_{n\to\infty} Q_n = \frac{250}{1-0.01} = 252.5.$$

59. (a) (i) On the night of December 31, 1999:

First deposit will have grown to $2(1.04)^7$ million dollars.
Second deposit will have grown to $2(1.04)^6$ million dollars.
. . .
Most recent deposit (Jan.1, 1999) will have grown to $2(1.04)$ million dollars.

Thus

$$\begin{aligned} \text{Total amount} &= 2(1.04)^7 + 2(1.04)^6 + \cdots + 2(1.04) \\ &= 2(1.04)\underbrace{(1 + 1.04 + \cdots + (1.04)^6)}_{\text{finite geometric series}} \\ &= 2(1.04)\left(\frac{1-(1.04)^7}{1-1.04}\right) \\ &= 16.43 \text{ million dollars.} \end{aligned}$$

(ii) Notice that if 10 payments are made, there are 9 years between the first and the last. On the day of the last payment:

First deposit will have grown to $2(1.04)^9$ million dollars.
Second deposit will have grown to $2(1.04)^8$ million dollars.
...
Last deposit will be 2 million dollars.

Therefore

$$\begin{aligned}\text{Total amount} &= 2(1.04)^9 + 2(1.04)^8 + \cdots + 2\\ &= 2(\underbrace{1 + 1.04 + (1.04)^2 + \cdots + (1.04)^9}_{\text{finite geometric series}})\\ &= 2\left(\frac{1-(1.04)^{10}}{1-1.04}\right)\\ &= 24.01 \text{ million dollars.}\end{aligned}$$

(b) In part (a) (ii) we found the future value of the contract 9 years in the future. Thus

$$\text{Present Value} = \frac{24.01}{(1.04)^9} = 16.87 \text{ million dollars.}$$

Alternatively, we can calculate the present value of each of the payments separately:

$$\begin{aligned}\text{Present Value} &= 2 + \frac{2}{1.04} + \frac{2}{(1.04)^2} + \cdots + \frac{2}{(1.04)^9}\\ &= 2\left(\frac{1-(1/1.04)^{10}}{1-1/1.04}\right) = 16.87 \text{ million dollars.}\end{aligned}$$

Notice that the present value of the contract (\$16.87 million) is considerably less than the face value of the contract, \$20 million.

60. A person should expect to pay the present value of the bond on the day it is bought.

$$\begin{aligned}\text{Present value of first payment} &= \frac{10}{1.04}\\ \text{Present value of second payment} &= \frac{10}{(1.04)^2}, \text{ etc.}\end{aligned}$$

Therefore,

$$\text{Total present value} = \frac{10}{1.04} + \frac{10}{(1.04)^2} + \frac{10}{(1.04)^3} + \cdots.$$

This is a geometric series with $a = \frac{10}{1.04}$ and $x = \frac{1}{1.04}$, so

$$\text{Total present value} = \frac{\frac{10}{1.04}}{1-\frac{1}{1.04}} = £250.$$

61. (a)

$$\begin{aligned}\text{Total amount of money deposited} &= 100 + 92 + 84.64 + \cdots\\ &= 100 + 100(0.92) + 100(0.92)^2 + \cdots\\ &= \frac{100}{1-0.92} = 1250 \quad \text{dollars}\end{aligned}$$

(b) Credit multiplier $= 1250/100 = 12.50$
The 12.50 is the factor by which the bank has increased its deposits, from \$100 to \$1250.

62. If the half-life is T hours, then the exponential decay formula $Q = Q_0 e^{-kt}$ gives $k = \ln 2/T$. If we start with $Q_0 = 1$ tablet, then the amount of drug present in the body after $5T$ hours is

$$Q = e^{-5kT} = e^{-5\ln 2} = 0.03125,$$

so 3.125% of a tablet remains. Thus, immediately after taking the first tablet, there is one tablet in the body. Five half-lives later, this has reduced to $1 \cdot 0.03125 = 0.03125$ tablets, and immediately after the second tablet there are $1 + 0.03125$ tablets in the body. Continuing this forever leads to

$$\text{Number of tablets in body } = 1 + 0.03125 + (0.03125)^2 + \cdots + (0.03125)^n + \cdots.$$

This is an infinite geometric series, with common ratio $x = 0.03125$, and sum $1/(1-x)$. Thus

$$\text{Number of tablets in body } = \frac{1}{1-0.03125} = 1.0323.$$

63. This series converges by the alternating series test, so we can use Theorem 9.9. The n^{th} partial sum of the series is given by

$$S_n = 1 - \frac{1}{6} + \frac{1}{120} - \cdots + \frac{(-1)^{n-1}}{(2n-1)!},$$

so the absolute value of the first term omitted is $1/(2n+1)!$. By Theorem 9.9, we know that the true value of the sum differs from S_n by less than $1/(2n+1)!$. Thus, we want to choose n large enough so that $1/(2n+1)! \le 0.01$. Substituting $n = 2$ into the expression $1/(2n+1)!$ yields $1/720$ which is less than 0.01, so $S_2 = 1 - (1/6) = 5/6$ approximates the sum to within 0.01 of the actual sum.

