CHAPTER ELEVEN

Solutions for Section 11.1

Exercises

1. (a) (III) An island can only sustain the population up to a certain size. The population will grow until it reaches this limiting value.
(b) (V) The ingot will get hot and then cool off, so the temperature will increase and then decrease.
(c) (I) The speed of the car is constant, and then decreases linearly when the breaks are applied uniformly.
(d) (II) Carbon-14 decays exponentially.
(e) (IV) Tree pollen is seasonal, and therefore cyclical.

2. Since $y = x^3$, we know that $y' = 3x^2$. Substituting $y = x^3$ and $y' = 3x^2$ into the differential equation we get

$$\begin{aligned} 0 &= xy' - 3y \\ &= x(3x^2) - 3(x^3) \\ &= 3x^3 - 3x^3 \\ &= 0. \end{aligned}$$

Since this equation is true for all x, we see that $y = x^3$ is in fact a solution.

3. In order to prove that $y = A + Ce^{kt}$ is a solution to the differential equation

$$\frac{dy}{dt} = k(y - A),$$

we must show that the derivative of y with respect to t is in fact equal to $k(y - A)$:

$$\begin{aligned} y &= A + Ce^{kt} \\ \frac{dy}{dt} &= 0 + (Ce^{kt})(k) \\ &= kCe^{kt} \\ &= k(Ce^{kt} + A - A) \\ &= k\left((Ce^{kt} + A) - A\right) \\ &= k(y - A). \end{aligned}$$

4. If $P = P_0e^t$, then

$$\frac{dP}{dt} = \frac{d}{dt}(P_0e^t) = P_0e^t = P.$$

5. We know that at time $t = 0$ the value of y is 8. Since we are told that $dy/dt = 0.5y$, we know that at time $t = 0$ the derivative of y is $.5(8) = 4$. Thus as t goes from 0 to 1, y will increase by 4, so at $t = 1$, $y = 8 + 4 = 12$.

Likewise, at $t = 1$, we get $dy/dt = 0.5(12) = 6$ so that at $t = 2$, we obtain $y = 12 + 6 = 18$.
At $t = 2$, we have $dy/dt = 0.5(18) = 9$ so that at $t = 3$, we obtain $y = 18 + 9 = 27$.
At $t = 3$, we have $dy/dt = 0.5(27) = 13.5$ so that at $t = 4$, we obtain $y = 27 + 13.5 = 40.5$.
Thus we get the values in the following table

t	0	1	2	3	4
y	8	12	18	27	40.5

6. Since $y = x^2 + k$, we know that $y' = 2x$. Substituting $y = x^2 + k$ and $y' = 2x$ into the differential equation, we get

$$\begin{aligned} 10 &= 2y - xy' \\ &= 2(x^2 + k) - x(2x) \\ &= 2x^2 + 2k - 2x^2 \\ &= 2k. \end{aligned}$$

Thus, $k = 5$ is the only solution.

7. If $Q = Ce^{kt}$, then

$$\frac{dQ}{dt} = Cke^{kt} = k(Ce^{kt}) = kQ.$$

We are given that $\frac{dQ}{dt} = -0.03Q$, so we know that $kQ = -0.03Q$. Thus we either have $Q = 0$ (in which case $C = 0$ and k is anything) or $k = -0.03$. Notice that if $k = -0.03$, then C can be any number.

8. If y satisfies the differential equation, then we must have

$$\begin{aligned} \frac{d\,(5+3e^{kx})}{dx} &= 10 - 2(5+3e^{kx}) \\ 3ke^{kx} &= 10 - 10 - 6e^{kx} \\ 3ke^{kx} &= -6e^{kx} \\ k &= -2. \end{aligned}$$

So, if $k = -2$ the formula for y solves the differential equation.

9. If $y = \sin 2t$, then $\frac{dy}{dt} = 2\cos 2t$, and $\frac{d^2y}{dt^2} = -4\sin 2t$.
Thus $\frac{d^2y}{dt^2} + 4y = -4\sin 2t + 4\sin 2t = 0$.

10. If $y = \cos\omega t$, then

$$\frac{dy}{dt} = -\omega\sin\omega t, \qquad \frac{d^2y}{dt^2} = -\omega^2\cos\omega t.$$

Thus, if $\frac{d^2y}{dt^2} + 9y = 0$, then

$$\begin{aligned} -\omega^2\cos\omega t + 9\cos\omega t &= 0 \\ (9-\omega^2)\cos\omega t &= 0. \end{aligned}$$

Thus $9 - \omega^2 = 0$, or $\omega^2 = 9$, so $\omega = \pm 3$.

11. Differentiating and using the fact that

$$\frac{d}{dt}(\cosh t) = \sinh t \quad \text{and} \quad \frac{d}{dt}(\sinh t) = \cosh t,$$

we see that

$$\begin{aligned} \frac{dx}{dt} &= \omega C_1\sinh\omega t + \omega C_2\cosh\omega t \\ \frac{d^2x}{dt^2} &= \omega^2 C_1\cosh\omega t + \omega^2 C_2\sinh\omega t \\ &= \omega^2\left(C_1\cosh\omega t + C_2\sinh\omega t\right). \end{aligned}$$

Therefore, we see that

$$\frac{d^2x}{dt^2} = \omega^2 x.$$

12. Differentiating $x^2 + y^2 = r^2$ implicitly, with r a constant, gives

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

Problems

13. (a) If $y = Cx^n$ is a solution to the given differential equation, then we must have

$$\begin{aligned} x\frac{d\,(Cx^n)}{dx} - 3(Cx^n) &= 0 \\ x(Cnx^{n-1}) - 3(Cx^n) &= 0 \\ Cnx^n - 3Cx^n &= 0 \\ C(n-3)x^n &= 0. \end{aligned}$$

Thus, if $C = 0$, we get $y = 0$ is a solution, for every n. If $C \neq 0$, then $n = 3$, and so $y = Cx^3$ is a solution.

(b) Because $y = 40$ for $x = 2$, we cannot have $C = 0$. Thus, by part (a), we get $n = 3$. The solution to the differential equation is

$$y = Cx^3.$$

To determine C if $y = 40$ when $x = 2$, we substitute these values into the equation.

$$40 = C \cdot 2^3$$
$$40 = C \cdot 8$$
$$C = 5.$$

So, now both C and n are fixed at specific values.

14. (a) $P = \frac{1}{1+e^{-t}} = (1 + e^{-t})^{-1}$

$\frac{dP}{dt} = -(1 + e^{-t})^{-2}(-e^{-t}) = \frac{e^{-t}}{(1+e^{-t})^2}$.

Then $P(1-P) = \frac{1}{1+e^{-t}}\left(1 - \frac{1}{1+e^{-t}}\right) = \left(\frac{1}{1+e^{-t}}\right)\left(\frac{e^{-t}}{1+e^{-t}}\right) = \frac{e^{-t}}{(1+e^{-t})^2} = \frac{dP}{dt}$.

(b) As t tends to ∞, e^{-t} goes to 0. Thus $\lim\limits_{t\to\infty} \frac{1}{1+e^{-t}} = 1$.

15.

$$\begin{array}{rlll}
\text{(I)} & y = 2\sin x, & dy/dx = 2\cos x, & d^2y/dx^2 = -2\sin x \\
\text{(II)} & y = \sin 2x, & dy/dx = 2\cos 2x, & d^2y/dx^2 = -4\sin 2x \\
\text{(III)} & y = e^{2x}, & dy/dx = 2e^{2x}, & d^2y/dx^2 = 4e^{2x} \\
\text{(IV)} & y = e^{-2x}, & dy/dx = -2e^{-2x}, & d^2y/dx^2 = 4e^{-2x}
\end{array}$$

and so:

(a) (IV)
(b) (III)
(c) (III), (IV)
(d) (II)

16. It is easiest to begin by writing down the first and second derivatives for each possible solution:

(I) $y = \cos x$, so $y' = -\sin x$, and $y'' = -\cos x$.
(II) $y = \cos(-x)$, so $y' = \sin(-x)$, and $y'' = -\cos(-x)$.
(III) $y = x^2$, so $y' = 2x$, and $y'' = 2$.
(IV) $y = e^x + e^{-x}$, so $y' = e^x - e^{-x}$, and $y'' = e^x + e^{-x}$.
(V) $y = \sqrt{2x}$, so $y' = \frac{1}{2}(2x)^{-1/2} \cdot 2 = 1/\sqrt{2x}$, and $y'' = -\frac{1}{2}(2x)^{-3/2} \cdot 2 = -(2x)^{-3/2}$.

By substituting these into the given differential equations, we get following solutions:

(a) (IV)
(b) None
(c) (V)
(d) (I), (II)
(e) (III)

Solutions for Section 11.2

Exercises

1. There are many possible answers. One possibility is shown in Figures 11.1 and 11.2.

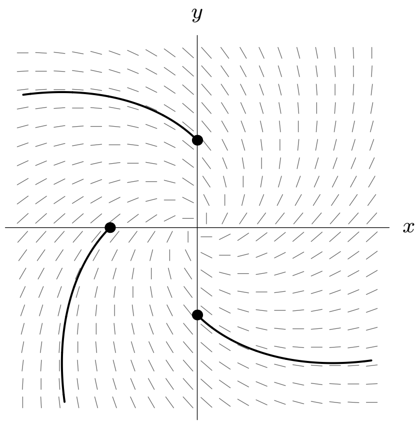


Figure 11.1

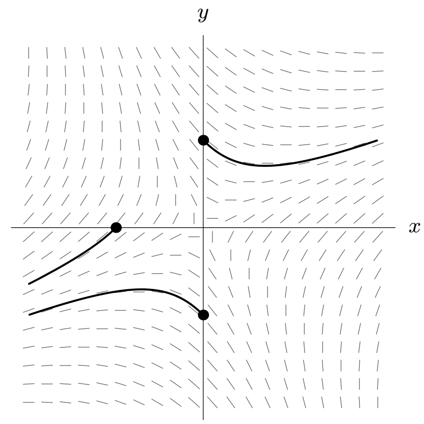


Figure 11.2

2. (a) See Figure 11.3.
(b) The solution is $y(x) = 1$.
(c) Since $y' = 0$ and $x(y-1) = 0$, this is a solution.

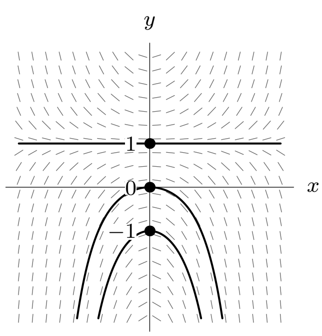


Figure 11.3

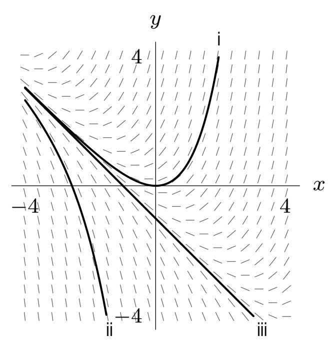


Figure 11.4

3. (a) See Figure 11.4.
(b) The solution through $(-1, 0)$ appears to be linear, so its equation is $y = -x - 1$.
(c) If $y = -x - 1$, then $y' = -1$ and $x + y = x + (-x - 1) = -1$, so this checks as a solution.

Problems

4. Notice that $y' = \dfrac{x+y}{x-y}$ is zero when $x = -y$ and is undefined when $x = y$. A solution curve will be horizontal (slope= 0) when passing through a point with $x = -y$, and will be vertical (slope undefined) when passing through a point with $x = y$. The only slope field for which this is true is slope field (b).

5. (a) See Figure 11.5.

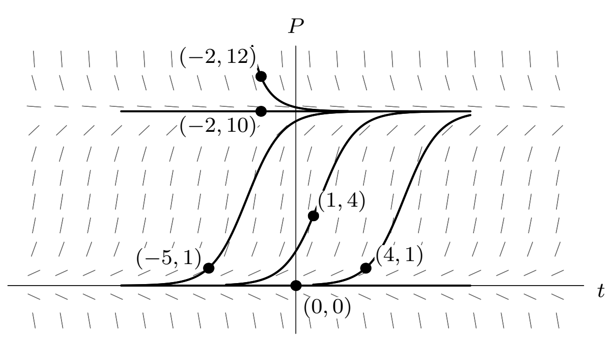


Figure 11.5

(b) If $0 < P < 10$, the solution is increasing; if $P > 10$, it is decreasing. So P tends to 10.

6. (a) and (b) See Figure 11.6

(c) Figure 11.6 shows that a solution will be increasing if its y-values fall in the range $-1 < y < 2$. This makes sense since if we examine the equation $y' = 0.5(1 + y)(2 - y)$, we will find that $y' > 0$ if $-1 < y < 2$. Notice that if the y-value ever gets to 2, then $y' = 0$ and the function becomes constant, following the line $y = 2$. (The same is true if ever $y = -1$.)

From the graph, the solution is decreasing if $y > 2$ or $y < -1$. Again, this also follows from the equation, since in either case $y' < 0$.

The curve has a horizontal tangent if $y' = 0$, which only happens if $y = 2$ or $y = -1$. This also can be seen on the graph in Figure 11.6.

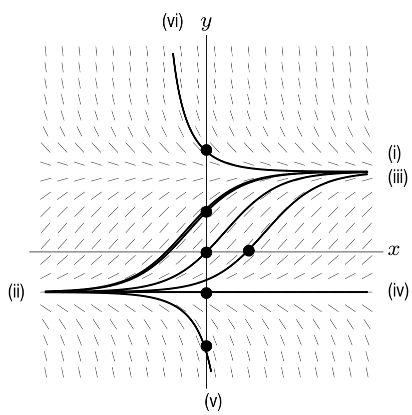


Figure 11.6

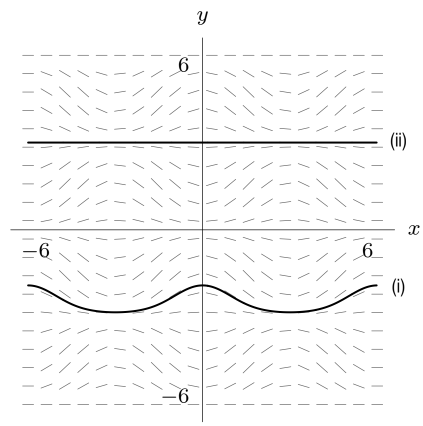


Figure 11.7

7. (a) See Figure 11.7.

(b) We can see that the slope lines are horizontal when y is an integer multiple of π. We conclude from Figure 11.7 that the solution is $y = n\pi$ in this case.

To check this, we note that if $y = n\pi$, then $(\sin x)(\sin y) = (\sin x)(\sin n\pi) = 0 = y'$. Thus $y = n\pi$ is a solution to $y' = (\sin x)(\sin y)$, and it passes through $(0, n\pi)$.

8. (a) Since $y' = -y$, the slope is negative above the x-axis (when y is positive) and positive below the x-axis (when y is negative). The only slope field for which this is true is II.

(b) Since $y' = y$, the slope is positive for positive y and negative for negative y. This is true of both I and III. As y get larger, the slope should get larger, so the correct slope field is I.

(c) Since $y' = x$, the slope is positive for positive x and negative for negative x. This corresponds to slope field V.

(d) Since $y' = \dfrac{1}{y}$, the slope is positive for positive y and negative for negative y. As y approaches 0, the slope becomes larger in magnitude, which correspond to solution curves close to vertical. The correct slope field is III.

(e) Since $y' = y^2$, the slope is always positive, so this must correspond to slope field IV.

9. (a) II (b) VI (c) IV (d) I (e) III (f) V

10. The slope fields in (I) and (II) appear periodic. (I) has zero slope at $x = 0$, so (I) matches $y' = \sin x$, whereas (II) matches $y' = \cos x$. The slope in (V) tends to zero as $x \to \pm\infty$, so this must match $y' = e^{-x^2}$. Of the remaining slope fields, only (III) shows negative slopes, matching $y' = xe^{-x}$. The slope in (IV) is zero at $x = 0$, so it matches $y' = x^2e^{-x}$. This leaves field (VI) to match $y' = e^{-x}$.

Solutions for Section 11.3

Exercises

1. (a)

Table 11.1 *Euler's method for $y' = x + y$ with $y(0) = 1$*

x	y	$\Delta y =$(slope)Δx
0	1	$0.1 = (1)(0.1)$
0.1	1.1	$0.12 = (1.2)(0.1)$
0.2	1.22	$0.142 = (1.42)(0.1)$
0.3	1.362	$0.1662 = (1.662)(0.1)$
0.4	1.5282	

So $y(0.4) \approx 1.5282$.

(b)

Table 11.2 *Euler's method for $y' = x + y$ with $y(-1) = 0$*

x	y	$\Delta y =$(slope)Δx
-1	0	$-0.1 = (-1)(0.1)$
-0.9	-0.1	$-0.1 = (-1)(0.1)$
-0.8	-0.2	$-0.1 = (-1)(0.1)$
-0.7	-0.3	
$\vdots$	$\vdots$	Notice that y
0	-1	decreases by 0.1
$\vdots$	$\vdots$	for every step
0.4	-1.4	

So $y(0.4) = -1.4$. (This answer is exact.)

2. (a) The results from Euler's method with $\Delta x = 0.1$ are in Table 11.3.

(b) We have

$$y(x) = \frac{x^4}{4} + C,$$

so that $y(0) = 0$ gives $C = 0$, and the required solution is therefore

$$y(x) = \frac{x^4}{4}.$$

This is shown in the 3rd column of Table 11.3.

(c) The computed solution underestimates the real solution since the solution is concave up and is approximated in every interval by the tangent which is beneath the curve. See Figure 11.8.

Table 11.3

Computed Solution		
x_n	Approx. $y(x_n)$	$y(x_n)$
0	0	0
0.1	0	0.000025
0.2	0.0001	0.0004
0.3	0.0009	0.002025
0.4	0.0036	0.0064
0.5	0.01	0.015625
0.6	0.0225	0.0324
0.7	0.0441	0.060025
0.8	0.0784	0.1024
0.9	0.1296	0.164025
1.0	0.2025	0.25

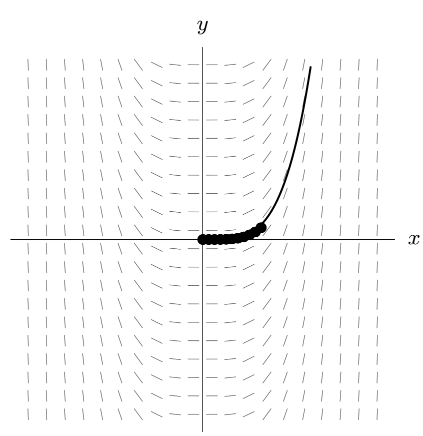


Figure 11.8

3. (a) See Figure 11.9.

(b) $y(0) = 1$,

$$y(0.1) \approx y(0) + 0.1y(0) = 1 + 0.1(1) = 1.1$$
$$y(0.2) \approx y(0.1) + 0.1y(0.1) = 1.1 + 0.1(1.1) = 1.21$$
$$y(0.3) \approx y(0.2) + 0.1y(0.2) = 1.21 + 0.1(1.21) = 1.331$$
$$y(0.4) \approx 1.4641$$
$$y(0.5) \approx 1.61051$$
$$y(0.6) \approx 1.77156$$
$$y(0.7) \approx 1.94872$$
$$y(0.8) \approx 2.14359$$
$$y(0.9) \approx 2.35795$$
$$y(1.0) \approx 2.59374$$

(c) See Figure 11.9. A smooth curve drawn through the solution points seems to match the slope field.