64. No. If the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges then, using Theorem 9.2, part 3, we have $\displaystyle\lim_{n\to\infty}(-1)^{n-1}a_n = 0$, which cannot happen if $\displaystyle\lim_{n\to\infty} a_n \neq 0$.

65. If $\sum(a_n + b_n)$ converged, then $\sum(a_n + b_n) - \sum a_n = \sum b_n$ would converge by Theorem 9.2. Since we know that $\sum b_n$ does not converge, we conclude that $\sum(a_n + b_n)$ diverges.

66. We have $0 \le a_n/n \le a_n$ for all $n \ge 1$. Therefore, since $\sum a_n$ converges, $\sum a_n/n$ converges by the Comparison Test.

67. Since $\sum a_n$ converge, we know that $\lim_{n\to\infty} a_n = 0$. Thus $\lim_{n\to\infty}(1/a_n)$ does not exist, and it follows that $\sum(1/a_n)$ diverges by Property 3 of Theorem 9.2.

68. There is not enough information to determine whether or not na_n converges. To see that this is the case, note that if $a_n = 1/n^2$, then $\sum na_n = \sum(1/n)$, which diverges. However, if $a_n = 1/n^3$ then $\sum na_n = \sum(1/n^2)$, which converges.

69. We have $a_n + (a_n/2) = (3/2)a_n$, so the series $\sum(a_n + a_n/2)$ converges since it is a constant multiple of the convergent series $\sum a_n$.

70. Since $\sum a_n$ converges, we know that $\lim_{n\to\infty} a_n = 0$. Therefore, we can choose a positive integer N large enough so that $|a_n| \le 1$ for all $n \ge N$, so we have $0 \le a_n^2 \le a_n$ for all $n \ge N$. Thus, by Property 2 of Theorem 9.2, $\sum a_n^2$ converges by comparison with the convergent series $\sum a_n$.

71. The series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} + \frac{1}{n}\right) = \sum_{n=1}^{\infty}\frac{2}{n}$$

diverges by Theorem 9.2 and the fact that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

The series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n}\right) = \sum_{n=1}^{\infty} 0 = 0$$

converges. But $\displaystyle\sum_{n=1}^{\infty} -\frac{1}{n}$ diverges by Theorem 9.2 and the fact that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Thus, if $a_n = 1/n$ and $b_n = 1/n$, so that $\sum a_n$ and $\sum b_n$ both diverge, we see that $\sum(a_n + b_n)$ may diverge.

If, on the other hand, $a_n = 1/n$ and $b_n = -1/n$, so that $\sum a_n$ and $\sum b_n$ both diverge, we see that $\sum(a_n + b_n)$ may converge.

Therefore, if $\sum a_n$ and $\sum b_n$ both diverge, we cannot tell whether $\sum(a_n + b_n)$ converges or diverges. Thus the statement is true.

72. We want to estimate $\sum_{k=1}^{100{,}000} \frac{1}{k}$ using left and right Riemann sum approximations to $f(x) = 1/x$ on the interval $1 \leq x \leq 100{,}000$. Figure 9.9 shows a left Riemann sum approximation with 99,999 terms. Since $f(x)$ is decreasing, the left Riemann sum overestimates the area under the curve. Figure 9.9 shows that the first term in the sum is $f(1) \cdot 1$ and the last is $f(99{,}999) \cdot 1$, so we have

$$\int_1^{100{,}000} \frac{1}{x}\, dx < \text{ LHS } = f(1) \cdot 1 + f(2) \cdot 1 + \cdots + f(99{,}999) \cdot 1.$$

Since $f(x) = 1/x$, the left Riemann sum is

$$\text{LHS } = \frac{1}{1} \cdot 1 + \frac{1}{2} \cdot 1 + \cdots + \frac{1}{99{,}999} \cdot 1 = \sum_{k=1}^{99{,}999} \frac{1}{k},$$

so

$$\int_1^{100{,}000} \frac{1}{x}\, dx < \sum_{k=1}^{99{,}999} \frac{1}{k}.$$

Since we want the sum to go $k = 100{,}000$ rather than $k = 99{,}999$, we add $1/100{,}000$ to both sides:

$$\int_1^{100{,}000} \frac{1}{x}\, dx + \frac{1}{100{,}000} < \sum_{k=1}^{99{,}999} \frac{1}{k} + \frac{1}{100{,}000} = \sum_{k=1}^{100{,}000} \frac{1}{k}.$$

The left Riemann sum has therefore given us an underestimate for our sum. We now use the right Riemann sum in Figure 9.10 to get an overestimate for our sum.