(d) For $y = e^x$, we have $y' = e^x = y$ and $y(0) = e^0 = 1$. See Table 11.4.

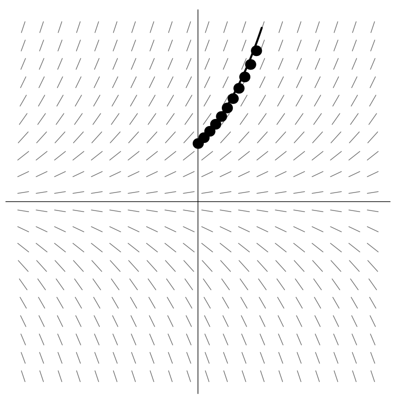


Figure 11.9

Table 11.4

Computed Solution		
x_n	Approx. $y(x_n)$	$y(x_n)$
0	1	1
0.1	1.1	1.10517
0.2	1.21	1.22140
0.3	1.331	1.34986
0.4	1.4641	1.49182
0.5	1.61051	1.64872
0.6	1.77156	1.82212
0.7	1.94872	2.01375
0.8	2.14359	2.22554
0.9	2.35795	2.45960
1.0	2.59374	2.71828

4. (a) See Table 11.5. At $x = 1$, $y \approx 0.16$.

(b) See Figure 11.10.

Table 11.5

x	y	$\Delta y =$ (slope)Δx
0	0	0
0.2	0	0.0016
0.4	0.0016	0.0128
0.6	0.0144	0.0432
0.8	0.0576	0.1024
1	0.1600	

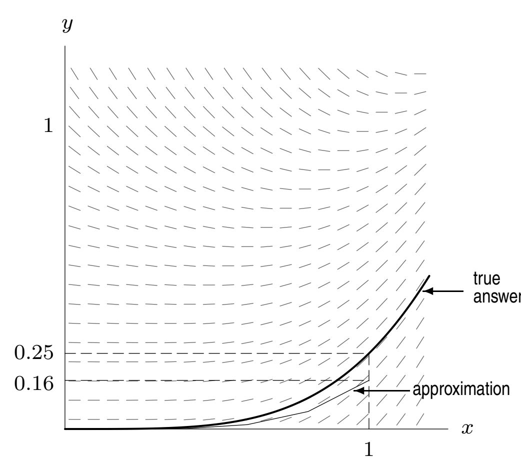


Figure 11.10

(c) Our answer to (a) appears to be an underestimate. This is as we would expect, since the curve is concave up.

Problems

5. (a) $\Delta x = 0.5$

Table 11.6 *Euler's method for $y' = 2x$, with $y(0) = 1$*

x	y	$\Delta y =$(slope)Δx
0	1	$0 = (2 \cdot 0)(0.5)$
0.5	1	$0.5 = (2 \cdot 0.5)(0.5)$
1	1.5	

$\Delta x = 0.25$

Table 11.7 *Euler's method for $y' = 2x$, with $y(0) = 1$*

x	y	$\Delta y =$(slope)Δx
0	1	$0 = (2 \cdot 0)(0.25)$
0.25	1	$0.125 = (2 \cdot 0.25)(0.25)$
0.50	1.125	$0.25 = (2 \cdot 0.5)(0.25)$
0.75	1.375	$0.375 = (2 \cdot 0.75)(0.25)$
1	1.75	

(b) General solution is $y = x^2 + C$, and $y(0) = 1$ gives $C = 1$. Thus, the solution is $y = x^2 + 1$. So the true value of y when $x = 1$ is $y = 1^2 + 1 = 2$.

(c) When $\Delta x = 0.5$, error $= 0.5$.
When $\Delta x = 0.25$, error $= 0.25$.
Thus, decreasing Δx by a factor of 2 has decreased the error by a factor of 2, as expected.

6. (a)

Table 11.8

t	y	slope $= \frac{1}{t}$	$\Delta y = $ (slope)$\Delta t = \frac{1}{t}(0.1)$
1	0	1	0.1
1.1	0.1	0.909	0.091
1.2	0.191	0.833	0.083
1.3	0.274	0.769	0.077
1.4	0.351	0.714	0.071
1.5	0.422	0.667	0.067
1.6	0.489	0.625	0.063
1.7	0.552	0.588	0.059
1.8	0.610	0.556	0.056
1.9	0.666	0.526	0.053
2	0.719		

(b) If $\frac{dy}{dt} = \frac{1}{t}$, then $y = \ln|t| + C$.
Starting at $(1, 0)$ means $y = 0$ when $t = 1$, so $C = 0$ and $y = \ln|t|$.
After ten steps, $t = 2$, so $y = \ln 2 \approx 0.693$.

(c) Approximate $y = 0.719$, Exact $y = 0.693$.
Thus the approximate answer is too big. This is because the solution curve is concave down, and so the tangent lines are above the curve. Figure 11.11 shows the slope field of $y' = 1/t$ with the solution curve $y = \ln t$ plotted on top of it.

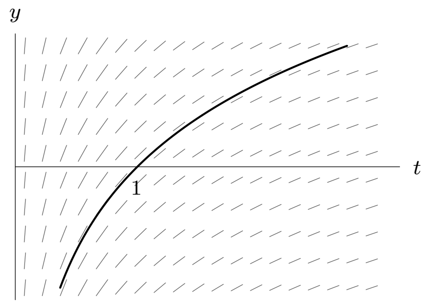


Figure 11.11

7. (a) (i)

Table 11.9 *Euler's method for $y' = (\sin x)(\sin y)$, starting at $(0, 2)$*

x	y	$\Delta y =$(slope)Δx
0	2	$0 = (\sin 0)(\sin 2)(0.1)$
0.1	2	$0.009 = (\sin 0.1)(\sin 2)(0.1)$
0.2	2.009	$0.018 = (\sin 0.2)(\sin 2.009)(0.1)$
0.3	2.027	

(ii)

Table 11.10 *Euler's method for $y' = (\sin x)(\sin y)$, starting at $(0, \pi)$*

x	y	$\Delta y =$(slope)Δx
0	π	$0 = (\sin 0)(\sin \pi)(0.1)$
0.1	π	$0 = (\sin 0.1)(\sin \pi)(0.1)$
0.2	π	$0 = (\sin 0.2)(\sin \pi)(0.1)$
0.3	π	

(b) The slope field shows that the slope of the solution curve through $(0, \pi)$ is always 0. Thus the solution curve is the horizontal line with equation $y = \pi$.

8. For $\Delta x = 0.2$, we get the following results.

$$y(1.2) \approx y(1) + 0.2 \sin(1 \cdot y(1)) = 1.168294$$
$$y(1.4) \approx y(1.2) + 0.2 \sin(1.2 \cdot y(1.2)) = 1.365450$$
$$y(1.6) \approx y(1.4) + 0.2 \sin(1.4 \cdot y(1.4)) = 1.553945$$
$$y(1.8) \approx y(1.6) + 0.2 \sin(1.6 \cdot y(1.6)) = 1.675822$$
$$y(2.0) \approx y(1.8) + 0.2 \sin(1.8 \cdot y(1.8)) = 1.700779$$

Repeating this with $\Delta x = 0.1$ and 0.05 gives the results in Table 11.11 below

Table 11.11

Computed Solution			
x-value	$\Delta x = 0.2$	$\Delta x = 0.1$	$\Delta x = 0.05$
1.0	1	1	1
1.1		1.084147	1.086501
1.2	1.168294	1.177079	1.181232
1.3		1.275829	1.280619
1.4	1.365450	1.375444	1.379135
1.5		1.469214	1.469885
1.6	1.553945	1.549838	1.546065
1.7		1.611296	1.602716
1.8	1.675822	1.650458	1.637809
1.9		1.667451	1.652112
2.0	1.700779	1.664795	1.648231

The computed approximations for $y(2)$ using step sizes $\Delta x = 0.2, 0.1, 0.05$ are 1.700779, 1.664795, and 1.648231, respectively. Plotting these points we see that they lie approximately on a straight line.

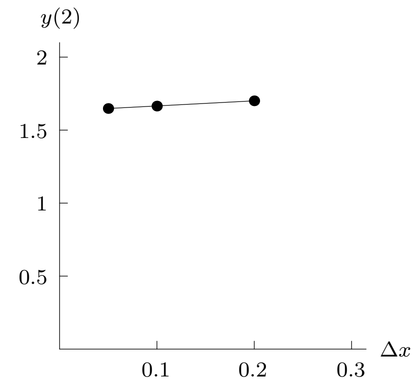


Figure 11.12

In the limit, as Δx tends to zero, the results produced by Euler's method should converge to the exact value of $y(2)$. This limiting value is the vertical intercept of the line drawn in Figure 11.12. This gives $y(2) \approx 1.632$.

9. (a) Using one step, $\frac{\Delta B}{\Delta t} = 0.05$, so $\Delta B = \left(\frac{\Delta B}{\Delta t}\right) \Delta t = 50$. Therefore we get an approximation of $B \approx 1050$ after one year.

(b) With two steps, $\Delta t = 0.5$ and we have

Table 11.12

t	B	$\Delta B = (0.05B)\Delta t$
0	1000	25
0.5	1025	25.63
1.0	1050.63	

(c) Keeping track to the nearest hundredth with $\Delta t = 0.25$, we have

Table 11.13

t	B	$\Delta B = (0.05B)\Delta t$
0	1000	12.5
0.25	1012.5	12.66
0.5	1025.16	12.81
0.75	1037.97	12.97
1	1050.94	

(d) In part (a), we get our approximation by making a single increment, ΔB, where ΔB is just $0.05B$. If we think in terms of interest, ΔB is just like getting one end of the year interest payment. Since ΔB is 0.05 times the balance B, it is like getting 5% interest at the end of the year.

(e) Part (b) is equivalent to computing the final amount in an account that begins with \$1000 and earns 5% interest compounded twice annually. Each step is like computing the interest after 6 months. When $t = 0.5$, for example, the interest is $\Delta B = (0.05B) \cdot \frac{1}{2}$, and we add this to \$1000 to get the new balance.

Similarly, part (c) is equivalent to the final amount in an account that has an initial balance of \$1000 and earns 5% interest compounded quarterly.

10. Assume that $x > 0$ and that we use n steps in Euler's method. Label the x-coordinates we use in the process $x_0, x_1, \ldots, x_n$, where $x_0 = 0$ and $x_n = x$. Then using Euler's method to find $y(x)$, we get

Table 11.14

	x	y	$\Delta y = \text{(slope)}\Delta x$
P_0	$0 = x_0$	0	$f(x_0)\Delta x$
P_1	x_1	$f(x_0)\Delta x$	$f(x_1)\Delta x$
P_2	x_2	$f(x_0)\Delta x + f(x_1)\Delta x$	$f(x_2)\Delta x$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
P_n	$x = x_n$	$\sum_{i=0}^{n-1} f(x_i)\Delta x$	

Thus the result from Euler's method is $\sum_{i=0}^{n-1} f(x_i)\Delta x$. We recognize this as the left-hand Riemann sum that approximates $\int_0^x f(t)\,dt$.

Solutions for Section 11.4

Exercises

1. Separating variables gives

$$\int \frac{1}{P} dP = -\int 2dt,$$

so

$$\ln|P| = -2t + C.$$

Therefore

$$P = \pm e^{-2t+C} = Ae^{-2t}.$$

The initial value $P(0) = 1$ gives $1 = A$, so

$$P = e^{-2t}.$$

2. Separating variables gives

$$\int \frac{dP}{P} = \int 0.02\,dt,$$

so

$$\ln|P| = 0.02t + C.$$

Thus

$$|P| = e^{0.02t+C}$$

and

$$P = Ae^{0.02t}, \text{ where } A = \pm e^C.$$

We are given $P(0) = 20$. Therefore, $P(0) = Ae^{(0.02)\cdot 0} = A = 20$. So the solution is

$$P = 20e^{0.02t}.$$

3. Separating variables and integrating both sides gives

$$\int \frac{1}{L} dL = \frac{1}{2}\int dp$$

or

$$\ln|L| = \frac{1}{2}p + C.$$

This can be written

$$L = \pm e^{(1/2)p+C} = Ae^{p/2}.$$

The initial condition $L(0) = 100$ gives $100 = A$, so

$$L = 100e^{p/2}.$$

4. Separating variables gives

$$\int \frac{dQ}{Q} = \int \frac{dt}{5},$$

so

$$\ln|Q| = \frac{1}{5}t + C.$$

So

$$|Q| = e^{\frac{1}{5}t+C} = e^{\frac{1}{5}t}e^{C}$$

and

$$Q = Ae^{\frac{1}{5}t}, \text{ where } A = \pm e^{C}.$$

From the initial conditions we know that $Q(0) = 50$, so $Q(0) = Ae^{(\frac{1}{5})\cdot 0} = A = 50$. Thus

$$Q = 50e^{\frac{1}{5}t}.$$

5. Separating variables gives

$$\int \frac{dy}{y} = -\int \frac{1}{3}\,dx$$
$$\ln|y| = -\frac{1}{3}x + C.$$

Solving for y, we have

$$y = Ae^{-\frac{1}{3}x}, \text{ where } A = \pm e^{C}.$$

Since $y(0) = A = 10$, we have

$$y = 10e^{-\frac{1}{3}x}.$$

6. Separating variables gives

$$\int P dP = \int dt$$

so that

$$\frac{P^2}{2} = t + C$$

or

$$P = \pm\sqrt{2t + D}$$

(where $D = 2C$).

The initial condition $P(0) = 1$ implies we must take the positive root and that $1 = D$, so

$$P = \sqrt{2t+1}.$$

7. Separating variables gives

$$\int \frac{dm}{m} = \int 3\,dt$$
$$\ln|m| = 3t + C$$
$$m = \pm e^{C}e^{3t} = Ae^{3t}.$$

Since $m = 5$ when $t = 1$, we have $5 = Ae^3$, so $A = 5/e^3$. Thus

$$m = \frac{5}{e^3}e^{3t} = 5e^{3t-3}.$$

8. Separating variables gives
$$\int \frac{dI}{I} = \int 0.2\,dx,$$
so
$$\ln|I| = 0.2x + C.$$
Thus,
$$I = Ae^{0.2x}, \text{ where } A = \pm e^C.$$
According to the given boundary condition, $I(-1) = 6$. Therefore, $I(-1) = Ae^{0.2(-1)} = Ae^{-0.2} = 6$, so $A = 6e^{0.2}$. Thus
$$I = 6e^{0.2}e^{0.2x} = 6e^{0.2(x+1)}.$$

9. Separating variables gives
$$\int \frac{dz}{z} = \int 5\,dt$$
$$\ln|z| = 5t + C.$$
Solving for z, we have
$$z = Ae^{5t}, \text{ where } A = \pm e^C.$$
Using the fact that $z(1) = 5$, we have $z(1) = Ae^5 = 5$, so $A = 5/e^5$. Therefore,
$$z = \frac{5}{e^5}e^{5t} = 5e^{5t-5}.$$

10. Separating variables gives
$$\int \frac{1}{m}dm = \int ds.$$
Hence
$$\ln|m| = s + C$$
which gives
$$m = \pm e^{s+C} = Ae^s.$$
The initial condition $m(1) = 2$ gives $2 = Ae^1$ or $A = 2/e$, so
$$m = \frac{2}{e}e^s = 2e^{s-1}.$$

11. Separating variables gives
$$\int \frac{1}{u^2}du = \int \frac{1}{2}dt$$
or
$$-\frac{1}{u} = \frac{1}{2}t + C.$$
The initial condition gives $C = -1$ and so
$$u = \frac{1}{1 - (1/2)t}.$$

12. Separating variables and integrating gives
$$\int \frac{1}{z}dz = \int y dy$$
which gives
$$\ln|z| = \frac{1}{2}y^2 + C$$
or
$$z = \pm e^{(1/2)y^2 + C} = Ae^{y^2/2}.$$
The initial condition $y = 0$, $z = 1$ gives $A = 1$. Therefore
$$z = e^{y^2/2}.$$

13. Separating variables gives

$$\begin{aligned}\int \frac{dy}{y-200} &= \int 0.5dt \\ \ln|y-200| &= 0.5t + C \\ y &= 200 + Ae^{0.5t}, \quad \text{where } A = \pm e^C.\end{aligned}$$

The initial condition, $y(0) = 50$, gives

$$50 = 200 + A, \quad \text{so} \quad A = -150.$$

Thus,

$$y = 200 - 150e^{0.5t}.$$

14. Separating variables gives

$$\int \frac{dP}{P+4} = \int dt,$$

so

$$\begin{aligned}\ln|P+4| &= t + C \\ P + 4 &= Ae^t \\ P &= Ae^t - 4.\end{aligned}$$

Since $P = 100$ when $t = 0$, we have $P(0) = Ae^0 - 4 = 100$, and $A = 104$. Therefore

$$P = 104e^t - 4.$$

15. Factoring out a 2 on the right makes the integration easier:

$$\begin{aligned}\frac{dy}{dx} &= 2y - 4 = 2(y-2) \\ \int \frac{dy}{y-2} &= \int 2\,dx,\end{aligned}$$

giving

$$\ln|y-2| = 2x + C.$$

Thus,

$$|y-2| = e^{2x+C},$$

so

$$y - 2 = Ae^{2x}, \text{ where } A = \pm e^C.$$

The curve passes through $(2, 5)$, which means $3 = Ae^4$, so $A = 3/e^4$. Thus,

$$y = 2 + \frac{3}{e^4}e^{2x} = 2 + 3e^{2x-4}.$$

16. Factoring and separating variables gives

$$\begin{aligned}\frac{dQ}{dt} &= 0.3(Q-400) \\ \int \frac{dQ}{Q-400} &= \int 0.3\,dt \\ \ln|Q-400| &= 0.3t + C \\ Q &= 400 + Ae^{0.3t}, \quad \text{where } A = \pm e^C.\end{aligned}$$