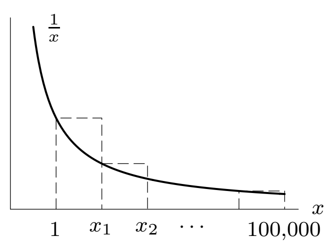


Figure 9.9

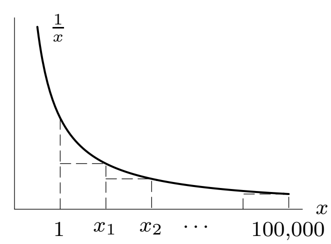


Figure 9.10

The right Riemann sum again has 99,999 terms, but this time the sum underestimates the area under the curve. Figure 9.10 shows that the first rectangle has area $f(2) \cdot 1$ and the last $f(100{,}000) \cdot 1$, so we have

$$\text{RHS } = f(2) \cdot 1 + f(3) \cdot 1 + \cdots + f(100{,}000) \cdot 1 < \int_1^{100{,}000} \frac{1}{x}\, dx.$$

Since $f(x) = 1/x$, the right Riemann sum is

$$\text{RHS } = \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \cdots + \frac{1}{100{,}000} \cdot 1 = \sum_{k=2}^{100{,}000} \frac{1}{k}.$$

So

$$\sum_{k=2}^{100{,}000} \frac{1}{k} < \int_1^{100{,}000} \frac{1}{x}\, dx.$$

Since we want the sum to start at $k = 1$, we add 1 to both sides:

$$\sum_{k=1}^{100{,}000} \frac{1}{k} = \frac{1}{1} + \sum_{k=2}^{100{,}000} \frac{1}{k} < 1 + \int_1^{100{,}000} \frac{1}{x}\, dx.$$

Putting these under- and overestimates together, we have

$$\int_1^{100{,}000} \frac{1}{x}\,dx + \frac{1}{100{,}000} < \sum_{k=1}^{100{,}000} \frac{1}{k} < 1 + \int_1^{100{,}000} \frac{1}{x}\,dx.$$

Since $\int_1^{100{,}000} \frac{1}{x}\,dx = \ln 100{,}000 - \ln 1 = 11.513$, we have

$$11.513 < \sum_{k=1}^{100{,}000} \frac{1}{k} < 12.513.$$

Therefore we have $\sum_{k=1}^{100{,}000} \frac{1}{k} \approx 12$.

73. Using a right-hand sum, we have

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} < \int_1^n \frac{dx}{x} = \ln n.$$

If a computer could add a million terms in one second, then it could add

$$60\frac{\text{sec}}{\text{min}} \cdot 60\frac{\text{min}}{\text{hour}} \cdot 24\frac{\text{hour}}{\text{day}} \cdot 365\frac{\text{days}}{\text{year}} \cdot 1\text{ million}\ \frac{\text{terms}}{\text{sec}}$$

terms per year. Thus,

$$1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n} < 1 + \ln n = 1 + \ln(60 \cdot 60 \cdot 24 \cdot 365 \cdot 10^6) \approx 32.082 < 33.$$

So the sum after one year is about 32.

74. The argument is false. Property 1 of Theorem 9.2 only applies to convergent series. In addition, by the limits comparison test with $b_n = 1/n^2$, the series converges.

CHECK YOUR UNDERSTANDING

1. False. The first 1000 terms could be the same for two different sequences and yet one sequence converges and the other diverges. For example, $s_n = 0$ for all n is a convergent sequence, but

$$t_n = \begin{cases} 0 & \text{if } n \le 1000 \\ n & \text{if } n > 1000 \end{cases}$$

is a divergent sequence.

2. False. The limit could be zero. For example, $s_n = 1/n$ is a convergent sequence of positive terms and $\lim_{n\to\infty} s_n = 0$.

3. True. If there is no term greater than a million, then the sequence is bounded by $0 < s_n < 10^6$ for all n.

4. True. If there is only a finite number of terms greater than a million, then we can choose the largest of them to be an upper bound M for the sequence. Thus the sequence is bounded by $0 < s_n \le M$ for all n.

5. False. The terms s_n tend to the limit of the sequence which may not be zero. For example, $s_n = 1 + 1/n$ is a convergent sequence and s_n tends to 1 as n increases.

6. True. The definition of convergence of a series is that its partial sums are a convergent sequence.

7. False. For example the sequence $-2, -1, 0, 1, 2, 3, \ldots$ with $s_n = n - 3$ is monotone increasing and has both positive and negative terms.

8. True. If a monotone sequence does not converge, then it is unbounded. If moreover the sequence contains only positive terms then it is bounded below by zero. Thus it is not bounded above, and in particular it is not bounded above by a million.

9. False. The sequence $-1, 1, -1, 1, \ldots$ given by $s_n = (-1)^n$ alternates in sign but does not converge.

10. False. The decreasing sequence $-1, -2, -3, \ldots$ has all terms less than a million, but it has no lower bound. Thus it is unbounded.

11. True. A geometric series, $a + ax + ax^2 + \cdots$, is a power series about $x = 0$ with all coefficients equal to a.

12. False. Writing out terms, we have

$$(x-1) + (x-2)^2 + (x-3)^3 + \cdots.$$

A power series is a sum of powers of $(x - a)$ for constant a. In this case, the value of a changes from term to term, so it is not a power series.

13. True. This power series has an interval of convergence about $x = 0$. If the power series converges for $x = 2$, the radius of convergence is 2 or more. Thus, $x = 1$ is well within the interval of convergence, so the series converges at $x = 1$.

14. False. This power series has an interval of convergence about $x = 0$. Knowing the power series converges for $x = 1$ does not tell us whether the series converges for $x = 2$. Since the series converges at $x = 1$, we know the radius of convergence is at least 1. However, we do not know whether the interval of convergence extends as far as $x = 2$, so we cannot say whether the series converges at $x = 2$.

For example, $\sum \frac{x^n}{2^n}$ converges for $x = 1$ (it is a geometric series with ratio of $1/2$), but does not converge for $x = 2$ (the terms do not go to 0).

Since this statement is not true for all C_n, the statement is false.

15. True. This power series has an interval of convergence centered on $x = 0$. If the power series does not converge for $x = 1$, then the radius of convergence is less than or equal to 1. Thus, $x = 2$ lies outside the interval of convergence, so the series does not converge there.

16. False. It does not tell us anything to know that b_n is larger than a convergent series. For example, if $a_n = 1/n^2$ and $b_n = 1$, then $0 \le a_n \le b_n$ and $\sum a_n$ converges, but $\sum b_n$ diverges. Since this statement is not true for all a_n and b_n, the statement is false.

17. True. This is one of the statements of the comparison test.

18. True. Consider the series $\sum(-b_n)$ and $\sum(-a_n)$. The series $\sum(-b_n)$ converges, since $\sum b_n$ converges, and

$$0 \le -a_n \le -b_n.$$

By the comparison test, $\sum(-a_n)$ converges, so $\sum a_n$ converges.

19. False. It is true that if $\sum |a_n|$ converges, then we know that $\sum a_n$ converges. However, knowing that $\sum a_n$ converges does *not* tell us that $\sum |a_n|$ converges.

For example, if $a_n = (-1)^{n-1}/n$, then $\sum a_n$ converges by the alternating series test. However, $\sum |a_n|$ is the harmonic series which diverges.

20. False. For example, if $a_n = 1/n$ and $b_n = -1/n$, then $|a_n + b_n| = 0$, so $\sum |a_n + b_n|$ converges. However $\sum |a_n|$ and $\sum |b_n|$ are the harmonic series, which diverge.

21. False. For example, if $a_n = 1/n^2$, then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1.$$

However, $\sum 1/n^2$ converges.

22. False, since if we write out the terms of the series, using the fact that $\cos 0 = 1, \cos \pi = -1, \cos(2\pi) = 1, \cos(3\pi) = -1$, and so on, we have

$$\begin{aligned}
&(-1)^0 \cos 0 + (-1)^1 \cos \pi + (-1)^2 \cos 2\pi + (-1)^3 \cos 3\pi + \cdots \\
&= (1)(1) + (-1)(-1) + (1)(1) + (-1)(-1) + \cdots \\
&= 1 + 1 + 1 + 1 + \cdots.
\end{aligned}$$

This is not an alternating series.

23. True. Writing out the terms of this series, we have

$$\begin{aligned}
&(1 + (-1)^1) + (1 + (-1)^2) + (1 + (-1)^3) + (1 + (-1)^4) + \cdots \\
&= (1-1) + (1+1) + (1-1) + (1+1) + \cdots \\
&= 0 + 2 + 0 + 2 + \cdots.
\end{aligned}$$

24. False. This is an alternating series, but since the terms do not go to zero, it does not converge.

25. False. The terms in the series do not go to zero:

$$2^{(-1)^1}+2^{(-1)^2}+2^{(-1)^3}+2^{(-1)^4}+2^{(-1)^5}+\cdots = 2^{-1}+2^1+2^{-1}+2^1+2^{-1}+\cdots$$
$$= 1/2+2+1/2+2+1/2+\cdots.$$

26. False. For example, if $a_n=(-1)^{n-1}/n$, then $\sum a_n$ converges by the alternating series test. But $(-1)^n a_n=(-1)^n(-1)^{n-1}/n=(-1)^{2n-1}/n=-1/n$. Thus, $\sum(-1)^n a_n$ is the negative of the harmonic series and does not converge.