The initial condition, $Q(0) = 50$, gives

$$50 = 400 + A \quad \text{so} \quad A = -350.$$

Thus

$$Q = 400 - 350e^{0.3t}.$$

17. Factoring out the 0.1 gives

$$\frac{dm}{dt} = 0.1m + 200 = 0.1(m + 2000)$$

$$\int \frac{dm}{m + 2000} = \int 0.1\, dt,$$

so

$$\ln|m + 2000| = 0.1t + C,$$

and

$$m = Ae^{0.1t} - 2000, \text{ where } A = \pm e^C.$$

Using the initial condition, $m(0) = Ae^{(0.1)\cdot 0} - 2000 = 1000$, gives $A = 3000$. Thus

$$m = 3000e^{0.1t} - 2000.$$

18. Rearrange and write

$$\int \frac{1}{1 - R} dR = \int dy$$

or

$$-\ln|1 - R| = y + C$$

which can be written as

$$1 - R = \pm e^{-C-y} = Ae^{-y}$$

or

$$R = 1 - Ae^{-y}.$$

The initial condition $R(1) = 0.1$ gives $0.1 = 1 - Ae^{-1}$ and so

$$A = 0.9e.$$

Therefore

$$R = 1 - 0.9e^{1-y}.$$

19. Rewriting gives

$$\frac{dB}{dt} + 2B = 50$$

and

$$\frac{dB}{dt} = -2B + 50 = -2(B - 25),$$

so

$$\int \frac{dB}{B - 25} = -\int 2\, dt$$

$$\ln|B - 25| = -2t + C.$$

Thus, we have

$$B - 25 = Ae^{-2t}, \text{ where } A = \pm e^C.$$

Using the initial condition, $B(1) = 100$, we have $75 = Ae^{-2}$, so $A = 75e^2$. Thus

$$B = 25 + 75e^2e^{-2t} = 25 + 75e^{2-2t}.$$

20. Write

$$\int \frac{1}{y} dy = \int \frac{1}{3 + t} dt$$

and so

$$\ln|y| = \ln|3 + t| + C$$

or

$$\ln|y| = \ln D|3 + t|$$

where $\ln D = C$. Therefore

$$y = D(3 + t).$$

The initial condition $y(0) = 1$ gives $D = \frac{1}{3}$, so

$$y = \frac{1}{3}(3 + t).$$

21. Separating variables gives

$$\begin{aligned}\frac{dz}{dt} &= te^z \\ e^{-z}dz &= tdt \\ \int e^{-z}\,dz &= \int t\,dt,\end{aligned}$$

so

$$-e^{-z} = \frac{t^2}{2} + C.$$

Since the solution passes through the origin, $z = 0$ when $t = 0$, we must have

$$-e^{-0} = \frac{0}{2} + C, \text{ so } C = -1.$$

Thus

$$-e^{-z} = \frac{t^2}{2} - 1,$$

or

$$z = -\ln\left(1 - \frac{t^2}{2}\right).$$

22. Separating variables gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{5y}{x} \\ \int \frac{dy}{y} &= \int \frac{5}{x}dx \\ \ln|y| &= 5\ln|x| + C.\end{aligned}$$

Thus

$$|y| = e^{5\ln|x|}e^C = e^C e^{\ln|x|^5} = e^C|x|^5,$$

giving

$$y = Ax^5, \quad \text{where} \quad A = \pm e^C.$$

Since $y = 3$ when $x = 1$, so $A = 3$. Thus

$$y = 3x^5.$$

23. Separating variables gives

$$\begin{aligned}\frac{dy}{dt} &= y^2(1+t) \\ \int \frac{dy}{y^2} &= \int (1+t)\,dt,\end{aligned}$$

so

$$-\frac{1}{y} = t + \frac{t^2}{2} + C,$$

giving

$$y = -\frac{1}{t + t^2/2 + C}.$$

Since $y = 2$ when $t = 1$, we have

$$2 = -\frac{1}{1 + 1/2 + C}, \quad \text{so} \quad 2C + 3 = -1, \quad \text{and} \quad C = -2.$$

Thus

$$y = -\frac{1}{t^2/2 + t - 2} = -\frac{2}{t^2 + 2t - 4}.$$

24. Separating variables gives

$$\frac{dz}{dt} = z + zt^2 = z(1+t^2)$$
$$\int \frac{dz}{z} = \int (1+t^2)dt,$$

so

$$\ln|z| = t + \frac{t^3}{3} + C,$$

giving

$$z = Ae^{t+t^3/3}.$$

We have $z = 5$ when $t = 0$, so $A = 5$ and

$$z = 5e^{t+t^3/3}.$$

25. Separating variables gives

$$\frac{dw}{d\theta} = \theta w^2 \sin\theta^2$$
$$\int \frac{dw}{w^2} = \int \theta \sin\theta^2 \, d\theta,$$

so

$$-\frac{1}{w} = -\frac{1}{2}\cos\theta^2 + C.$$

According to the initial conditions, $w(0) = 1$, so $-1 = -\frac{1}{2} + C$ and $C = -\frac{1}{2}$. Thus,

$$-\frac{1}{w} = -\frac{1}{2}\cos\theta^2 - \frac{1}{2}$$
$$\frac{1}{w} = \frac{\cos\theta^2 + 1}{2}$$
$$w = \frac{2}{\cos\theta^2 + 1}.$$

26. Separating variables and integrating gives

$$\int \frac{1}{w^2} dw = -\int \tan\psi d\psi.$$

To integrate the right side, write $\tan\psi = \sin\psi/\cos\psi$ and use the substitution $w = \cos\psi$ giving

$$-\frac{1}{w} = \ln|\cos\psi| + C$$

so

$$w = \frac{-1}{\ln|\cos\psi| + C}.$$

Using the initial condition $w(0) = 2$ we have

$$2 = \frac{-1}{\ln|\cos 0| + C} = \frac{-1}{\ln 1 + C} = \frac{-1}{0 + C}$$

so

$$C = -\frac{1}{2}.$$

Thus the solution is

$$w = \frac{-1}{\ln|\cos\psi| - 1/2}.$$

27. Separating variables gives

$$x(x+1)\frac{du}{dx} = u^2$$
$$\int \frac{du}{u^2} = \int \frac{dx}{x(x+1)} = \int \left(\frac{1}{x} - \frac{1}{1+x}\right) dx,$$

so

$$-\frac{1}{u} = \ln|x| - \ln|x+1| + C.$$

We have $u(1) = 1$, so $-\frac{1}{1} = \ln|1| - \ln|1+1| + C$. So $C = \ln 2 - 1$. Solving for u yields

$$-\frac{1}{u} = \ln|x| - \ln|x+1| + \ln 2 - 1 = \ln\frac{2|x|}{|x+1|} - 1,$$

so

$$u = \frac{-1}{\ln|\frac{2x}{x+1}| - 1}.$$

28.

(a) Yes	**(b)** No	**(c)** Yes
(d) No	**(e)** Yes	**(f)** Yes
(g) No	**(h)** Yes	**(i)** No
(j) Yes	**(k)** Yes	**(l)** No

Problems

29. Separating variables gives

$$\int \frac{dR}{R} = \int k\,dt.$$

Integrating gives

$$\ln|R| = kt + C,$$

so

$$|R| = e^{kt+C} = e^{kt}e^C$$
$$R = Ae^{kt}, \quad \text{where } A = \pm e^C \quad \text{or} \quad A = 0.$$

30. Separating variables gives

$$\frac{dQ}{dt} = \frac{Q}{k}$$
$$\int \frac{dQ}{Q} = \int \frac{1}{k}\,dt$$

Integrating gives

$$\ln|Q| = \frac{t}{k} + C$$
$$Q = Ae^{t/k}, \quad \text{where } A = \pm e^C \quad \text{or} \quad A = 0.$$

31. Separating variables gives

$$\int \frac{dP}{P-a} = \int dt.$$

Integrating yields

$$\ln|P-a| = t + C,$$

so

$$|P-a| = e^{t+C} = e^te^C$$
$$P = = a + Ae^t, \quad \text{where } A = \pm e^C \quad \text{or } A = 0.$$

32. Separating variables gives

$$\int \frac{dQ}{b-Q} = \int dt.$$

Integrating yields

$$-\ln|b-Q| = t + C,$$

so

$$\begin{aligned} |b-Q| &= e^{-(t+C)} = e^{-t}e^{-C} \\ Q &= b - Ae^{-t}, \quad \text{where } A = \pm e^{-C} \quad \text{or } A = 0. \end{aligned}$$

33. Separating variables gives

$$\int \frac{dP}{P-a} = \int k\,dt.$$

Integrating yields

$$\ln|P-a| = kt + C,$$

so

$$P = a + Ae^{kt} \quad \text{where } A = \pm e^{C} \quad \text{or } A = 0.$$

34. Factoring and separating variables gives

$$\begin{aligned} \frac{dR}{dt} &= a\left(R + \frac{b}{a}\right) \\ \int \frac{dR}{R + b/a} &= \int a\,dt \\ \ln\left|R + \frac{b}{a}\right| &= at + C \\ R &= -\frac{b}{a} + Ae^{at}, \quad \text{where } A \text{ can be any constant.} \end{aligned}$$

35. Separating variables and integrating gives

$$\int \frac{1}{aP+b}dP = \int dt.$$

This gives

$$\begin{aligned} \frac{1}{a}\ln|aP+b| &= t + C \\ \ln|aP+b| &= at + aC \\ aP + b &= \pm e^{at+aC} = Ae^{at}, \quad \text{where } A = \pm e^{aC} \quad \text{or} \quad A = 0, \end{aligned}$$

or

$$P = \frac{1}{a}(Ae^{at} - b).$$

36. Separating variables and integrating gives

$$\int \frac{1}{y^2}dy = \int k(1+t^2)dt$$

or

$$-\frac{1}{y} = k\left(t + \frac{1}{3}t^3\right) + C.$$

Hence,

$$y = \frac{-1}{k(t + \frac{1}{3}t^3) + C}.$$

37. Separating variables and integrating gives

$$\int \frac{1}{R^2+1}dR = \int a dx$$

or

$$\arctan R = ax + C$$

so that

$$R = \tan(ax + C).$$

38. Separating variables and integrating gives

$$\int \frac{1}{L-b}dL = \int k(x+a)dx$$

or

$$\ln|L-b| = k\left(\frac{1}{2}x^2 + ax\right) + C.$$

Solving for L gives

$$L = b + Ae^{k(\frac{1}{2}x^2+ax)}, \quad \text{where } A \text{ can be any constant.}$$

39. Separating variables gives

$$\frac{dy}{dt} = y(2-y),$$

so

$$\int \frac{dy}{y(y-2)} = -\int dt,$$

so

$$-\frac{1}{2}\int\left(\frac{1}{y} - \frac{1}{y-2}\right)dy = -\int dt.$$

Integrating yields

$$\frac{1}{2}(\ln|y-2| - \ln|y|) = -t + C,$$

so

$$\ln\frac{|y-2|}{|y|} = -2t + 2C.$$

Exponentiating both sides yields

$$\begin{aligned}\left|1 - \frac{2}{y}\right| &= e^{-2t+2C}\\ \frac{2}{y} &= 1 - Ae^{-2t}, \quad \text{where } A = \pm e^{2C}\\ y &= \frac{2}{1-Ae^{-2t}}.\end{aligned}$$

But

$$y(0) = \frac{2}{1-A} = 1,$$

so $A = -1$, and

$$y = \frac{2}{1+e^{-2t}}.$$

40. Separating variables gives

$$\begin{aligned}t\frac{dx}{dt} &= (1 + 2\ln t)\tan x\\ \frac{dx}{\tan x} &= \left(\frac{1+2\ln t}{t}\right)dt\\ \int\frac{\cos x}{\sin x}\,dx &= \int\left(\frac{1}{t} + \frac{2\ln t}{t}\right)dt.\end{aligned}$$

Integrating gives

$$\begin{aligned}\ln|\sin x| &= \ln t + (\ln t)^2 + C\\ |\sin x| &= e^{\ln t + (\ln t)^2 + C} = t(e^{\ln t})^{\ln t}e^C = t(t^{\ln t})e^C.\end{aligned}$$

So

$$\sin x = At^{(\ln t)+1}, \quad \text{where } A = \pm e^C \quad \text{or} \quad A = 0.$$

Therefore

$$x = \arcsin(At^{(\ln t)+1}).$$

41. Separating variables gives

$$\frac{dx}{dt} = \frac{x\ln x}{t},$$

so

$$\int \frac{dx}{x\ln x} = \int \frac{dt}{t},$$

and thus

$$\ln|\ln x| = \ln t + C,$$

so

$$|\ln x| = e^C e^{\ln t} = e^C t.$$

Therefore

$$\ln x = At, \quad \text{where } A = \pm e^C \quad \text{or} \quad A = 0, \quad \text{so} \quad x = e^{At}.$$

42. Since

$$\frac{dy}{dt} = -y\ln\left(\frac{y}{2}\right),$$

we have

$$\frac{dy}{y\ln(y/2)} = -\,dt,$$

so that

$$\int \frac{dy}{y\ln(y/2)} = \int (-\,dt).$$

Substituting $w = \ln(y/2),\ dw = \dfrac{1}{y}\,dy$ gives:

$$\int \frac{dw}{w} = -\int\,dt$$

so

$$\begin{aligned} \ln|w| &= -t + C \\ \ln\left|\ln\left(\frac{y}{2}\right)\right| &= -t + C. \end{aligned}$$

Since $y(0) = 1$, we have $C = \ln|\ln\frac{1}{2}| = \ln(|-\ln 2|) = \ln(\ln 2)$. Thus

$$\ln\left|\ln\left(\frac{y}{2}\right)\right| = -t + \ln(\ln 2),$$

or

$$\left|\ln\left(\frac{y}{2}\right)\right| = e^{-t+\ln(\ln 2)}$$

Since $e^{\ln(\ln 2)} = \ln 2$, this simplifies to

$$\left|\ln\left(\frac{y}{2}\right)\right| = (\ln 2)e^{-t},$$

so

$$\ln\left(\frac{y}{2}\right) = \pm(\ln 2)e^{-t}.$$

Since $y(0) = 1$, and $\ln(1/2) = -\ln 2$, we take the $-$ sign, giving

$$\ln\left(\frac{y}{2}\right) = -(\ln 2)e^{-t}.$$

Thus,

$$\begin{aligned} y &= 2e^{-(\ln 2)e^{-t}} \\ y &= 2(e^{-\ln 2})^{e^{-t}} = 2(2^{-1})^{e^{-t}} \\ y &= 2(2^{-e^{-t}}). \end{aligned}$$

(Note that $\ln(y/2) = (\ln 2)e^{-t}$ does not satisfy $y(0) = 1$.)

43. (a) Separating variables and integrating gives

$$\int \frac{1}{100-y}dy = \int dt$$

so that

$$-\ln|100-y| = t + C$$

or

$$y(t) = 100 - Ae^{-t}.$$

(b) See Figure 11.13.

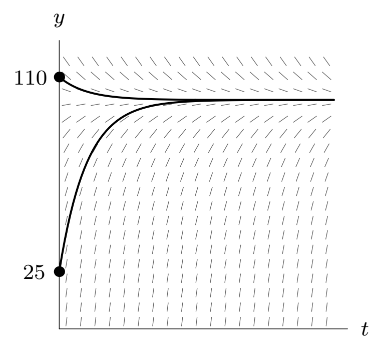


Figure 11.13

(c) The initial condition $y(0) = 25$ gives $A = 75$, so the solution is

$$y(t) = 100 - 75e^{-t}.$$

The initial condition $y(0) = 110$ gives $A = -10$ so the solution is

$$y(t) = 100 + 10e^{-t}.$$

(d) The increasing function, $y(t) = 100 - 75e^{-t}$.

44. (a) The slope field for $dy/dx = xy$ is in Figure 11.14.

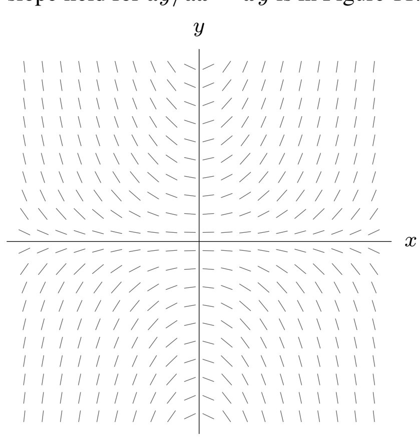


Figure 11.14

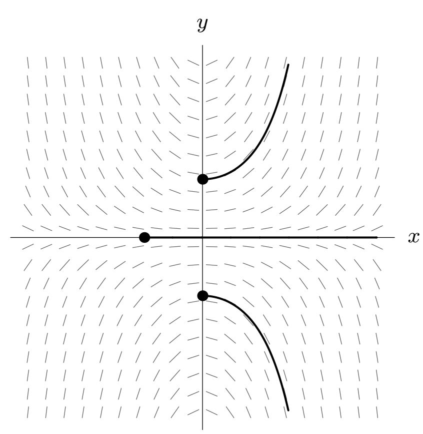


Figure 11.15

(b) Some solution curves are shown in Figure 11.15.

(c) Separating variables gives

$$\int \frac{1}{y}dy = \int xdx$$

or

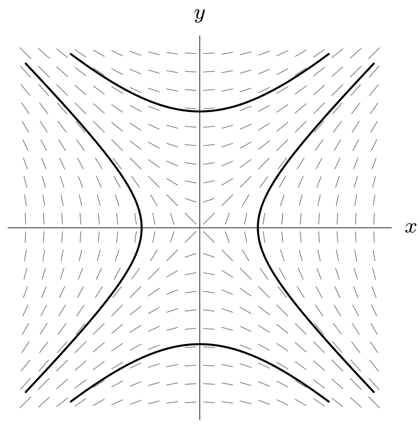
$$\ln|y| = \frac{1}{2}x^2 + C.$$

Solving for y gives

$$y(x) = Ae^{x^2/2}$$

where $A = \pm e^C$. In addition, $y(x) = 0$ is a solution. So $y(x) = Ae^{x^2/2}$ is a solution for any A.

45. (a), (b)



(c) Since $dy/dx = x/y$, we have

$$\int y\,dy = \int x\,dx,$$

and thus

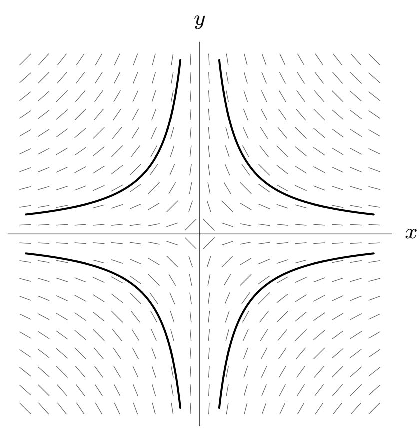
$$\frac{y^2}{2} = \frac{x^2}{2} + C,$$

or

$$y^2 - x^2 = 2C.$$

This is the equation of the hyperbolas in part (b).

46. (a), (b)



(c) Since

$$\frac{dy}{dx} = -\frac{y}{x},$$

we have

$$\int \frac{dy}{y} = -\int \frac{dx}{x},$$

so

$$\ln|y| = -\ln|x| + C,$$

giving

$$|y| = e^{-\ln|x|+C} = (|x|)^{-1}e^{C}.$$

Thus,

$$y = \frac{A}{x}, \quad \text{where } A = \pm e^{C} \quad \text{or} \quad A = 0.$$

47. By looking at the slope fields, we see that any solution curve of $y' = x/y$ intersects any solution curve to $y' = -y/x$. Now if the two curves intersect at (x, y), then the two slopes at (x, y) are negative reciprocals of each other, because

$$-\frac{1}{x/y} = -\frac{y}{x}.$$

Hence, the two curves intersect at right angles.

Solutions for Section 11.5

Exercises

1. (a) (I)
(b) (IV)
(c) (II) and (IV)
(d) (II) and (III)

2. (a) = (I), (b) = (IV), (c) = (III). Graph (II) represents an egg originally at 0° C which is moved to the kitchen table (20° C) two minutes after the egg in part (a) is moved.

3. (a) Separating variables, we have $\frac{dH}{H-200} = -k\,dt$, so $\int \frac{dH}{H-200} = \int -k\,dt$, whence $\ln|H - 200| = -kt + C$, and $H - 200 = Ae^{-kt}$, where $A = \pm e^C$. The initial condition is that the yam is $20°C$ at the time $t = 0$. Thus $20 - 200 = A$, so $A = -180$. Thus $H = 200 - 180e^{-kt}$.
(b) Using part (a), we have $120 = 200 - 180e^{-k(30)}$. Solving for k, we have $e^{-30k} = \frac{-80}{-180}$, giving

$$k = \frac{\ln\frac{4}{9}}{-30} \approx 0.027.$$

Note that this k is correct if t is given in *minutes*. (If t is given in hours, $k = \frac{\ln\frac{4}{9}}{-\frac{1}{2}} \approx 1.62$.)