27. This is true. It is a restatement of Theorem 9.9.

28. This statement is false. The statement is true if the series converges by the alternating series test, but not in general. Consider, for example, the alternating series

$$S=10-0.01+0.8-0.7-0+0-0+\cdots.$$

Since the later terms are all 0, we can find the sum exactly:

$$S=10.69.$$

If we approximated the sum by the first term, $S_1=10$, the magnitude of the first term omitted would be 0.01. Thus, if the statement in this problem were true, we would say that the true value of the sum lay between $10+0.01=10.01$ and $10-0.01=9.99$ which it does not.

29. True. Let $c_n=(-1)^n|a_n|$. Then $|c_n|=|a_n|$ so $\sum|c_n|$ converges, and therefore $\sum c_n=\sum(-1)^n|a_n|$ converges.

30. True. Since the series is alternating, Theorem 9.9 gives the error bound. Summing the first 100 terms gives S_{100}, and if the true sum is S,

$$|S-S_{100}|<a_{101}=\frac{1}{101}<0.01.$$

31. True. The radius of convergence, R, is given by $\lim_{n\to\infty}|C_{n+1}|/|C_n|=1/R$, if this limit exists, and since these series have the same coefficients, C_n, the radii of convergence are the same.

32. False. Two series can have the same radius of convergence without having the same coefficients. For example, $\sum x^n$ and $\sum nx^n$ both have radius of convergence of 1:

$$\lim_{n\to\infty}\frac{C_{n+1}}{C_n}=\lim_{n\to\infty}\frac{1}{1}=1 \quad \text{and} \quad \lim_{n\to\infty}\frac{B_{n+1}}{B_n}=\lim_{n\to\infty}\frac{n+1}{n}=1.$$

33. True. If the terms do not tend to zero, the partial sums do not tend to a limit. For example, if the terms are all greater than 0.1, the partial sums will grow without bound.

34. False. Consider the series $\displaystyle\sum_{n=1}^{\infty}1/n$. This series does not converge, but $1/n\to 0$ as $n\to\infty$.

35. False. If $a_n=b_n=1/n$, then $\sum a_n$ and $\sum b_n$ do not converge. However, $a_nb_n=1/n^2$, so $\sum a_nb_n$ does converge.

36. False. If $a_nb_n=1/n^2$ and $a_n=b_n=1/n$, then $\sum a_nb_n$ converges, but $\sum a_n$ and $\sum b_n$ do not converge.

37. True. If $\sum|a_n|$ is convergent, then so is $\sum a_n$.

38. False. The alternating harmonic series $\displaystyle\sum\frac{(-1)^n}{n}$ is conditionally convergent because it converges by the Alternating Series test, but the harmonic series $\displaystyle\sum\left|\frac{(-1)^n}{n}\right|=\sum\frac{1}{n}$ is divergent. The alternating harmonic series is not absolutely convergent.

39. False. There are power series, such as $\sum x^n/n$, which converge at one endpoint, -1, but not at the other, 1.

40. True. By the comparison test, if $\sum a_n$ is larger term-by-term than a divergent series, then $\sum a_n$ diverges. If $\sum b_n$ diverges, then so does $\sum 0.5b_n$.

41. True. The power series $\sum C_n(x-a)^n$ converges at $x=a$.

42. True. Since the power series converges at $x=10$, the radius of convergence is at least 10. Thus, $x=-9$ must be within the interval of convergence.

43. False. If $\sum C_n x^n$ converges at $x = 10$, the radius of convergence is at least 10. However, if the radius of convergence were exactly 10, then $x = 10$ is the endpoint of the interval of convergence and convergence there does not guarantee convergence at the other endpoint.

44. True. Intervals of convergence can be of any length and centered at any point and can include one endpoint and not the other.

45. False. The interval of convergence of $\sum C_n x^n$ is centered at the origin.

46. True. The interval of convergence is centered on $x = a$, so $a = (-11 + 1)/2 = -5$.

PROJECTS FOR CHAPTER NINE

1. (a) To show f is decreasing for $x > 1$, we look at $f'(x)$:

$$f'(x) = n(n+1)x^{n-1} - n(n+1)x^n = n(n+1)x^{n-1}(1-x).$$

Thus, for $x > 1$, we have $f'(x) < 0$, so f is decreasing. Since $f(1) = 1$, this means $f(x) < 1$ for $x > 1$. Factoring x^n out of $f(x)$, we get

$$f(x) = (n+1)x^n - nx^{n+1} = x^n(n+1-nx) < 1.$$

(b) We simplify the value of x

$$x = \frac{1+1/n}{1+1/(n+1)} = \frac{(n+1)/n}{(n+2)/(n+1)} = \frac{(n+1)^2}{n(n+2)}.$$

Before substituting into $x^n(n+1-nx) < 1$, we calculate

$$\begin{aligned} n+1-nx &= n+1-n\frac{(n+1)^2}{n(n+2)} \\ &= \frac{(n+1)(n+2)-(n+1)^2}{n+2} \\ &= \frac{(n+1)(n+2-(n+1))}{n+2} = \frac{n+1}{n+2}. \end{aligned}$$

Thus, substituting into the inequality from part (a), $x^n(n+1-nx) < 1$, gives

$$x^n\left(\frac{n+1}{n+2}\right) < 1.$$

(c) We want to show $s_n < s_{n+1}$. Since $s_n = (1+1/n)^n$ and $s_{n+1} = (1+1/(n+1))^{n+1}$, using the definition of x, we have

$$\begin{aligned} \frac{s_n}{s_{n+1}} &= \frac{(1+1/n)^n}{(1+1/(n+1))^n} \cdot \frac{1}{(1+1/(n+1))} \\ &= x^n\left(\frac{n+1}{n+2}\right). \end{aligned}$$

Thus, by part (b), we have

$$\frac{s_n}{s_{n+1}} = x^n\left(\frac{n+1}{n+2}\right) < 1,$$

so

$$s_n < s_{n+1}.$$

Thus, the sequence is increasing.

(d) Substituting $x = 1 + 1/2n$ into the inequality from part (a) gives

$$\left(1+\frac{1}{2n}\right)^n \left(n+1-n\left(1+\frac{1}{2n}\right)\right) = \left(1+\frac{1}{2n}\right)^n \left(1-\frac{1}{2}\right) = \frac{1}{2}\left(1+\frac{1}{2n}\right)^n < 1.$$

Thus

$$\left(1+\frac{1}{2n}\right)^n < 2.$$

(e) When we square this inequality, we get

$$\left(1+\frac{1}{2n}\right)^{2n} < 4,$$

that is, for all n

$$s_{2n} < 4.$$

Thus, the even terms are bounded above by 4. Because we have shown the sequence is increasing, for each odd term, we have

$$s_{2n-1} < 2_{2n} < 4,$$

so the odd terms are also bounded above by 4. Since all terms are bounded below by 0, the sequence is bounded.

(f) From parts (c) and (e), we know that the sequence is increasing and bounded, and therefore, by Theorem 9.1, it has a limit.

2. (a) (i) p^2

(ii) There are two ways to do this. One way is to compute your opponent's probability of winning two in a row, which is $(1-p)^2$. Then the probability that neither of you win the next points is:

$$\begin{aligned}
1 - (\text{Probability you win next two} &+ \text{Probability opponent wins next two}) \\
&= 1-(p^2+(1-p)^2) \\
&= 1-(p^2+1-2p+p^2) \\
&= 2p^2-2p \\
&= 2p(1-p).
\end{aligned}$$

The other way to compute this is to observe either you win the first point and lose the second or vice versa. Both have probability $p(1-p)$, so the probability you split the points is $2p(1-p)$.

(iii)

$$\begin{aligned}
\text{Probability} &= (\text{Probability of splitting next two}) \cdot (\text{Probability of winning two after that}) \\
&= 2p(1-p)p^2
\end{aligned}$$

(iv)

$$\begin{aligned}
\text{Probability} &= (\text{Probability of winning next two}) + (\text{Probability of splitting next two,} \\
&\qquad \text{winning two after that}) \\
&= p^2 + 2p(1-p)p^2
\end{aligned}$$

(v) The probability is:

$$\begin{aligned}
w &= (\text{Probability of winning first two}) \\
&\quad + (\text{Probability of splitting first two})\cdot(\text{Probability of winning next two}) \\
&\quad + (\text{Prob. of split. first two})\cdot(\text{Prob. of split. next two})\cdot(\text{Prob. of winning next two}) \\
&\quad + \cdots \\
&= p^2 + 2p(1-p)p^2 + (2p(1-p))^2\, p^2 + \cdots.
\end{aligned}$$

This is an infinite geometric series with a first term of p^2 and a ratio of $2p(1-p)$. Therefore the probability of winning is

$$w = \frac{p^2}{1-2p(1-p)}.$$

(vi) For $p = 0.5$, $w = \frac{(0.5)^2}{1-2(0.5)(1-(0.5))} = 0.5$. This is what we would expect. If you and your opponent are equally likely to score the next point, you and your opponent are equally likely to win the next game.

For $p = 0.6$, $w = \frac{(0.6)^2}{1-2(0.6)(0.4)} = 0.69$. Here your probability of winning the next point has been magnified to a probability 0.69 of winning the game. Thus it gives the better player an advantage to have to win by two points, rather than the "sudden death" of winning by just one point. This makes sense: when you have to win by two, the stronger player always gets a second chance to overcome the weaker player's winning the first point on a "fluke."

For $p = 0.7$, $w = \frac{(0.7)^2}{1-2(0.7)(0.3)} = 0.84$. Again, the stronger player's probability of winning is magnified.

For $p = 0.4$, $w = \frac{(0.4)^2}{1-2(0.4)(0.6)} = 0.31$. We already computed that for $p = 0.6$, $w = 0.69$. Thus the value for w when $p = 0.4$, should be the same as the probability of your opponent winning for $p = 0.6$, namely $1 - 0.69 = 0.31$.