4. (a) To find the equilibrium solutions, we must set

$$dy/dx = 0.5y(y - 4)(2 + y) = 0$$

which gives three solutions: $y = 0$, $y = 4$, and $y = -2$.
(b) From Figure 11.16, we see that $y = 0$ is stable and $y = 4$ and $y = -2$ are both unstable.

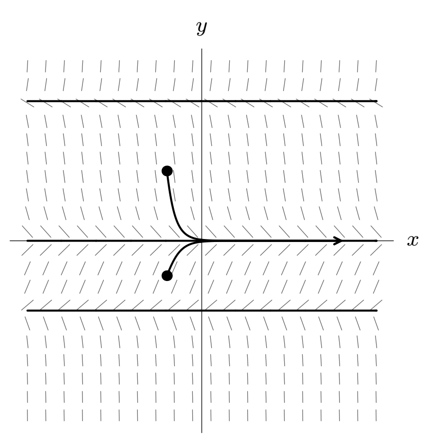


Figure 11.16

5. (a) The equilibrium solutions occur where the slope $y' = 0$, which occurs on the slope field where the lines are horizontal, or (looking at the equation) at $y = 2$ and $y = -1$. Looking at the slope field, we can see that $y = 2$ is stable, since the slopes at nearby values of y point toward it, whereas $y = -1$ is unstable.

(b) Draw solution curves passing through the given points by starting at these points and following the flow of the slopes, as shown in Figure 11.17.

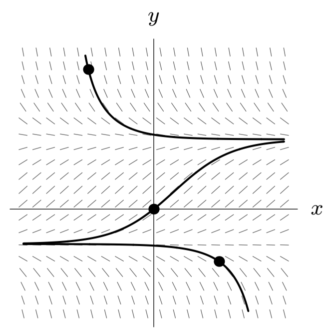


Figure 11.17

6. The equilibrium solutions of a differential equation are those functions satisfying the differential equation whose derivative is everywhere 0. Graphically, this means that a function is an equilibrium solution if it is a horizontal line that lies on the slope field. Looking at the figure in the problem, it appears that the equilibrium solutions for this problem are at $y = 1$ and $y = 3$. An equilibrium solution is stable if a small change in the initial value conditions gives a solution which tends toward equilibrium as $t \to \infty$. we see that $y = 3$ is a stable solution, while $y = 1$ is an unstable solution. See Figure 11.18.

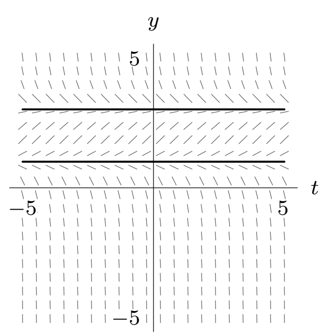


Figure 11.18

Problems

7. (a) Since the growth rate of the tumor is proportional to its size, we should have

$$\frac{dS}{dt} = kS.$$

(b) We can solve this differential equation by separating variables and then integrating:

$$\begin{aligned} \int \frac{dS}{S} &= \int k\,dt \\ \ln|S| &= kt + B \\ S &= Ce^{kt}. \end{aligned}$$

(c) This information is enough to allow us to solve for C:

$$5 = Ce^{0t}$$
$$C = 5.$$

(d) Knowing that $C = 5$, this second piece of information allows us to solve for k:

$$8 = 5e^{3k}$$
$$k = \frac{1}{3}\ln\left(\frac{8}{5}\right) \approx 0.1567.$$

So the tumor's size is given by

$$S = 5e^{0.1567t}.$$

8. (a) The rate of growth of the money in the account is proportional to the amount of money in the account. Thus

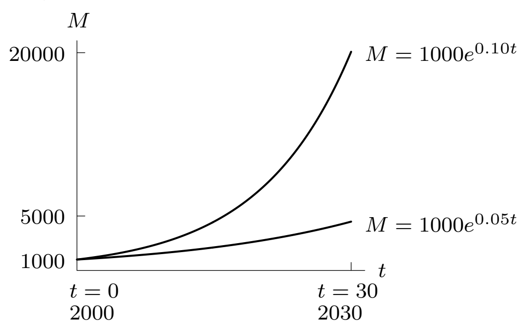
$$\frac{dM}{dt} = rM.$$

(b) Solving, we have $dM/M = r\,dt$.

$$\int \frac{dM}{M} = \int r\,dt$$
$$\ln|M| = rt + C$$
$$M = e^{rt+C} = Ae^{rt}, \qquad A = e^C.$$

When $t = 0$ (in 2000), $M = 1000$, so $A = 1000$ and $M = 1000e^{rt}$.

(c)



9. (a) Since we are told that the rate at which the quantity of the drug decreases is proportional to the amount of the drug left in the body, we know the differential equation modeling this situation is

$$\frac{dQ}{dt} = kQ.$$

Since we are told that the quantity of the drug is decreasing, we know that $k < 0$.

(b) We know that the general solution to the differential equation

$$\frac{dQ}{dt} = kQ$$

is

$$Q = Ce^{kt}.$$

(c) We are told that the half life of the drug is 3.8 hours. This means that at $t = 3.8$, the amount of the drug in the body is half the amount that was in the body at $t = 0$, or, in other words,

$$0.5Q(0) = Q(3.8).$$

Solving this equation gives

$$\begin{aligned}0.5Q(0) &= Q(3.8)\\ 0.5Ce^{k(0)} &= Ce^{k(3.8)}\\ 0.5C &= Ce^{k(3.8)}\\ 0.5 &= e^{k(3.8)}\\ \ln(0.5) &= k(3.8)\\ \frac{\ln(0.5)}{3.8} &= k\\ k &\approx -0.182.\end{aligned}$$

(d) From part (c) we know that the formula for Q is

$$Q = Ce^{-0.182t}.$$

We are told that initially there are 10 mg of the drug in the body. Thus at $t = 0$, we get

$$10 = Ce^{-0.182(0)}$$

so

$$C = 10.$$

Thus our equation becomes

$$Q(t) = 10e^{-0.182t}.$$

Substituting $t = 12$, we get

$$\begin{aligned}Q(t) &= 10e^{-0.182t}\\ Q(12) &= 10e^{-0.182(12)}\\ &= 10e^{-2.184}\\ Q(12) &\approx 1.126 \text{ mg}.\end{aligned}$$

10. (a) A very hot cup of coffee cools faster than one near room temperature. The differential equation given says that the rate at which the coffee cools is proportional to the difference between the temperature of the surrounding air and the temperature of the coffee. Since $\frac{dT}{dt} < 0$ (the coffee is cooling) and $T - 20 > 0$ (the coffee is warmer than room temperature), k must be positive.

(b) Separating variables gives

$$\int \frac{1}{T-20}dT = \int -kdt$$

and so

$$\ln|T-20| = -kt + C$$

and

$$T(t) = 20 + Ae^{-kt}.$$

If the coffee is initially boiling (100° C), then $A = 80$ and so

$$T(t) = 20 + 80e^{-kt}.$$

When $t = 2$, the coffee is at $90°C$ and so $90 = 20 + 80e^{-2k}$ so that $k = \frac{1}{2}\ln\frac{8}{7}$.
Let the time when the coffee reaches $60°C$ be T_d, so that

$$60 = 20 + 80e^{-kT_d}$$

$$e^{-kT_d} = \frac{1}{2}.$$

Therefore, $T_d = \frac{1}{k}\ln 2 = \frac{2\ln 2}{\ln\frac{8}{7}} \approx 10$ minutes.

11. (a) Letting k be the constant of proportionality, by Newton's Law of Cooling, we have

$$\frac{dH}{dt} = k(68 - H).$$

(b) We solve this equation by separating variables:

$$\begin{aligned} \int \frac{dH}{68 - H} &= \int k\, dt \\ -\ln|68 - H| &= kt + C \\ 68 - H &= \pm e^{C-kt} \\ H &= 68 - Ae^{-kt}. \end{aligned}$$

(c) We are told that $H = 40$ when $t = 0$; this tells us that

$$\begin{aligned} 40 &= 68 - Ae^{-k(0)} \\ 40 &= 68 - A \\ A &= 28. \end{aligned}$$

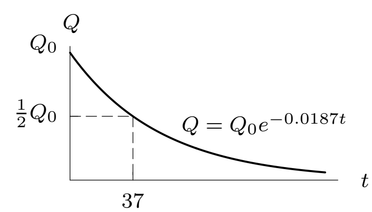
Knowing A, we can solve for k using the fact that $H = 48$ when $t = 1$:

$$\begin{aligned} 48 &= 68 - 28e^{-k(1)} \\ \frac{20}{28} &= e^{-k} \\ k &= -\ln\left(\frac{20}{28}\right) = 0.33647. \end{aligned}$$

So the formula is $H(t) = 68 - 28e^{-0.33647t}$. We calculate H when $t = 3$, by

$$H(3) = 68 - 28e^{-0.33647(3)} = 57.8^\circ\text{F}.$$

12. (a)



(b) $\dfrac{dQ}{dt} = -kQ$

(c) Since $25\% = 1/4$, it takes two half-lives $= 74$ hours for the drug level to be reduced to 25%. Alternatively, $Q = Q_0e^{-kt}$ and $\frac{1}{2} = e^{-k(37)}$, we have

$$k = -\frac{\ln(1/2)}{37} \approx 0.0187.$$

Therefore $Q = Q_0e^{-0.0187t}$. We know that when the drug level is 25% of the original level that $Q = 0.25Q_0$. Setting these equal, we get

$$0.25 = e^{-0.0187t}.$$

giving

$$t = -\frac{\ln(0.25)}{0.0187} \approx 74 \text{ hours} \approx 3 \text{ days}.$$

13. According to Newton's Law of Cooling, the temperature, T, of the roast as a function of time, t, satisfies

$$\begin{aligned} T'(t) &= k(350 - T) \\ T(0) &= 40. \end{aligned}$$

Solving this differential equation, we get that $T = 350 - 310e^{-kt}$ for some $k > 0$. To find k, we note that at $t = 1$ we have $T = 90$, so

$$\begin{aligned} 90 &= 350 - 310e^{-k(1)} \\ \frac{260}{310} &= e^{-k} \\ k &= -\ln\left(\frac{260}{310}\right) \\ &\approx 0.17589. \end{aligned}$$

Thus, $T = 350 - 310e^{-0.17589t}$. Solving for t when $T = 140$, we have

$$\begin{aligned} 140 &= 350 - 310e^{-0.17589t} \\ \frac{210}{310} &= e^{-0.17589t} \\ t &= \frac{\ln(210/310)}{-0.17589} \\ t &\approx 2.21 \text{ hours.} \end{aligned}$$

14. (a) Since the amount leaving the blood is proportional to the quantity in the blood,

$$\frac{dQ}{dt} = -kQ \quad \text{for some positive constant } k.$$

Thus $Q = Q_0e^{-kt}$, where Q_0 is the initial quantity in the bloodstream. Only 20% is left in the blood after 3 hours. Thus $0.20 = e^{-3k}$, so $k = \frac{\ln 0.20}{-3} \approx 0.5365$. Therefore $Q = Q_0e^{-0.5365t}$.

(b) Since 20% is left after 3 hours, after 6 hours only 20% of that 20% will be left. Thus after 6 hours only 4% will be left, so if the patient is given 100 mg, only 4 mg will be left 6 hours later.

15. (a) We know that the rate at which morphine leaves the body is proportional to the amount of morphine in the body at that particular instant. If we let Q be the amount of morphine in the body, we get that

$$\text{Rate of morphine leaving the body} = kQ,$$

where k is the rate of proportionality. The solution is $Q = Q_0e^{kt}$ (neglecting the continuously incoming morphine). Since the half-life is 2 hours, we have

$$\frac{1}{2}Q_0 = Q_0e^{k\cdot 2},$$

so

$$k = \frac{\ln(1/2)}{2} = -0.347.$$

(b) Since

$$\text{Rate of change of quantity } = \text{Rate in } - \text{Rate out},$$

we have

$$\frac{dQ}{dt} = -0.347Q + 2.5.$$

(c) Equilibrium occurs when $dQ/dt = 0$, that is, when $0.347Q = 2.5$ or $Q = 7.2$ mg.

16. Since it takes 6 years to reduce the pollution to 10%, another 6 years would reduce the pollution to 10% of 10%, which is equivalent to 1% of the original. Therefore it takes 12 years for 99% of the pollution to be removed. (Note that the value of Q_0 does not affect this.) Thus the second time is double the first because the fraction remaining, 0.01, in the second instance is the square of the fraction remaining, 0.1, in the first instance.

17. Michigan:

$$\frac{dQ}{dt} = -\frac{r}{V}Q = -\frac{158}{4.9 \times 10^3}Q \approx -0.032Q$$

so

$$Q = Q_0e^{-0.032t}.$$

We want to find t such that

$$0.1Q_0 = Q_0e^{-0.032t}$$

so

$$t = \frac{-\ln(0.1)}{0.032} \approx 72 \text{ years.}$$

Ontario:

$$\frac{dQ}{dt} = -\frac{r}{V}Q = \frac{-209}{1.6 \times 10^3}Q = -0.131Q$$

so

$$Q = Q_0 e^{-0.131t}.$$

We want to find t such that

$$0.1Q_0 = Q_0 e^{-0.131t}$$

so

$$t = \frac{-\ln(0.1)}{0.131} \approx 18 \text{ years.}$$

Lake Michigan will take longer because it is larger (4900 km^3 compared to 1600 km^3) and water is flowing through it at a slower rate (158 km^3/year compared to 209 km^3/year).

18. Lake Superior will take the longest, because the lake is largest (V is largest) and water is moving through it most slowly (r is smallest). Lake Erie looks as though it will take the least time because V is smallest and r is close to the largest. For Erie, $k = r/V = 175/460 \approx 0.38$. The lake with the largest value of r is Ontario, where $k = r/V = 209/1600 \approx 0.13$. Since e^{-kt} decreases faster for larger k, Lake Erie will take the shortest time for any fixed fraction of the pollution to be removed.

For Lake Superior,

$$\frac{dQ}{dt} = -\frac{r}{V}Q = -\frac{65.2}{12{,}200}Q \approx -0.0053Q$$

so

$$Q = Q_0 e^{-0.0053t}.$$

When 80% of the pollution has been removed, 20% remains so $Q = 0.2Q_0$. Substituting gives us

$$0.2Q_0 = Q_0 e^{-0.0053t}$$

so

$$t = -\frac{\ln(0.2)}{0.0053} \approx 301 \text{ years.}$$

(Note: The 301 is obtained by using the exact value of $\frac{r}{V} = \frac{65.2}{12{,}200}$, rather than 0.0053. Using 0.0053 gives 304 years.)
For Lake Erie, as in the text

$$\frac{dQ}{dt} = -\frac{r}{V}Q = -\frac{175}{460}Q \approx -0.38Q$$

so

$$Q = Q_0 e^{-0.38t}.$$

When 80% of the pollution has been removed

$$0.2Q_0 = Q_0 e^{-0.38t}$$
$$t = -\frac{\ln(0.2)}{0.38} \approx 4 \text{ years.}$$

So the ratio is

$$\frac{\text{Time for Lake Superior}}{\text{Time for Lake Erie}} \approx \frac{301}{4} \approx 75.$$

In other words it will take about 75 times as long to clean Lake Superior as Lake Erie.

19. (a) Suppose $Y(t)$ is the quantity of oil in the well at time t. We know that the oil in the well decreases at a rate proportional to $Y(t)$, so

$$\frac{dY}{dt} = -kY.$$

Integrating, and using the fact that initially $Y = Y_0 = 10^6$, we have

$$Y = Y_0 e^{-kt} = 10^6 e^{-kt}.$$

In six years, $Y = 500,000 = 5 \cdot 10^5$, so

$$5 \cdot 10^5 = 10^6 e^{-k \cdot 6}$$

so

$$\begin{aligned}0.5 &= e^{-6k}\\ k &= -\frac{\ln 0.5}{6} = 0.1155.\end{aligned}$$

When $Y = 600,000 = 6 \cdot 10^5$,

$$\text{Rate at which oil decreasing } = \left|\frac{dY}{dt}\right| = kY = 0.1155(6 \cdot 10^5) = 69{,}300 \text{ barrels/year.}$$

(b) We solve the equation

$$\begin{aligned}5 \cdot 10^4 &= 10^6 e^{-0.1155t}\\ 0.05 &= e^{-0.1155t}\\ t &= \frac{\ln 0.05}{-0.1155} = 25.9 \text{ years.}\end{aligned}$$

20. (a) Assuming that the world's population grows exponentially, satisfying $dP/dt = cP$, and that the land in use for crops is proportional to the population, we expect A to satisfy $dA/dt = kA$.

(b) We have $A(t) = A_0e^{kt} = (1\times10^9)e^{kt}$, where t is the number of years after 1950. Since $2\times10^9 = (1\times10^9)e^{k(30)}$, we have $e^{30k} = 2$, so $k = \frac{\ln 2}{30} \approx 0.023$. Thus, $A \approx (1 \times 10^9)e^{0.023t}$. We want to find t such that $3.2 \times 10^9 = A(t) = (1 \times 10^9)e^{0.023t}$. Taking logarithms yields

$$t = \frac{\ln(3.2)}{0.023} \approx 50.6 \text{ years.}$$

Thus this model predicts land will have run out by the year 2001.

21. (a) $\dfrac{dT}{dt} = -k(T - A)$, where $A = 68°$F is the temperature of the room, and t is time since 9 am.

(b)

$$\begin{aligned}\int \frac{dT}{T-A} &= -\int k\,dt\\ \ln|T-A| &= -kt + C\\ T &= A + Be^{-kt}.\end{aligned}$$

Using $A = 68$, and $T(0) = 90.3$, we get $B = 22.3$. Thus

$$T = 68 + 22.3e^{-kt}.$$

At $t = 1$, we have

$$\begin{aligned}89.0 &= 68 + 22.3e^{-k}\\ 21 &= 22.3e^{-k}\\ k &= -\ln\frac{21}{22.3} \approx 0.06.\end{aligned}$$

Thus $T = 68 + 22.3e^{-0.06t}$.

We want to know when T was equal to $98.6°$F, the temperature of a live body, so

$$\begin{aligned}98.6 &= 68 + 22.3e^{-0.06t}\\ \ln\frac{30.6}{22.3} &= -0.06t\\ t &= \left(-\frac{1}{0.06}\right)\ln\frac{30.6}{22.3}\\ t &\approx -5.27.\end{aligned}$$

The victim was killed approximately $5\frac{1}{4}$ hours prior to 9 am, at 3:45 am.

22. (a) The differential equation is

$$\frac{dT}{dt} = -k(T - A),$$

where $A = 10^\circ$F is the outside temperature.

(b) Integrating both sides yields

$$\int \frac{dT}{T - A} = -\int k\,dt.$$

Then $\ln|T - A| = -kt + C$, so $T = A + Be^{-kt}$. Thus

$$T = 10 + 58e^{-kt}.$$

Since 10:00 pm corresponds to $t = 9$,

$$\begin{aligned}
57 &= 10 + 58e^{-9k} \\
\frac{47}{58} &= e^{-9k} \\
\ln\frac{47}{58} &= -9k \\
k &= -\frac{1}{9}\ln\frac{47}{58} \approx 0.0234.
\end{aligned}$$

At 7:00 the next morning $(t = 18)$ we have

$$\begin{aligned}
T &\approx 10 + 58e^{18(-0.0234)} \\
&= 10 + 58(0.66) \\
&\approx 48^\circ\text{F},
\end{aligned}$$

so the pipes won't freeze.