(b) (i)

$$\begin{aligned}
S = &\text{ (Prob. you score first point)}\\
&+\text{(Prob. you lose first point, your opponent loses the next,}\\
&\quad\text{you win the next)}\\
&+\text{(Prob. you lose a point, opponent loses, you lose,}\\
&\quad\text{opponent loses, you win)}\\
&+\cdots\\
= &\text{ (Prob. you score first point)}\\
&+\text{(Prob. you lose)}\cdot\text{(Prob. opponent loses)}\cdot\text{(Prob. you win)}\\
&+\text{(Prob. you lose)}\cdot\text{(Prob. opponent loses)}\cdot\text{(Prob. you lose)}\\
&\quad\cdot\text{(Prob. opponent loses)}\cdot\text{(Prob. you win)}+\cdots\\
= &\; p + (1-p)(1-q)p + ((1-p)(1-q))^2 p + \cdots\\
= &\; \frac{p}{1-(1-p)(1-q)}
\end{aligned}$$

(ii) Since S is your probability of winning the next point, we can use the formula computed in part (v) of (a) for winning two points in a row, thereby winning the game:

$$w = \frac{S^2}{1-2S(1-S)}.$$

- When $p = 0.5$ and $q = 0.5$,

$$S = \frac{0.5}{1-(0.5)(0.5)} = 0.67.$$

Therefore

$$w = \frac{S^2}{1-2S(1-S)} = \frac{(0.67)^2}{1-2(0.67)(1-0.67)} = 0.80.$$

- When $p = 0.6$ and $q = 0.5$,

$$S = \frac{0.6}{1-(0.4)(0.5)} = 0.75 \quad \text{and} \quad w = \frac{(0.75)^2}{1-2(0.75)(1-0.75)} = 0.9.$$

3. (a) Let k by the relative rate of decay, per minute, of quinine. Since quinine's half-life is 11.5 hours, we have

$$\frac{1}{2} = e^{-k(11.5)(60)},$$

so

$$k = \frac{\ln 2}{(11.5)(60)} \approx 0.001.$$

Hence, $k = 0.1\%$/min.

(b) Just prior to 8 am of the first day the patient has no quinine in her body. Assuming the drug mixes rapidly in the patient's body, she has about $50/70 \approx 0.714$ mg/kg of the drug soon after 8 am. Suppose we represent the concentration of quinine in the patient (in mg/kg) by x and represent time since 8 am (in minutes) by t. Then

$$x = Ae^{-0.001t},$$

where A is the initial concentration and $k = -0.001$ is the rate at which quinine is metabolized per minute. There are $24 \cdot 60 = 1440$ minutes in a day. On the first day, the patient begins with 0.714 mg/kg in her system, so just before 8 am of the second day the patient's system holds

$$0.714e^{-0.001 \cdot 1440} \approx 0.169 \text{ mg/kg}.$$

After the patient's second dose of quinine, her system contains $0.714 + 0.169 = 0.883$ mg/kg of quinine.

(c) By continuing in a similar manner, we see that just prior to 8 am on the third day, she has $0.883e^{-0.001 \cdot 1440} \approx 0.209$ mg/kg; just after 8 am, she has $0.209 + 0.714 = 0.923$ mg/kg. Just prior to 8 am on the fourth day, she has $0.923e^{-0.001 \cdot 1440} \approx 0.218$ mg/kg; just after 8 am, she has $0.228 + 0.714 = 0.932$ mg/kg. We can keep going with these calculations: just prior to 8 am on the fifth day, the concentration is 0.221 mg/kg; on the sixth day, it is 0.222 mg/kg; on the seventh day, it is 0.222 mg/kg, and so on forever.

We find a formula for the concentration just after the n^{th} dose as follows. The last dose contributes 0.714 mg/kg. The previous dose contributes $0.714e^{-0.001(1440)}$ mg/kg. The dose before that contributes $0.714e^{-0.001(2)(1440)}$ mg/kg, and so on, back to $0.714e^{-0.001(n-1)(1440)}$ mg/kg from the initial dose. So

$$\begin{array}{c}\text{Concentration just}\\ \text{after } n \text{ doses}\end{array} = 0.714 + 0.714e^{-1.44} + 0.714\left(e^{-1.44}\right)^2 + \cdots + 0.714\left(e^{-1.44}\right)^{n-1}.$$

We notice that this is a geometric series, with sum given by

$$\begin{array}{c}\text{Concentration just}\\ \text{after } n \text{ doses}\end{array} = 0.714\left(\frac{1 - e^{-1.44n}}{1 - e^{-1.44}}\right) = 0.936(1 - e^{-1.44n}).$$

Although the concentration of quinine does not reach an equilibrium it does fall into a steady-state pattern which repeats over and over again. This makes sense; at some point the patient must metabolize the daily dosage exactly. If we let $n \to \infty$ in our formula, we have $e^{-1.44n} \to 0$, which means that the concentration just after the n^{th} dose gets very close to 0.936. So the concentration just before the n^{th} dose is $0.936 - 0.714 = 0.222$, as we found in our calculations for the first few days.