(c) We assumed that the temperature outside the house stayed constant at 10°F. This is probably incorrect because the temperature was most likely warmer during the day (between 1 pm and 10 pm) and colder after (between 10 pm and 7 am). Thus, when the temperature in the house dropped from 68°F to 57°F between 1 pm and 10 pm, the outside temperature was probably higher than 10°F, which changes our calculation of the value of the constant k. The house temperature will most certainly be lower than 48°F at 7 am, but not by much—not enough to freeze.

23. (a) If $C' = -kC$, and then $C = C_0e^{-kt}$. Since the half-life is 5730 years, $\frac{1}{2}C_0 = C_0e^{-5730k}$. Solving for k, we have $-5730k = \ln(1/2)$ so $k = \frac{-\ln(1/2)}{5730} \approx 0.000121$.

(b) From the given information, we have $0.91 = e^{-kt}$, where t is the age of the shroud. Solving for t, we have $t = \frac{-\ln 0.91}{k} \approx 779.4$ years.

24. (a) Since speed is the derivative of distance, Galileo's mistaken conjecture was $\frac{dD}{dt} = kD$.

(b) We know that if Galileo's conjecture were true, then $D(t) = D_0e^{kt}$, where D_0 would be the initial distance fallen. But if we drop an object, it starts out not having traveled any distance, so $D_0 = 0$. This would lead to $D(t) = 0$ for all t.

Solutions for Section 11.6

Exercises

1. (a) If $B = f(t)$, where t is in years,

$$\begin{aligned}
\frac{dB}{dt} &= \text{Rate of money earned from interest} + \text{Rate of money deposited} \\
\frac{dB}{dt} &= 0.10B + 1000.
\end{aligned}$$

(b) We use separation of variables to solve the differential equation

$$\frac{dB}{dt} = 0.1B + 1000.$$

$$\begin{aligned}\int \frac{1}{0.1B+1000}dB &= \int dt\\ \frac{1}{0.1}\ln|0.1B+1000| &= t+C_1\\ 0.1B+1000 &= C_2e^{0.1t}\\ B &= Ce^{0.1t}-10{,}000\end{aligned}$$

For $t=0$, $B=0$, hence $C=10{,}000$. Therefore, $B=10{,}000e^{0.1t}-10{,}000$.

2. (a) There are two factors that are affecting B: the money leaving the account, which is at a constant rate of -2000 per year, and the interest accumulating in it, which accrues at a rate of $(0.08)B$. Since

$$\text{Rate of change of balance} = \text{Rate in} - \text{Rate out},$$

the differential equation for B is

$$\frac{dB}{dt} = 0.08B - 2000.$$

(b) We solve the differential equation by separating variables and then integrating:

$$\begin{aligned}\int \frac{dB}{0.08B-2000} &= \int dt\\ 12.5\ln|0.08B-2000| &= t+C\\ \ln|0.08B-2000| &= \frac{t}{12.5}+C\\ 0.08B-2000 &= \pm e^{0.08t+C}\\ B &= 25{,}000+Ae^{0.08t}.\end{aligned}$$

(c) (i) If the initial deposit is 20,000, then we have $B=20{,}000$ when $t=0$, which leads to $A=-5000$. Knowing A, we can find $B(5)$ as:

$$B(5) = 25{,}000-5000e^{0.08(5)} = \$17{,}540.88.$$

(ii) Now $B=30{,}000$ when $t=0$ leads to $A=5000$, giving $B(5)=\$32{,}459.12$.

3. (a) By Newton's Law of Cooling, we have

$$\frac{dH}{dt} = k(H-50)$$

for some k. Furthermore, we know the juice's original temperature $H(0)=90$.

(b) Separating variables, we get

$$\int \frac{dH}{(H-50)} = \int k\,dt.$$

We then integrate:

$$\begin{aligned}\ln|H-50| &= kt+C\\ H-50 &= e^{kt}\cdot A\\ H &= 50+Ae^{kt}.\end{aligned}$$

Thus, $H(0)=90$ gives $A=40$, and $H(5)=80$ gives

$$\begin{aligned}50+40e^{5k} &= 80\\ e^{5k} &= \frac{30}{40}\\ 5k &= \ln(0.75)\\ k &= \frac{1}{5}\ln(0.75)\approx -0.05754.\end{aligned}$$

Therefore

$$H(t) = 50+40e^{-0.05754t}.$$

(c) We now solve for t at which $H(t) = 60$:

$$\begin{aligned} 60 &= 50 + 40e^{-0.05754t} \\ \frac{1}{4} &= e^{-0.05754t} \\ \ln(0.25) &= -0.05754t \\ t &= 24 \text{ minutes.} \end{aligned}$$

4. Since mg is constant and $a = dv/dt$, differentiating $ma = mg - kv$ gives

$$m\frac{da}{dt} = -k\frac{dv}{dt} = -ma.$$

Thus, the differential equation is

$$\frac{da}{dt} = -\frac{k}{m}a.$$

Solving for a gives

$$a = a_0 e^{-kt/m}.$$

At $t = 0$, we have $a = g$, the acceleration due to gravity. Thus, $a_0 = g$, so

$$a = ge^{-kt/m}.$$

Problems

5. (a) Since the rate of change is proportional to the amount present, $dy/dt = ky$ for some constant k.

(b) Solving the differential equation, we have $y = Ae^{kt}$, where A is the initial amount. Since 100 grams become 54.9 grams in one hour, $54.9 = 100e^k$, so $k = \ln(54.9/100) \approx -0.5997$.
Thus, after 10 hours, there remains $100e^{(-0.5997)10} \approx 0.2486$ grams.

6. Let $C(t)$ be the current flowing in the circuit at time t, then

$$\frac{dC}{dt} = -\alpha C$$

where $\alpha > 0$ is the constant of proportionality between the rate at which the current decays and the current itself.

The general solution of this differential equation is $C(t) = Ae^{-\alpha t}$ but since $C(0) = 30$, we have that $A = 30$, and so we get the particular solution $C(t) = 30e^{-\alpha t}$.

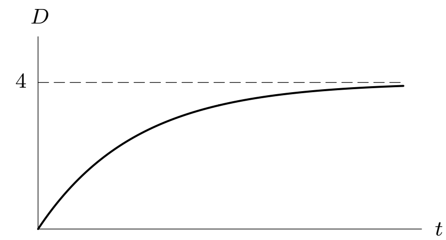
When $t = 0.01$, the current has decayed to 11 amps so that $11 = 30e^{-\alpha 0.01}$ which gives $\alpha = -100\ln(11/30) = 100.33$ so that,

$$C(t) = 30e^{-100.33t}.$$

7. Let $D(t)$ be the quantity of dead leaves, in grams per square centimeter. Then $\frac{dD}{dt} = 3 - 0.75D$, where t is in years. We factor out -0.75 and then separate variables.

$$\begin{aligned} \frac{dD}{dt} &= -0.75(D-4) \\ \int \frac{dD}{D-4} &= \int -0.75\,dt \\ \ln|D-4| &= -0.75t + C \\ |D-4| &= e^{-0.75t+C} = e^{-0.75t}e^C \\ D &= 4 + Ae^{-0.75t}, \text{ where } A = \pm e^C. \end{aligned}$$

If initially the ground is clear, the solution looks like the following graph:



The equilibrium level is 4 grams per square centimeter, regardless of the initial condition.

8. (a) If I is intensity and l is the distance traveled through the water, then for some $k > 0$,

$$\frac{dI}{dl} = -kI.$$

(The proportionality constant is negative because intensity decreases with distance). Thus $I = Ae^{-kl}$. Since $I = A$ when $l = 0$, A represents the initial intensity of the light.

(b) If 50% of the light is absorbed in 10 feet, then $0.50A = Ae^{-10k}$, so $e^{-10k} = \frac{1}{2}$, giving

$$k = \frac{-\ln\frac{1}{2}}{10} = \frac{\ln 2}{10}.$$

In 20 feet, the percentage of light left is

$$e^{-\frac{\ln 2}{10}\cdot 20} = e^{-2\ln 2} = (e^{\ln 2})^{-2} = 2^{-2} = \frac{1}{4},$$

so $\frac{3}{4}$ or 75% of the light has been absorbed. Similarly, after 25 feet,

$$e^{-\frac{\ln 2}{10}\cdot 25} = e^{-2.5\ln 2} = (e^{\ln 2})^{-\frac{5}{2}} = 2^{-\frac{5}{2}} \approx 0.177.$$

Approximately 17.7% of the light is left, so 82.3% of the light has been absorbed.

9. Let the depth of the water at time t be y. Then $\dfrac{dy}{dt} = -k\sqrt{y}$, where k is a positive constant. Separating variables,

$$\int \frac{dy}{\sqrt{y}} = -\int k\,dt,$$

so

$$2\sqrt{y} = -kt + C\,.$$

When $t = 0$, $y = 36$; $2\sqrt{36} = -k\cdot 0 + C$, so $C = 12$.
When $t = 1$, $y = 35$; $2\sqrt{35} = -k + 12$, so $k \approx 0.17$.
Thus, $2\sqrt{y} \approx -0.17t + 12$. We are looking for t such that $y = 0$; this happens when $t \approx \frac{12}{0.17} \approx 71$ hours, or about 3 days.

10. We are given that the rate of change of pressure with respect to volume, dP/dV is proportional to P/V, so that

$$\frac{dP}{dV} = k\frac{P}{V}.$$

Using separation of variables and integrating gives

$$\int \frac{dP}{P} = k\int \frac{dV}{V}.$$

Evaluating these integral gives

$$\ln P = k\ln V + c$$

or equivalently,

$$P = AV^k.$$

11. (a) Since the rate of change of the weight is equal to

$$\frac{1}{3500}(\text{Intake} - \text{Amount to maintain weight})$$

we have

$$\frac{dW}{dt} = \frac{1}{3500}(I - 20W).$$

(b) Starting off with the equation

$$\frac{dW}{dt} = -\frac{2}{350}(W - \frac{I}{20}),$$

we separate variables and integrate:

$$\int \frac{dW}{W - \frac{I}{20}} = -\int \frac{2}{350}\,dt.$$

Thus we have

$$\ln|W - \frac{I}{20}| = -\frac{2}{350}t + C$$

so that

$$W - \frac{I}{20} = Ae^{-\frac{2}{350}t}$$

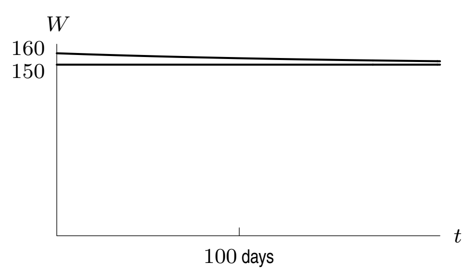
or in other words

$$W = \frac{I}{20} + Ae^{-\frac{2}{350}t}.$$

Let us call the person's initial weight W_0 at $t = 0$. Then $W_0 = \frac{I}{20} + Ce^0$, so $C = W_0 - \frac{I}{20}$. Thus

$$W = \frac{I}{20} + \left(W_0 - \frac{I}{20}\right)e^{-\frac{2}{350}t}.$$

(c) Using part (b), we have $W = 150 + 10e^{-\frac{2}{350}t}$. This means that $W \to 150$ as $t \to \infty$. See the following figure.



12. Since the rate at which the volume, V, is decreasing is proportional to the surface area, A, we have

$$\frac{dV}{dt} = -kA,$$

where the negative sign reflects the fact that V is decreasing. Suppose the radius of the sphere is r. Then $V = \frac{4}{3}\pi r^3$ and, using the chain rule, $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. The surface area of a sphere is given by $A = 4\pi r^2$. Thus

$$4\pi r^2\frac{dr}{dt} = -k4\pi r^2$$

so

$$\frac{dr}{dt} = -k.$$

Since the radius decreases from 1 cm to 0.5 cm in 1 month, we have $k = 0.5$ cm/month. Thus

$$\frac{dr}{dt} = -0.5$$

so

$$r = -0.5t + r_0.$$

Since $r = 1$ when $t = 0$, we have $r_0 = 1$, so

$$r = -0.5t + 1.$$

We want to find t when $r = 0.2$, so

$$0.2 = -0.5t + 1$$

and

$$t = \frac{0.8}{0.5} = 1.6 \text{ months}.$$

13. Let $V(t)$ be the volume of water in the tank at time t, then

$$\frac{dV}{dt} = k\sqrt{V}$$

This is a separable equation which has the solution

$$V(t) = (\frac{kt}{2} + C)^2$$

Since $V(0) = 200$ this gives $200 = C^2$ so

$$V(t) = (\frac{kt}{2} + \sqrt{200})^2.$$

However, $V(1) = 180$ therefore

$$180 = (\frac{k}{2} + \sqrt{200})^2,$$

so that $k = 2\left(\sqrt{180} - \sqrt{200}\right) = -1.45146$. Therefore,

$$V(t) = (-0.726t + \sqrt{200})^2.$$

The tank will be half-empty when $V(t) = 100$, so we solve

$$100 = (-0.726t + \sqrt{200})^2$$

to obtain $t = 5.7$ days. The tank will be half empty in 5.7 days.

The volume after 4 days is $V(4)$ which is approximately 126.32 liters.

14. (a) If P = pressure and h = height, $\frac{dP}{dh} = -3.7 \times 10^{-5}P$, so $P = P_0e^{-3.7\times10^{-5}h}$. Now $P_0 = 29.92$, since pressure at sea level (when $h = 0$) is 29.92, so $P = 29.92e^{-3.7\times10^{-5}h}$. At the top of Mt. Whitney, the pressure is

$$P = 29.92e^{-3.7\times10^{-5}(14500)} \approx 17.50 \text{ inches of mercury}.$$

At the top of Mt. Everest, the pressure is

$$P = 29.92e^{-3.7\times10^{-5}(29000)} \approx 10.23 \text{ inches of mercury}.$$

(b) The pressure is 15 inches of mercury when

$$15 = 29.92e^{-3.7\times10^{-5}h}$$

Solving for h gives $h = \frac{-1}{3.7\times10^{-5}}\ln(\frac{15}{29.92}) \approx$ 18,661.5 feet.

15. We are given that

$$BC = 2OC.$$

If the point A has coordinates (x, y) then $OC = x$ and $AC = y$. The slope of the tangent line, y', is given by

$$y' = \frac{AC}{BC} = \frac{y}{BC},$$

so

$$BC = \frac{y}{y'}.$$

Substitution into $BC = 2OC$ gives

$$\frac{y}{y'} = 2x,$$

so

$$\frac{y'}{y} = \frac{1}{2x}.$$

Separating variables to integrate this differential equation gives

$$\begin{aligned}\int \frac{dy}{y} &= \int \frac{dx}{2x} \\ \ln|y| &= \frac{1}{2}\ln|x| + C = \ln\sqrt{|x|} + \ln A \\ |y| &= A\sqrt{|x|} \\ y &= \pm(A\sqrt{x}).\end{aligned}$$

Thus, in the first quadrant, the curve has equation $y = A\sqrt{x}$.

16. (a) For this situation,

$$\begin{pmatrix}\text{Rate money added}\\ \text{to account}\end{pmatrix} = \begin{pmatrix}\text{Rate money added}\\ \text{via interest}\end{pmatrix} + \begin{pmatrix}\text{Rate money}\\ \text{deposited}\end{pmatrix}$$

Translating this into an equation yields

$$\frac{dB}{dt} = 0.1B + 1200.$$

(b) Solving this equation via separation of variables gives

$$\begin{aligned}\frac{dB}{dt} &= 0.1B + 1200 \\ &= (0.1)(B + 12000)\end{aligned}$$

So

$$\int \frac{dB}{B + 12000} = \int 0.1\,dt$$

and

$$\ln|B + 12000| = 0.1t + C$$

solving for B,

$$|B + 12000| = e^{(0.1)t+C} = e^C e^{(0.1)t}$$

or

$$B = Ae^{0.1t} - 12000, \text{ (where } A = e^c\text{)}$$

We may find A using the initial condition $B_0 = f(0) = 0$

$$A - 12000 = 0 \quad \text{or} \quad A = 12000$$

(c) After 5 years, the balance is

$$\begin{aligned}B = f(5) &= 12{,}000(e^{(0.1)(5)} - 1) \\ &\approx 7784.66 \text{ dollars.}\end{aligned}$$

17. (a) The balance in the account at the beginning of the month is given by the following sum

$$\begin{pmatrix}\text{balance in}\\ \text{account}\end{pmatrix} = \begin{pmatrix}\text{previous month's}\\ \text{balance}\end{pmatrix} + \begin{pmatrix}\text{interest on}\\ \text{previous month's balance}\end{pmatrix} + \begin{pmatrix}\text{monthly deposit}\\ \text{of \$100}\end{pmatrix}$$

Denote month i's balance by B_i. Assuming the interest is compounded continuously, we have

$$\begin{pmatrix}\text{previous month's}\\ \text{balance}\end{pmatrix} + \begin{pmatrix}\text{interest on previous}\\ \text{month's balance}\end{pmatrix} = B_{i-1}e^{0.1/12}.$$

Since the interest rate is $10\% = 0.1$ per year, interest is $\frac{0.1}{12}$ per month. So at month i, the balance is

$$B_i = B_{i-1}e^{\frac{0.1}{12}} + 100$$

Explicitly, we have for the five years (60 months) the equations:

$$\begin{aligned} B_0 &= 0 \\ B_1 &= B_0 e^{\frac{0.1}{12}} + 100 \\ B_2 &= B_1 e^{\frac{0.1}{12}} + 100 \\ B_3 &= B_2 e^{\frac{0.1}{12}} + 100 \\ &\vdots \\ B_{60} &= B_{59} e^{\frac{0.1}{12}} + 100 \end{aligned}$$

In other words,

$$\begin{aligned} B_1 &= 100 \\ B_2 &= 100e^{\frac{0.1}{12}} + 100 \\ B_3 &= (100e^{\frac{0.1}{12}} + 100)e^{\frac{0.1}{12}} + 100 \\ &= 100e^{\frac{(0.1)2}{12}} + 100e^{\frac{0.1}{12}} + 100 \\ B_4 &= 100e^{\frac{(0.1)3}{12}} + 100e^{\frac{(0.1)2}{12}} + 100e^{\frac{(0.1)}{12}} + 100 \\ &\vdots \\ B_{60} &= 100e^{\frac{(0.1)59}{12}} + 100e^{\frac{(0.1)58}{12}} + \cdots + 100e^{\frac{(0.1)1}{12}} + 100 \\ B_{60} &= \sum_{k=0}^{59} 100e^{\frac{(0.1)k}{12}} \end{aligned}$$

(b) The sum $B_{60} = \sum_{k=0}^{59} 100e^{\frac{(0.1)k}{12}}$ can be written as $B_{60} = \sum_{k=0}^{59} 1200e^{\frac{(0.1)k}{12}}(\frac{1}{12})$ which is the left Riemann sum for $\int_0^5 1200e^{0.1t}dt$, with $\Delta t = \frac{1}{12}$ and $N = 60$. Evaluating the sum on a calculator gives $B_{60} = 7752.26$.

(c) The situation described by this problem is almost the same as that in Problem 16, except that here the money is being deposited once a month rather than continuously; however the nominal yearly rates are the same. Thus we would expect the balance after 5 years to be approximately the same in each case. This means that the answer to part (b) of this problem should be approximately the same as the answer to part (c) to Problem 16. Since the deposits in this problem start at the end of the first month, as opposed to right away, we would expect the balance after 5 years to be slightly smaller than in Problem 16, as is the case.

Alternatively, we can use the Fundamental Theorem of Calculus to show that the integral can be computed exactly

$$\int_0^5 1200e^{0.1t}dt = 12000(e^{(0.1)5} - 1) = 7784.66$$

Thus $\int_0^5 1200e^{0.1t}dt$ represents the exact solution to Problem 16. Since $1200e^{0.1t}$ is an increasing function, the left hand sum we calculated in part (b) of this problem underestimates the integral. Thus the answer to part (b) of this problem should be less than the answer to part (c) of Problem 16.

18. (a) The quantity and the concentration both increase with time. As the concentration increases, the rate at which the drug is excreted also increases, and so the rate at which the drug builds up in the blood decreases; thus the graph of concentration against time is concave down. The concentration rises until the rate of excretion exactly balances the rate at which the drug is entering; at this concentration there is a horizontal asymptote. (See Figure 11.19.)

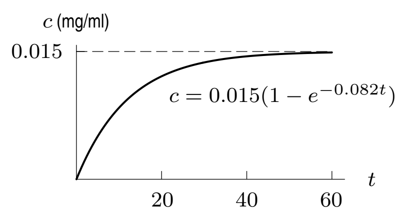


Figure 11.19

(b) Let's start by writing a differential equation for the quantity, $Q(t)$.

$$\text{Rate at which quantity of drug changes} = \text{Rate in} - \text{Rate out}$$
$$\frac{dQ}{dt} = 43.2 - 0.082Q$$

where Q is measured in mg. We want an equation for concentration $c(t) = Q(t)/v$, where $c(t)$ is measured in mg/ml and v is volume, so $v = 35,000$ ml.

$$\frac{1}{v}\frac{dQ}{dt} = \frac{43.2}{v} - 0.082\frac{Q}{v},$$

giving

$$\frac{dc}{dt} = \frac{43.2}{35,000} - 0.082c.$$

(c) Factor out -0.082 and separate variables to solve.

$$\begin{aligned}\frac{dc}{dt} &= -0.082(c - 0.015)\\ \int \frac{dc}{c - 0.015} &= -0.082\int dt\\ \ln|c - 0.015| &= -0.082t + B\\ c - 0.015 &= Ae^{-0.082t} \quad \text{where} \quad A = \pm e^B\end{aligned}$$

Since $c = 0$ when $t = 0$, we have $A = -0.015$, so

$$c = 0.015 - 0.015e^{-0.082t} = 0.015(1 - e^{-0.082t}).$$

Thus $c \to 0.015$ mg/ml as $t \to \infty$.

19. (a) $\dfrac{dy}{dt} = -k(y - a)$, where $k > 0$ and a are constants.

(b) $\displaystyle\int \frac{dy}{y - a} = \int -k\,dt$, so $\ln|y-a| = \ln(y-a) = -kt+C$. Thus, $y-a = Ae^{-kt}$ where $A = e^C$. Initially nothing has been forgotten, so $y(0) = 1$. Therefore, $1 - a = Ae^0 = A$, so $y - a = (1 - a)e^{-kt}$ or $y = (1 - a)e^{-kt} + a$.

(c) As $t \to \infty$, $e^{-kt} \to 0$, so $y \to a$.
Thus, a represents the fraction of material which is remembered in the long run. The constant k tells us about the rate at which material is forgotten.

20. (a) We have

$$\frac{dp}{dt} = -k(p - p^*),$$

where k is constant. Notice that $k > 0$, since if $p > p^*$ then dp/dt should be negative, and if $p < p^*$ then dp/dt should be positive.

(b) Separating variables, we have

$$\int \frac{dp}{p - p^*} = \int -k\,dt.$$

Solving, we find $p = p^* + (p_0 - p^*)e^{-kt}$, where p_0 is the initial price.

(c) See Figure 11.20.

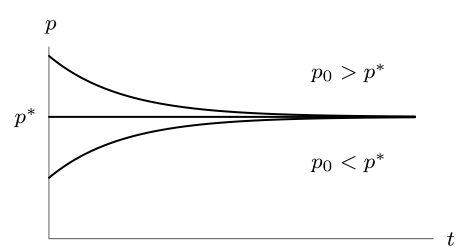


Figure 11.20

(d) As $t \to \infty$, $p \to p^*$. We see this in the solution in part (b), since as $t \to \infty$, $e^{-kt} \to 0$. In other words, as $t \to \infty$, p approaches the equilibrium price p^*.

21. (a)

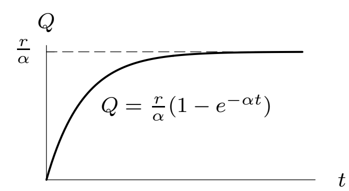
$$\begin{aligned}\frac{dQ}{dt} &= r - \alpha Q = -\alpha(Q - \frac{r}{\alpha})\\ \int \frac{dQ}{Q - r/\alpha} &= = -\alpha \int dt\\ \ln\left|Q - \frac{r}{\alpha}\right| &= -\alpha t + C\\ Q - \frac{r}{\alpha} &= Ae^{-\alpha t}\end{aligned}$$

When $t = 0$, $Q = 0$, so $A = -\frac{r}{\alpha}$ and

$$Q = \frac{r}{\alpha}(1 - e^{-\alpha t})$$

So,

$$Q_\infty = \lim_{t\to\infty} Q = \frac{r}{\alpha}.$$



(b) Doubling r doubles Q_∞. Since $Q_\infty = r/\alpha$, the time to reach $\frac{1}{2}Q_\infty$ is obtained by solving

$$\begin{aligned}\frac{r}{2\alpha} &= \frac{r}{\alpha}(1 - e^{-\alpha t})\\ \frac{1}{2} &= 1 - e^{-\alpha t}\\ e^{-\alpha t} &= \frac{1}{2}\\ t &= -\frac{\ln(1/2)}{\alpha} = \frac{\ln 2}{\alpha}.\end{aligned}$$

So altering r doesn't alter the time it takes to reach $\frac{1}{2}Q_\infty$. See Figure 11.21.

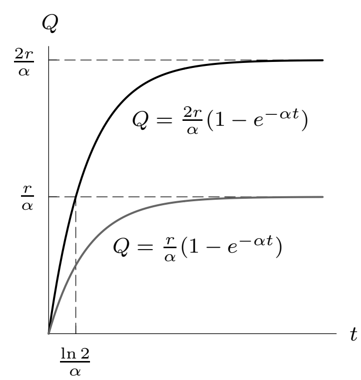


Figure 11.21

(c) Q_∞ is halved by doubling α, and so is the time, $t = \frac{\ln 2}{\alpha}$, to reach $\frac{1}{2}Q_\infty$.

22. (a) Concentration of carbon monoxide $= \dfrac{\text{Quantity in room}}{\text{Volume}}$.
If $Q(t)$ represents the quantity of carbon monoxide in the room at time t, $c(t) = Q(t)/60$.

$$\begin{array}{c}\text{Rate quantity of}\\ \text{carbon monoxide in room}\\ \text{changes}\end{array} = \text{rate in} - \text{rate out}$$

Now

$$\text{Rate in} = 5\%(0.002\text{m}^3/\text{min}) = 0.05(0.002) = 0.0001\text{m}^3/\text{min}.$$

Since smoky air is leaving at $0.002\text{m}^3/\text{min}$, containing a concentration $c(t) = Q(t)/60$ of carbon monoxide

$$\text{Rate out} = 0.002\frac{Q(t)}{60}$$

Thus

$$\frac{dQ}{dt} = 0.0001 - \frac{0.002}{60}Q$$

Since $c = Q/60$, we can substitute $Q = 60c$, giving

$$\begin{aligned}\frac{d(60c)}{dt} &= 0.0001 - \frac{0.002}{60}(60c)\\ \frac{dc}{dt} &= \frac{0.0001}{60} - \frac{0.002}{60}c\end{aligned}$$

(b) Factoring the right side of the differential equation and separating gives

$$\begin{aligned}\frac{dc}{dt} &= -\frac{0.0001}{3}(c - 0.05) \approx 3 \times 10^{-5}(c - 0.05)\\ \int \frac{dc}{c - 0.05} &= -\int 3 \times 10^{-5} dt\\ \ln|c - 0.05| &= -3 \times 10^{-5}t + K\\ c - 0.05 &= Ae^{-3\times 10^{-5}t} \quad \text{where} A = \pm e^K.\end{aligned}$$

Since $c = 0$ when $t = 0$, we have $A = -0.05$, so

$$c = 0.05 - 0.05e^{-3\times 10^{-5}t}$$

(c) As $t \to \infty$, $e^{-3\times 10^{-5}t} \to 0$ so $c \to 0.05$.
Thus in the long run, the concentration of carbon monoxide tends to 5%, the concentration of the incoming air.

23. $c = 0.05 - 0.05e^{-3\times 10^{-5}t}$
We want to solve for t when $c = 0.001$

$$\begin{aligned}0.001 &= 0.05 - 0.05e^{-3\times 10^{-5}t}\\ -0.049 &= -0.05e^{-3\times 10^{-5}t}\\ e^{-3\times 10^{-5}t} &= 0.98\\ t &= \frac{-\ln(0.98)}{3 \times 10^{-5}} = 673 \text{ min} \approx 11 \text{ hours } 13 \text{ min}.\end{aligned}$$

24. (a) Now

$$\frac{dS}{dt} = (\text{Rate at which salt enters the pool}) - (\text{Rate at which salt leaves the pool}),$$

and, for example,

$$\begin{pmatrix}\text{Rate at which salt}\\ \text{enters the pool}\end{pmatrix} = \begin{pmatrix}\text{Concentration of}\\ \text{salt solution}\end{pmatrix} \times \begin{pmatrix}\text{Flow rate of}\\ \text{salt solution}\end{pmatrix}$$
$$(\text{grams/minute}) = (\text{grams/liter}) \times (\text{liters/minute})$$

so

$$\text{Rate at which salt enters the pool} = (10 \text{ grams/liter}) \times (60 \text{ liters/minute}) = (600 \text{ grams/minute})$$

The rate at which salt leaves the pool depends on the concentration of salt in the pool. At time t, the concentration is $\frac{S(t)}{2\times 10^6 \text{ liters}}$, where $S(t)$ is measured in grams.
Thus

$$\text{Rate at which salt leaves the pool} = \frac{S(t) \text{ grams}}{2 \times 10^6 \text{ liters}} \times \frac{60 \text{ liters}}{\text{minute}} = \frac{3S(t) \text{ grams}}{10^5 \text{ minutes}}.$$

Thus

$$\frac{dS}{dt} = 600 - \frac{3S}{100{,}000}.$$

(b) $\frac{dS}{dt} = -\frac{3}{100{,}000}(S - 20{,}000{,}000)$
$\int \frac{dS}{S-20{,}000{,}000} = \int -\frac{3}{100{,}000}\, dt$
$\ln|S - 20{,}000{,}000| = -\frac{3}{100{,}000}t + C$
$S = 20{,}000{,}000 - Ae^{-\frac{3}{100{,}000}t}$
Since $S = 0$ at $t = 0$, $A = 20{,}000{,}000$. Thus $S(t) = 20{,}000{,}000 - 20{,}000{,}000e^{-\frac{3}{100{,}000}t}$.

(c) As $t \to \infty$, $e^{-\frac{3}{100{,}000}t} \to 0$, so $S(t) \to 20{,}000{,}000$ grams. The concentration approaches 10 grams/liter. Note that this makes sense; we'd expect the concentration of salt in the pool to become closer and closer to the concentration of salt being poured into the pool as $t \to \infty$.

25. (a) Newton's Law of Motion says that

$$\text{Force} = (\text{mass}) \times (\text{acceleration}).$$

Since acceleration, dv/dt, is measured upward and the force due to gravity acts downward,

$$-\frac{mgR^2}{(R+h)^2} = m\frac{dv}{dt}$$

so

$$\frac{dv}{dt} = -\frac{gR^2}{(R+h)^2}.$$

(b) Since $v = \frac{dh}{dt}$, the chain rule gives

$$\frac{dv}{dt} = \frac{dv}{dh} \cdot \frac{dh}{dt} = \frac{dv}{dh} \cdot v.$$

Substituting into the differential equation in part (a) gives

$$v\frac{dv}{dh} = -\frac{gR^2}{(R+h)^2}.$$

(c) Separating variables gives

$$\int v\, dv = -\int \frac{gR^2}{(R+h)^2}\, dh$$
$$\frac{v^2}{2} = \frac{gR^2}{(R+h)} + C$$

Since $v = v_0$ when $h = 0$,

$$\frac{{v_0}^2}{2} = \frac{gR^2}{(R+0)} + C \quad \text{gives} \quad C = \frac{{v_0}^2}{2} - gR,$$

so the solution is

$$\frac{v^2}{2} = \frac{gR^2}{(R+h)} + \frac{{v_0}^2}{2} - gR$$
$$v^2 = {v_0}^2 + \frac{2gR^2}{(R+h)} - 2gR$$

(d) The escape velocity v_0 ensures that $v^2 \geq 0$ for all $h \geq 0$. Since the positive quantity $\dfrac{2gR^2}{(R+h)} \to 0$ as $h \to \infty$, to ensure that $v^2 \geq 0$ for all h, we must have

$$v_0{}^2 \geq 2gR.$$

When $v_0{}^2 = 2gR$ so $v_0 = \sqrt{2gR}$, we say that v_0 is the escape velocity.

Solutions for Section 11.7

Exercises

1. A continuous growth rate of 0.2% means that

$$\frac{1}{P}\frac{dP}{dt} = 0.2\% = 0.002.$$

Separating variables and integrating gives

$$\int \frac{dP}{P} = \int 0.002\, dt$$
$$P = P_0 e^{0.002t} = (6.6 \times 10^6)e^{0.002t}.$$

2. (a) At $t = 0$, which corresponds to 1935, we have

$$P = \frac{1}{1 + 2.968e^{-0.0275(0)}} = 0.252$$

showing that about 25% of the land was in use in 1935.

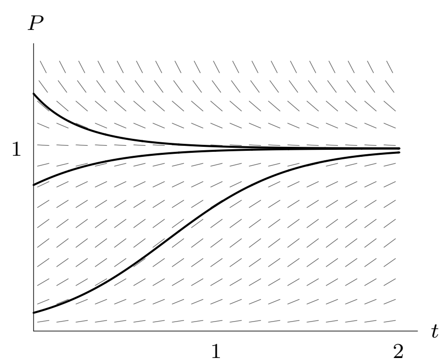
(b) This model predicts that as t gets very large, P will approach 1. That is, the model predicts that in the long run, all the land will be used for farming.

(c) To solve this graphically, enter the function into a graphing calculator and trace the resulting curve until it reaches a height of 0.5, which occurs when $t \approx 39.6$. Since $t = 0$ corresponds to 1935, $t = 39.6$ corresponds to $1935+39.6 = 1974.6$. According to this model, the Tojolobal were using half their land in 1974. Alternatively, we solve for t:

$$\begin{aligned}
\frac{1}{1 + 2.968e^{-0.0275t}} &= 0.5 \\
1 + 2.968e^{-0.0275t} &= 2 \\
2.968e^{-0.0275t} &= 1 \\
e^{-0.0275t} &= \frac{1}{2.968} \\
t &= \frac{\ln(1/2.968)}{-0.0275} = 39.6 \text{ years.}
\end{aligned}$$

(d) The inflection point occurs when $P = L/2$ or at one-half the carrying capacity. In this case, $P = \frac{1}{2}$ in 1974, as shown in part (c).

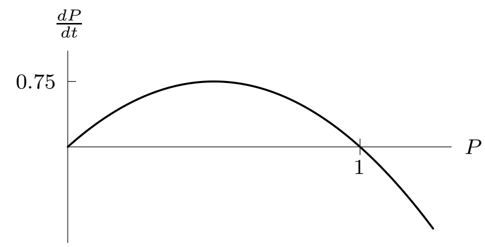
3. (a)



(b) The value $P = 1$ is a stable equilibrium. (See part (d) below for a more detailed discussion.)

(c) Looking at the solution curves, we see that P is increasing for $0 < P < 1$ and decreasing for $P > 1$. The values of $P = 0$, $P = 1$ are equilibria. In the long run, P tends to 1, unless you start with $P = 0$. The solution curves with initial populations of less than $P = \frac{1}{2}$ have inflection points at $P = \frac{1}{2}$. (This will be demonstrated algebraically in part (d) below.) At the inflection point, the population is growing fastest.

(d)



Since $\frac{dP}{dt} = 3P - 3P^2 = 3P(1 - P)$, the graph of $\frac{dP}{dt}$ against P is a parabola, opening downward with P intercepts at 0 and 1. The quantity $\frac{dP}{dt}$ is positive for $0 < P < 1$, negative for $P > 1$ (and $P < 0$). The quantity $\frac{dP}{dt}$ is 0 at $P = 0$ and $P = 1$, and maximum at $P = \frac{1}{2}$. The fact that $\frac{dP}{dt} = 0$ at $P = 0$ and $P = 1$ tells us that these are equilibria. Further, since $\frac{dP}{dt} > 0$ for $0 < P < 1$, we see that solution curves starting here will increase toward $P = 1$.

If the population starts at a value $P < \frac{1}{2}$, it increases at an increasing rate up to $P = \frac{1}{2}$. After this, P continues to increase, but at a decreasing rate. The fact that $\frac{dP}{dt}$ has a maximum at $P = \frac{1}{2}$ tells us that there is a point of inflection when $P = \frac{1}{2}$. Similarly, since $\frac{dP}{dt} < 0$ for $P > 1$, solution curves starting with $P > 1$ will decrease to $P = 1$. Thus, $P = 1$ is a stable equilibrium.

Problems

4. The US population in 1860 was 31.4 million. If between 1860 and 1870 the population had increased at the same rate as previous decades, 34.7%, the population in 1870 would have been $(31.4 \text{ million})(1.347) = 42.3$ million. In actuality the US population in 1870 was only 38.6 million. This is a shortfall of 3.7 million people.

History records that about $618{,}000$ soldiers died (total, both sides) during the Civil War (according to Collier's Encyclopedia, 1968). This accounts for only $\frac{1}{6}$ (roughly) of the shortfall. The rest of the shortfall can be attributed to civilian deaths and a decrease in the birth rate caused by absent males and an unwillingness to have babies under harsh economic conditions and political uncertainty.

5. (a) The logistic model is a reasonable one because at first very few houses have a VCR. As movie rentals become popular and as VCRs get cheaper, more people will buy VCRs. However, we know that the rate of VCR buying will start slowing down at some point as it is impossible for more than 100% of houses to have VCRs.

(b) To find the point of inflection, we must find the year at which the rate of VCR buying changes from increasing to decreasing. The following table shows the rate of change in the years from 1978 to 1990.