(d)

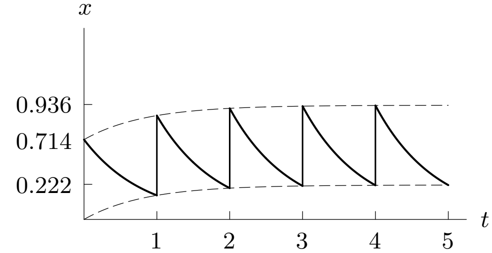


Figure 9.11

If we keep setting the clock back to 0 minutes each day at 8 am, then we have that at $t = 0$ each day, the concentration (starting on the fifth day or so) is 0.936 mg/kg. As the day progresses, we have

$$x = 0.936e^{-0.001 \cdot t}.$$

(e) The average concentration of quinine in the patient is given by the integral of the concentration over a day, divided by the time in a day:

$$\begin{aligned}\text{Average concentration } &= \frac{1}{1440}\int_0^{1440} x\,dt = \frac{1}{1440}\int_0^{1440} 0.936e^{-0.001t}dt \\ &= \frac{0.936}{1440}\left(\frac{-e^{-0.001t}}{0.001}\right)\Bigg|_0^{1440} = \frac{0.936}{1.44}(1-e^{-1.44}) \\ &\approx 0.496 \text{ mg/kg.}\end{aligned}$$

(f) Since the average concentration is 0.496 mg/kg and the minimum effective average concentration is 0.4 mg/kg, this treatment is effective. It is also safe—the highest concentration (0.936 mg/kg, achieved shortly after 8 am) is less than the toxic concentration of 3.0 mg/kg.

(g) Each dose of 25 mg corresponds to $25/70 = 0.357$ mg/kg. Let x_s be the steady-state concentration just before each 0.357 mg/kg dose. Then $x_s + 0.357$ will be the concentration just after the dose. Since we are in a steady-state, this concentration decays to exactly x_s just before the next dose. So

$$x_s = (x_s + 0.357)e^{-0.001(12)(60)}.$$

This means

$$x_s = \frac{0.357e^{-0.001(12)(60)}}{1-e^{-0.001(12)(60)}} \approx 0.339 \text{ mg/kg,}$$

so $x_s + 0.357 = 0.696$ mg/kg is the concentration just after each dose. At t minutes after a dose, for $0 \le t \le (12)(60)$, there is a steady-state concentration of

$$x = 0.696e^{-0.001t} \text{ mg/kg.}$$

This means

$$\begin{aligned}\text{Average concentration } &= \frac{1}{720}\int_0^{720} x\,dt \approx \frac{1}{720}\int_0^{720} 0.696e^{-0.001t}dt \\ &= \frac{0.696}{720}\left[\frac{-e^{-0.001t}}{0.001}\right]\Bigg|_0^{720} = \frac{0.696}{0.72}\left[1-0.487\right] \\ &\approx 0.496 \text{ mg/kg.}\end{aligned}$$

This treatment is also effective and safe. The average concentration of 0.496 mg/kg is greater than 0.4 mg/kg, and the highest concentration of 0.696 mg/kg is less than 3 mg/kg.

(h) For an exponentially decaying function, the average value between two points (x_0, y_0) and (x_1, y_1) is $\frac{(y_0-y_1)}{(x_1-x_0)r}$, where r is the relative rate of decay and A_0 is the initial concentration. The reason is as follows.

$$\begin{aligned}\text{Average } &= \frac{1}{x_1-x_0}\int_{x_0}^{x_1} A_0e^{-rt}dt \\ &= \frac{A_0}{x_1-x_0}\left[\frac{e^{-rt}}{r}\right]\Bigg|_{x_0}^{x_1} \\ &= \frac{y_0-y_1}{(x_1-x_0)\cdot r}\end{aligned}$$

(i) Since a steady state has been reached, y_0 is the concentration right after a dose and y_1 is the concentration just prior to a dose. Thus, $y_0 - y_1$ represents the increase in concentration from each dose. Furthermore, $x_1 - x_0$ is the time between doses. When we go to the new protocol, we halve both the numerator and the denominator of the equation for the average concentration, and so the average remains unchanged. Similarly, if we were to double the dose to 100 mg and give it every 48 hours we would simply be doubling both the numerator and the denominator; again the average concentration would not change.

(j) We want the final concentration to be 10^{-10} kg/kg $= 10^{-4}$ mg/kg. We therefore need to solve for t in $10^{-4} = 0.883 \cdot e^{-0.001\cdot t}$. Doing so yields $t \approx 9086$ min ≈ 6.3 days.