Year	1978	1979	1980	1981	1982	1983	1984
% Change per year	0.2	0.6	0.7	1.3	2.4	5.1	10.2
Year	1985	1986	1987	1988	1989	1990	1991
% Change per year	15.2	12.7	9.3	6.6	7.3	0	

Looking at the table, we see that the rate of percent change per year changes from increasing to decreasing in the year 1986. At this time 36% of households own VCRs giving $P = (1986, 36)$. Since at the inflection point we expect the vertical coordinate to be $L/2$, we get

$$L/2 = 36$$
$$L = 72\%.$$

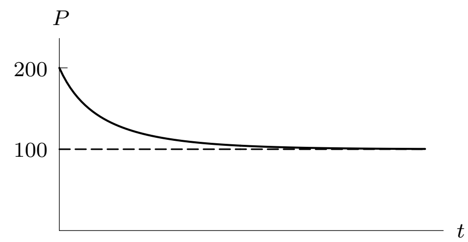
Thus we expect the limiting value to be 72%. This fits in well with the data that we have for 1990 and 1991.

(c) Since the general form of a logistic equation is

$$P = \frac{L}{1 + Ce^{-kt}}$$

where L is the limiting value, we have that in our case $L = 75$ and the limiting value is 75%.

6. Rewriting the equation as $\frac{1}{P}\frac{dP}{dt} = \frac{(100-P)}{1000}$, we see that this is a logistic equation. Before looking at its solution, we explain why there must always be at least 100 individuals. Since the population begins at 200, $\frac{dP}{dt}$ is initially negative, so the population decreases. It continues to do so while $P > 100$. If the population ever reached 100, however, then $\frac{dP}{dt}$ would be 0. This means the population would stop changing – so if the population ever decreased to 100, that's where it would stay. The fact that $\frac{dP}{dt}$ will always be negative also shows that the population will always be under 200, as shown below.



The solution, as given by the formula derived in the chapter, is

$$P = \frac{20000}{200 - 100e^{-t/10}}$$

7.

Table 11.15

Year	P	$\frac{dP}{dt} \approx \frac{P(t+10)-P(t-10)}{20}$
1790	3.9	
1800	5.3	$(7.2 - 3.9)/20 = 0.165$
1810	7.2	$(9.6 - 5.3)/20 = 0.215$
1820	9.6	$(12.9 - 7.2)/20 = 0.285$
1830	12.9	$(17.1 - 9.6)/20 = 0.375$
1840	17.1	$(23.2 - 12.9)/20 = 0.515$
1850	23.2	$(31.4 - 17.1)/20 = 0.715$
1860	31.4	$(38.6 - 23.2)/20 = 0.770$
1870	38.6	$(50.2 - 31.4)/20 = 0.940$
1880	50.2	$(62.9 - 38.6)/20 = 1.215$
1890	62.9	$(76.0 - 50.2)/20 = 1.290$
1900	76.0	$(92.0 - 62.9)/20 = 1.455$
1910	92.0	$(105.7 - 76.0)/20 = 1.485$
1920	105.7	$(122.8 - 92.0)/20 = 1.540$
1930	122.8	$(131.7 - 105.7)/20 = 1.300$
1940	131.7	$(150.7 - 122.8)/20 = 1.395$
1950	150.7	

According to these calculations, the largest value of dP/dt occurs in 1920 when the rate of change is $\frac{dP}{dt} = 1.540$ million people/year. The population in 1920 was 105.7 million. If we assume that the limiting value, L, is twice the population when it is changing most quickly, then $L = 2 \times 105.7 = 211.4$ million. This is greater than the estimate of 187 million computed in the text and closer to the actual 1990 population of 248.7 million.

8. (a) The equilibrium population will be reached when dP/dt approaches zero. Solving $1 - 0.0004P = 0$ gives $P = 2500$ fish as the equilibrium population.

(b) The solution of the differential equation is

$$P(t) = \frac{2500}{(1 + Ae^{-0.25t})}$$

subject to $P(-10) = 1000$ if $t = 0$ represents the present time. So we have

$$1000 = \frac{2500}{(1 + Ae^{2.5})}$$

from which $A = 0.123127$ and

$$P(0) = \frac{2500}{(1 + 0.123127)} \approx 2230.$$

Therefore, the current population is approximately 2230 fish.

(c) The effect of losing 10% of the fish each year gives the revised differential equation

$$\frac{dP}{dt} = (0.25 - 0.0001P)P - 0.1P$$

or

$$\frac{dP}{dt} = (0.15 - 0.0001P)P.$$

The revised equilibrium population is therefore about 1500 fish.

9. (a) We know that a logistic curve can be modeled by the function

$$P = \frac{L}{1 + Ce^{-kt}}$$

where $C = (L - P_0)/(P_0)$ and P is the number of people infected by the virus at a particular time t. We know that L is the limiting value, or the maximal number of people infected with the virus, so in our case

$$L = 5000.$$

We are also told that initially there are only ten people infected with the virus so that we get

$$P_0 = 10.$$

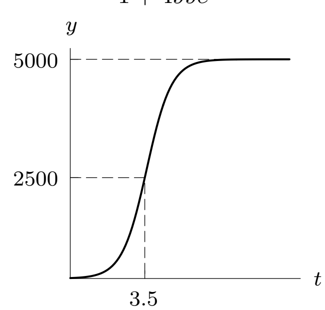
Thus we have

$$\begin{aligned} C &= \frac{L - P_0}{P_0} \\ &= \frac{5000 - 10}{10} \\ &= 499. \end{aligned}$$

We are also told that in the early stages of the virus, infection grows exponentially with $k = 1.78$. Thus we get that the logistic function for people infected is

$$P = \frac{5000}{1 + 499e^{-1.78t}}.$$

(b)



(c) Looking at the graph we see that the the point at which the rate changes from increasing to decreasing, the inflection point, occurs at roughly $t = 3.5$ giving a value of $P = 2500$. Thus after roughly 2500 people have been infected, the rate of infection starts dropping. See above.

10. (a) Let the population at time t be $P(t)$ and the relative growth rate be $G = \alpha - \beta P$. When $P = 600$, $G = 35 - 15 = 20\%$, and when $P = 800$, $G = 30 - 20 = 10\%$ so

$$\alpha - 600\beta = 0.20$$

$$\alpha - 800\beta = 0.10.$$

Therefore, $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2000}$, hence

$$\frac{1}{P}\frac{dP}{dt} = \frac{1}{2} - \frac{1}{2000}P.$$

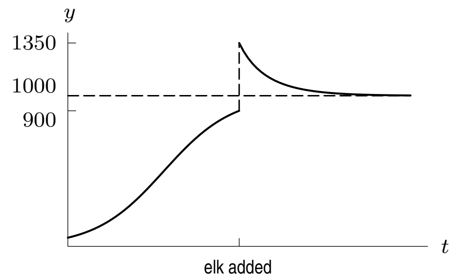
(b) The differential equation is a logistic equation

$$\frac{dP}{dt} = \frac{1}{2000}P(1000 - P)$$

and so the equilibrium population is $P = 1000$. We expect the population of 900 to increase to the equilibrium value of 1000.

(c) If the additional elk are added, the population of 1350 elk is above the equilibrium value, and the population will decrease to about 1000.

(d)

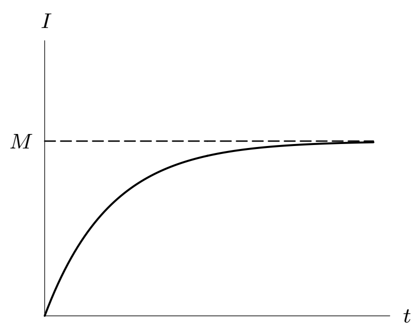


Importing more elk would be ecologically unsound, as the new population is in excess of the equilibrium population that Reading Island can support.

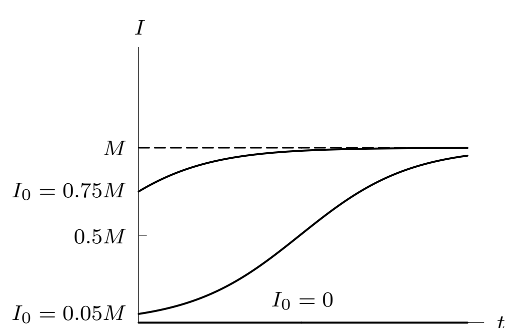
11. (a) Let I be the number of informed people at time t, and I_0 the number who know initially. Then this model predicts that $\frac{dI}{dt} = k(M - I)$ for some positive constant k. Solving this, we find the solution is

$$I = M - (M - I_0)e^{-kt}.$$

We sketch the solution with $I_0 = 0$. Notice that $\frac{dI}{dt}$ is largest when I is smallest, so the information spreads fastest in the beginning, at $t = 0$. In addition, the graph below shows that $I \to M$ as $t \to \infty$, meaning that everyone gets the information eventually.



(b) In this case, the model suggests that $\frac{dI}{dt} = kI(M - I)$ for some positive constant k. This is a logistic model with carrying capacity M. We sketch the solutions for three different values of I_0 below.



(i) If $I_0 = 0$ then $I = 0$ for all t. In other words, if nobody knows something, it does not spread by word of mouth!

(ii) If $I_0 = 0.05M$, then $\frac{dI}{dt}$ is increasing up to $I = \frac{M}{2}$. Thus, the information is spreading fastest at $I = \frac{M}{2}$.

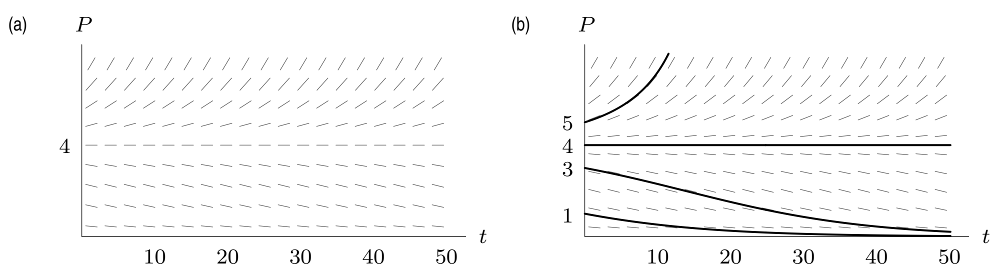
(iii) If $I_0 = 0.75M$, then $\frac{dI}{dt}$ is always decreasing for $I > \frac{M}{2}$, so $\frac{dI}{dt}$ is largest when $t = 0$.

12. (a) By the chain rule
$$\frac{dP}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = \frac{d}{du}\left(\frac{1}{u}\right)\cdot\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

(b) Substituting for $P = 1/u$ in the equation
$$\frac{dP}{dt} = kP\left(1-\frac{P}{L}\right)$$
gives
$$-\frac{1}{u^2}\frac{du}{dt} = k\frac{1}{u}\left(1-\frac{1}{Lu}\right).$$
Simplifying leads to
$$\frac{du}{dt} = -k\left(u-\frac{1}{L}\right)$$
and separating variables gives
$$\begin{aligned}\int\frac{du}{u-1/L} &= -\int k dt\\ \ln\left|u-\frac{1}{L}\right| &= -kt + C\\ u-\frac{1}{L} &= Ae^{-kt}\quad\text{where } A=\pm e^C\\ u &= \frac{1}{L}+Ae^{-kt}\end{aligned}$$

(c) Since $u = 1/P$, we have
$$\frac{1}{P} = \frac{1}{L}+Ae^{-kt} = \frac{1+LAe^{-kt}}{L}$$
so
$$P = \frac{L}{1+LAe^{-kt}}\quad\text{where } A \text{ is an arbitrary constant.}$$

13.



(c) There are two equilibrium values, $P = 0$, and $P = 4$. The first, representing extinction, is stable. The equilibrium value $P = 4$ is unstable because the populations increase if greater than 4, and decrease if less than 4. Notice that the equilibrium values can be obtained by setting $dP/dt = 0$:
$$\frac{dP}{dt} = 0.02P^2 - 0.08P = 0.02P(P-4) = 0$$
so
$$P = 0 \text{ or } P = 4.$$

14. (a)

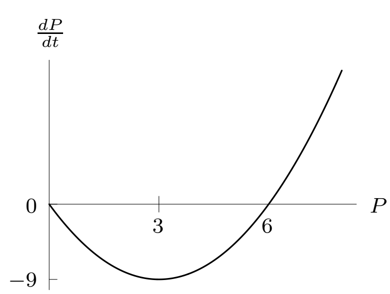


Figure 11.22

(b) Figure 11.22 shows that for $0 < P < 6$, the sign of dP/dt is negative. This means that P is decreasing over the interval $0 < P < 6$. As P decreases from $P(0) = 5$, the value of dP/dt gets more and more negative until $P = 3$. Thus the graph of P against t is concave down while P is decreasing from 5 to 3. As P decreases below 3, the slope of dP/dt increases toward 0, so the graph of P against t is concave up and asymptotic to the t-axis. At $P = 3$, there is an inflection point. See Figure 11.23.

(c) Figure 11.22 shows that for $P > 6$, the slope of dP/dt is positive and increases with P. Thus the graph of P against t is increasing and concave up. See Figure 11.23.

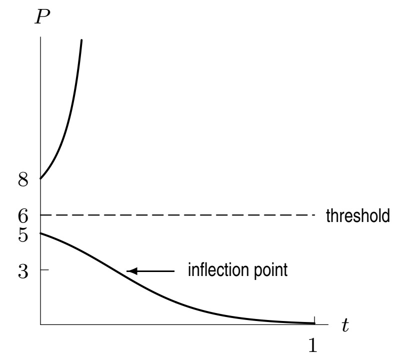


Figure 11.23

(d) For initial populations greater than the threshold value $P = 6$, the population increases without bound. Populations with initial value less than $P = 6$ decrease asymptotically toward 0, i.e. become extinct. Thus the initial population $P = 6$ is the dividing line, or threshold, between populations which grow without bound and those which die out.

15. (a)

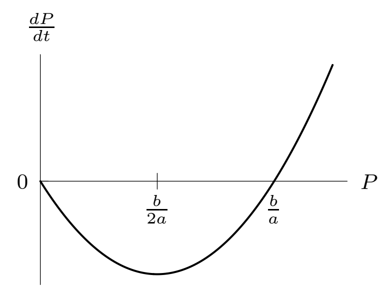


Figure 11.24

(b)

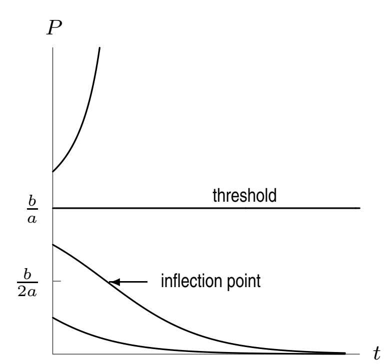


Figure 11.25

Figure 11.24 shows that dP/dt is negative for $P < \frac{b}{a}$, making P a decreasing function when $P(0) < \frac{b}{a}$. When $P > \frac{b}{a}$, the sign of dP/dt is positive, so P is an increasing function. Thus solution curves starting above $\frac{b}{a}$ are increasing, and those starting below $\frac{b}{a}$ are decreasing. See Figure 11.25.

For $P > \frac{b}{a}$, the slope, $\frac{dP}{dt}$, increases with P, so the graph of P against t is concave up. For $0 < P < \frac{b}{a}$, the value of P decreases with time. As P decreases, the slope $\frac{dP}{dt}$ decreases for $\frac{b}{2a} < P < \frac{b}{a}$, and increases toward 0 for $0 < P < \frac{b}{2a}$. Thus solution curves starting just below the threshold value of $\frac{b}{a}$ are concave down for $\frac{b}{2a} < P < \frac{b}{a}$ and concave up and asymptotic to the t-axis for $0 < P < \frac{b}{2a}$. See Figure 11.25.

(c) $P = \frac{b}{a}$ is called the threshold population because for populations greater than $\frac{b}{a}$, the population will increase without bound. For populations less than $\frac{b}{a}$, the population will go to zero, i.e. to extinction.

Solutions for Section 11.8

Exercises

1. Since

$$\frac{dS}{dt} = -aSI,$$
$$\frac{dI}{dt} = aSI - bI,$$
$$\frac{dR}{dt} = bI$$

we have

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = -aSI + aSI - bI + bI = 0.$$

Thus $\frac{d}{dt}(S + I + R) = 0$, so $S + I + R =$ constant.

2. This is an example of a predator-prey relationship. Normally, we would expect the worm population, in the absence of predators, to increase without bound. As the number of worms w increases, so would the rate of increase dw/dt; in other words, the relation $dw/dt = w$ might be a reasonable model for the worm population in the absence of predators.

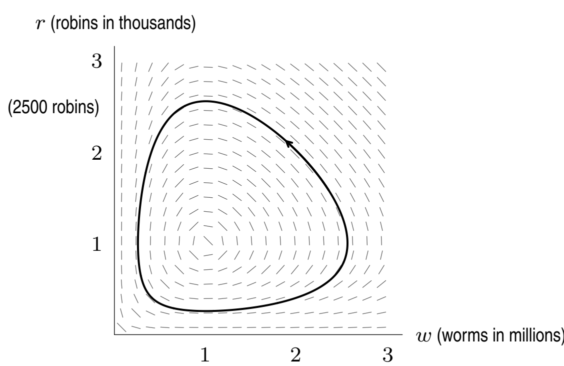
However, since there are predators (robins), dw/dt won't be that big. We must lessen dw/dt. It makes sense that the more interaction there is between robins and worms, the more slowly the worms are able to increase their numbers. Hence we lessen dw/dt by the amount wr to get $dw/dt = w - wr$. The term $-wr$ reflects the fact that more interactions between the species means slower reproduction for the worms.

Similarly, we would expect the robin population to decrease in the absence of worms. We'd expect the population decrease at a rate related to the current population, making $dr/dt = -r$ a reasonable model for the robin population in absence of worms. The negative term reflects the fact that the greater the population of robins, the more quickly they are dying off. The wr term in $dr/dt = -r + wr$ reflects the fact that the more interactions between robins and worms, the greater the tendency for the robins to increase in population.

3. If there are no worms, then $w = 0$, and $\frac{dr}{dt} = -r$ giving $r = r_0 e^{-t}$, where r_0 is the initial robin population. If there are no robins, then $r = 0$, and $\frac{dw}{dt} = w$ giving $w = w_0 e^t$, where w_0 is the initial worm population.

4. There is symmetry across the line $r = w$. Indeed, since $\frac{dr}{dw} = \frac{r(w-1)}{w(1-r)}$, if we switch w and r we get $\frac{dw}{dr} = \frac{w(r-1)}{r(1-w)}$, so $\frac{dr}{dw} = \frac{r(1-w)}{w(r-1)}$. Since switching w and r changes nothing, the slope field must be symmetric across the line $r = w$. The slope field shows that the solution curves are either spirals or closed curves. Since there is symmetry about the line $r = w$, the solutions must in fact be closed curves.

5. If $w = 2$ and $r = 2$, then $\frac{dw}{dt} = -2$ and $\frac{dr}{dt} = 2$, so initially the number of worms decreases and the number of robins increases. In the long run, however, the populations will oscillate; they will even go back to $w = 2$ and $r = 2$.



6. Sketching the trajectory through the point $(2, 2)$ on the slope field given shows that the maximum robin population is about 2500, and the minimum robin population is about 500. When the robin population is at its maximum, the worm population is about 1,000,000.

7.

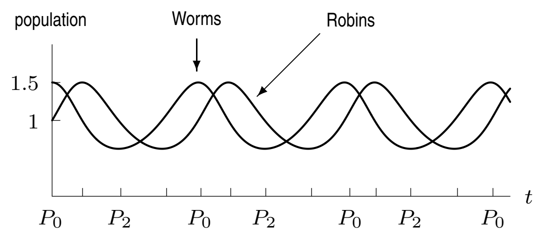


Figure 11.26

8. It will work somewhat; the maximum number the robins reach will increase. However, the minimum number the robins reach will decrease as well. (See graph of slope field.) In the long term, the robin-worm populations will again fall into a cycle. Notice, however, if the extra robins are added during the part of the cycle where there are the fewest robins, the new cycle will have smaller variation. See Figure 11.27.

Note that if too many robins are added, the minimum number may get so small the model may fail, since a small number of robins are more susceptible to disaster.

Figure 11.27

9. The numbers of robins begins to increase while the number of worms remains approximately constant. See Figure 11.28.

The numbers of robins and worms oscillate periodically between 0.2 and 3, with the robin population lagging behind the worm population.

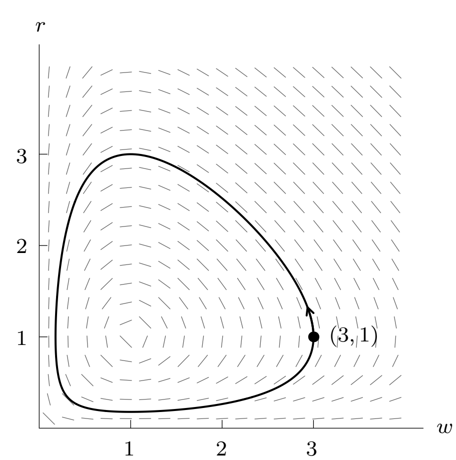


Figure 11.28

10. Estimating from the phase plane, we have

$$0.18 < r < 3$$

so the robin population lies between 180 and 3000. Similarly

$$0.2 < w < 3,$$

so the worm population lies between 200,000 and 3,000,000.

When the robin population is at its minimum $r \approx 0.2$, then $w \approx 0.87$, so that there are approximately 870,000 worms.

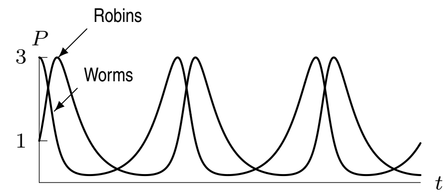


Figure 11.29

11. Here x and y both increase at about the same rate.

12. Initially $x = 0$, so we start with only y. Then y decreases while x increases. Then x continues to increase while y starts to increase as well. Finally y continues to increase while x decreases.

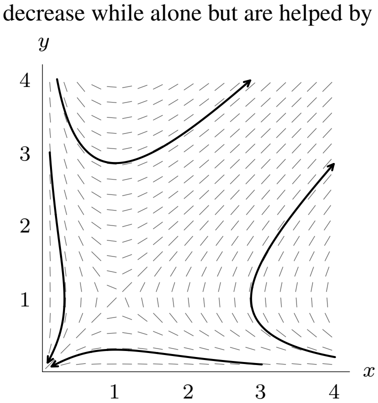
13. x decreases quickly while y increases more slowly.

14. The closed trajectory represents populations which oscillate repeatedly.

Problems

15. (a) Symbiosis, because both populations decrease while alone but are helped by the presence of the other.

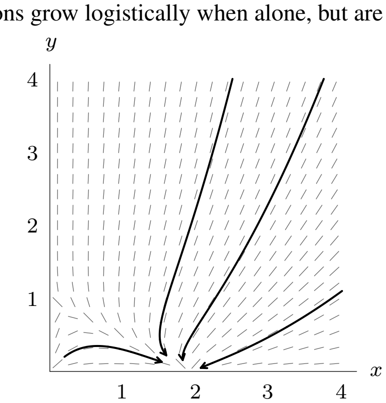
(b)



Both populations tend to infinity or both tend to zero.

16. (a) Competition, because both populations grow logistically when alone, but are harmed by the presence of the other.

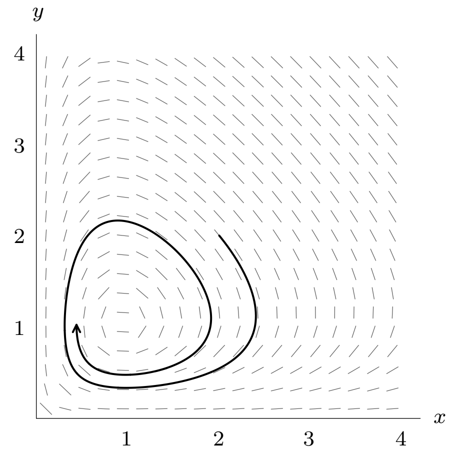
(b)



In the long run, $x \to 2$, $y \to 0$. In other words, y becomes extinct.

17. (a) Predator-prey, because x decreases while alone, but is helped by y, whereas y increases logistically when alone, and is harmed by x. Thus x is predator, y is prey.

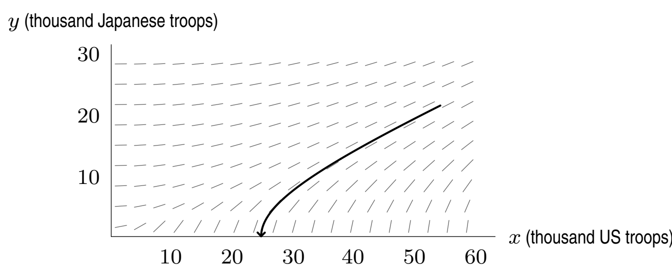
(b)



Provided neither initial population is zero, both populations tend to about 1. If x is initially zero, but y is not, then $y \to \infty$. If y is initially zero, but x is not, then $x \to 0$.

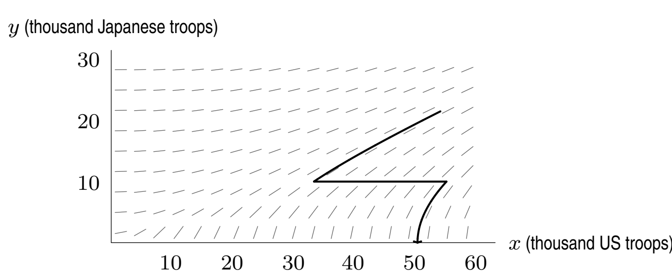
18. (a) Thinking of y as a function of x and x as a function of t, then by the chain rule: $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$, so:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-0.01x}{-0.05y} = \frac{x}{5y}$$



(b) The figure above shows the slope field for this differential equation and the trajectory starting at $x_0 = 54,\ y_0 = 21.5$. The trajectory goes to the x-axis, where $y = 0$, meaning that the Japanese troops were all killed or wounded before the US troops were, and thus predicts the US victory (which did occur). Since the trajectory meets the x-axis at $x \approx 25$, the differential equation predicts that about $25{,}000$ US troops would survive the battle.

(c) The fact that the US got reinforcements, while the Japanese did not, does not alter the predicted outcome (a US victory). The US reinforcements have the effect of changing the trajectory, altering the number of troops surviving the battle. See the graph below.



19. (a) Thinking of y as a function of x and x as a function of t, then by the chain rule: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, so:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-bx}{-ay} = \frac{bx}{ay}$$

(b) Separating variables,

$$\int ay\,dy = \int bx\,dx$$
$$a\frac{y^2}{2} = b\frac{x^2}{2} + k$$
$$ay^2 - bx^2 = C \quad \text{where } C = 2k$$

20. (a) Lanchester's square law for the battle of Iwo Jima is

$$0.05y^2 - 0.01x^2 = C.$$

If we measure x and y in thousands, $x_0 = 54$ and $y_0 = 21.5$, so $0.05(21.5)^2 - 0.01(54)^2 = C$ giving $C = -6.0475$. Thus the equation of the trajectory is

$$0.05y^2 - 0.01x^2 = -6.0475$$

giving

$$x^2 - 5y^2 = 604.75.$$

(b) Assuming that the battle did not end until all the Japanese were dead or wounded, that is, $y = 0$, then the number of US soldiers remaining is given by $x^2 - 5(0)^2 = 604.75$. This gives $x = 24.59$, or about 25,000 troops. This is approximately what happened.

21. (a) Since the guerrillas are hard to find, the rate at which they are put out of action is proportional to the number of chance encounters between a guerrilla and a conventional soldier, which is in turn proportional to the number of guerrillas and to the number of conventional soldiers. Thus the rate at which guerrillas are put out of action is proportional to the product of the strengths of the two armies.

(b)

$$\frac{dx}{dt} = -xy$$
$$\frac{dy}{dt} = -x$$

(c) Thinking of y as a function of x and x a function of of t, then by the chain rule: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$ so:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-x}{-xy} = \frac{1}{y}$$

Separating variables:

$$\int y\,dy = \int dx$$
$$\frac{y^2}{2} = x + C$$

The value of C is determined by the initial strengths of the two armies.

(d) The sign of C determines which side wins the battle. Looking at the general solution $\frac{y^2}{2} = x + C$, we see that if $C > 0$ the y-intercept is at $\sqrt{2C}$, so y wins the battle by virtue of the fact that it still has troops when $x = 0$. If $C < 0$ then the curve intersects the axes at $x = -C$, so x wins the battle because it has troops when $y = 0$. If $C = 0$, then the solution goes to the point $(0, 0)$, which represents the case of mutual annihilation.

(e) We assume that an army wins if the opposing force goes to 0 first. Figure 11.30 shows that the conventional force wins if $C > 0$ and the guerrillas win if $C < 0$. Neither side wins if $C = 0$ (all soldiers on both sides are killed in this case).

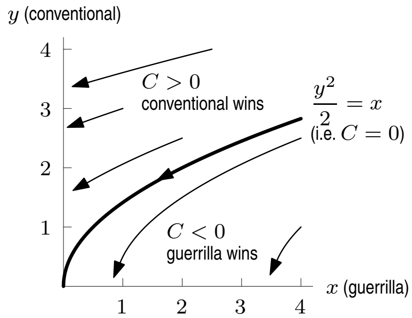


Figure 11.30

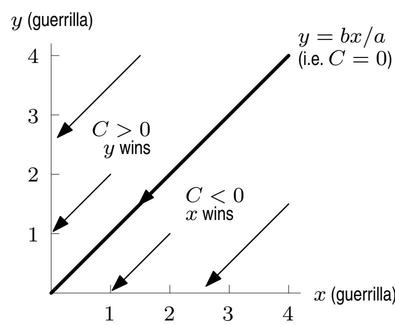
22. (a) Taking the constants of proportionality to be a and b, with $a > 0$ and $b > 0$, the equations are

$$\frac{dx}{dt} = -axy$$
$$\frac{dy}{dt} = -bxy$$

(b) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-bxy}{-axy} = \dfrac{b}{a}$. Solving the differential equation gives $y = \dfrac{b}{a}x + C$, where C depends on the initial sizes of the two armies.

(c) The sign of C determines which side wins the battle. Looking at the general solution $y = \dfrac{b}{a}x + C$, we see that if $C > 0$ the y-intercept is at C, so y wins the battle by virtue of the fact that it still has troops when $x = 0$. If $C < 0$ then the curve intersects the axes at $x = -\frac{a}{b}C$, so x wins the battle because it has troops when $y = 0$. If $C = 0$, then the solution goes to the point $(0,\ 0)$, which represents the case of mutual annihilation.

(d) We assume that an army wins if the opposing force goes to 0 first.



23. (a) We have

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{-3y - xy}{-2x - xy} = \frac{y(x+3)}{x(y+2)}.$$

Thus,

$$\left(\frac{y+2}{y}\right) dy = \left(\frac{x+3}{x}\right) dx$$

so

$$\int (1 + \frac{2}{y})\, dy = \int (1 + \frac{3}{x})\, dx.$$

So,

$$y + 2\ln|y| = x + 3\ln|x| + C.$$

Since x and y are non-negative,

$$y + 2\ln y = x + 3\ln x + C.$$

This is as far as we can go with this equation – we cannot solve for y in terms of x, for example. We can, however, put it in the form

$$e^{y+2\ln y} = e^{x+3\ln x+C}, \quad \text{or} \quad y^2e^y = Ax^3e^x.$$

(b) An equilibrium state satisfies

$$\frac{dx}{dt} = -2x - xy = 0 \quad \text{and} \quad \frac{dy}{dt} = -3y - xy = 0.$$

Solving the first equation, we have

$$-x(y+2) = 0, \quad \text{so} \quad x = 0 \quad \text{or} \quad y = -2.$$

The second equation has solutions

$$y = 0 \quad \text{or} \quad x = -3.$$

Since $x, y \geq 0$, the only equilibrium point is $(0, 0)$.

(c) We can use either of our forms for the solution. Looking at

$$y^2e^y = Ax^3e^x,$$

we see that if x and y are very small positive numbers, then

$$e^x \approx e^y \approx 1.$$

Thus,

$$y^2 \approx Ax^3, \quad \text{or} \quad \frac{y^2}{x^3} \approx A, \text{ a constant.}$$

Looking at

$$y + 2\ln y = x + 3\ln x + C,$$

we note that if x and y are small, then they are negligible compared to $\ln y$ and $\ln x$. Thus,

$$2\ln y \approx 3\ln x + C,$$

giving

$$\ln y^2 - \ln x^3 \approx C,$$

so

$$\ln \frac{y^2}{x^3} \approx C$$

and therefore

$$\frac{y^2}{x^3} \approx e^C, \text{ a constant.}$$

(d) If

$$x(0) = 4 \quad \text{and} \quad y(0) = 8,$$

then

$$8 + 2\ln 8 = 4 + 3\ln 4 + C.$$

Note that

$$2\ln 8 = 3\ln 4 = \ln 64,$$

giving

$$4 = C.$$

So the phase trajectory is

$$y + 2\ln y = x + 3\ln x + 4.$$

(Or equivalently, $y^2e^y = e^4x^3e^x = x^3e^{x+4}$.)

(e) If the concentrations are equal, then

$$y + 2\ln y = y + 3\ln y + 4,$$

giving

$$-\ln y = 4 \quad \text{or} \quad y = e^{-4}.$$

Thus, they are equal when $y = x = e^{-4} \approx 0.0183$.

(f) Using part (c), we have that if x is small,

$$\frac{y^2}{x^3} \approx e^4.$$

Since $x = e^{-10}$ is certainly small,

$$\frac{y^2}{e^{-30}} \approx e^4, \quad \text{and} \quad y \approx e^{-13}.$$

Solutions for Section 11.9

Exercises

1. (a) To find the equilibrium points we set

$$20x - 10xy = 0$$
$$25y - 5xy = 0.$$

So, $x = 0$, $y = 0$ is an equilibrium point. Another one is given by

$$10y = 20$$
$$5x = 25.$$

Therefore, $x = 5$, $y = 2$ is the other equilibrium point.

(b) At $x = 2$, $y = 4$,

$$\frac{dx}{dt} = 20x - 10xy = 40 - 80 = -40$$
$$\frac{dy}{dt} = 25y - 5xy = 100 - 40 = 60.$$

Since these are not both zero, this point is not an equilibrium point.

2. (a) $dS/dt = 0$ where $S = 0$ or $I = 0$ (both axes).
$dI/dt = 0.0026I(S - 192)$, so $dI/dt = 0$ where $I = 0$ or $S = 192$.
Thus every point on the S axis is an equilibrium point (corresponding to no one being sick).

(b) In region I, where $S > 192$, $\frac{dS}{dt} < 0$ and $\frac{dI}{dt} > 0$.
In region II, where $S < 192$, $\frac{dS}{dt} < 0$ and $\frac{dI}{dt} < 0$. See Figure 11.31.

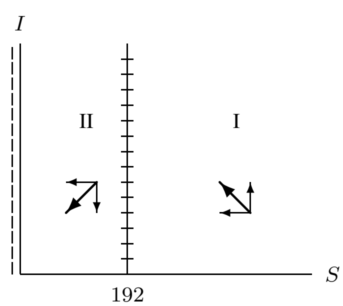


Figure 11.31

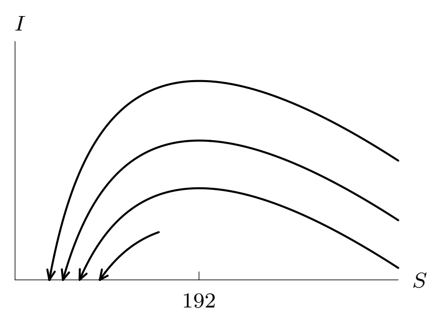


Figure 11.32

(c) If the trajectory starts with $S_0 > 192$, then I increases to a maximum when $S = 192$. If $S_0 < 192$, then I always decreases. See Figure 11.31. Regardless of the initial conditions, the trajectory always goes to a point on the S-axis (where $I = 0$). The S-intercept represents the number of students who never get the disease. See Figure 11.32.

3. The nullclines are where $\frac{dw}{dt} = 0$ or $\frac{dr}{dt} = 0$.
$\frac{dw}{dt} = 0$ when $w - wr = 0$, so $w(1 - r) = 0$ giving $w = 0$ or $r = 1$.
$\frac{dr}{dt} = 0$ when $-r + rw = 0$, so $r(w - 1) = 0$ giving $r = 0$ or $w = 1$.

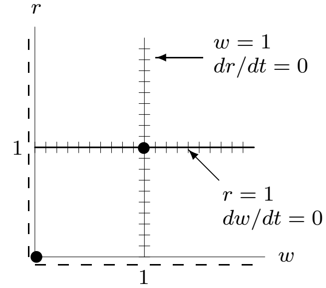


Figure 11.33: Nullclines and equilibrium points (dots)

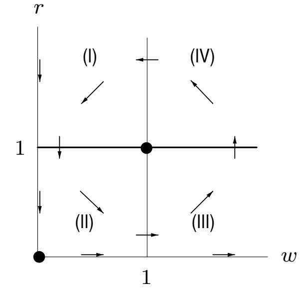


Figure 11.34

The equilibrium points are where the nullclines intersect: $(0,0)$ and $(1,1)$. The nullclines split the first quadrant into four sectors. See Figure 11.33. We can get a feel for how the populations interact by seeing the direction of the trajectories in each sector. See Figure 11.34. If the populations reach an equilibrium point, they will stay there. If the worm population dies out, the robin population will also die out, too. However, if the robin population dies out, the worm population will continue to grow.

Otherwise, it seems that the populations cycle around the equilibrium $(1,1)$. The trajectory moves from sector to sector: trajectories in sector (I) move to sector (II); trajectories in sector (II) move to sector (III); trajectories in sector (III) move to sector (IV); trajectories in sector (IV) move back to sector (I). The robins keep the worm population down by feeding on them, but the robins need the worms (as food) to sustain the population. These conflicting needs keep the populations moving in a cycle around the equilibrium.

Problems

4. We first find the nullclines. Again, we assume $x,\ y \geq 0$.
Vertical nullclines occur where $dx/dt = 0$, which happens when $\frac{dx}{dt} = x(2-x-y) = 0$,
i.e. when $x = 0$ or $x + y = 2$.
Horizontal nullclines occur where $dy/dt = 0$, which happens when $\frac{dy}{dt} = y(1-x-y) = 0$, i.e. when $y = 0$ or $x+y = 1$.
These nullclines are shown in Figure 11.35.

Equilibrium points (also shown in Figure 11.35) occur where both dy/dt and dx/dt are 0, i.e. at the intersections of vertical and horizontal nullclines. There are three such points for these equations: $(0,0)$, $(0,1)$, and $(2,0)$.

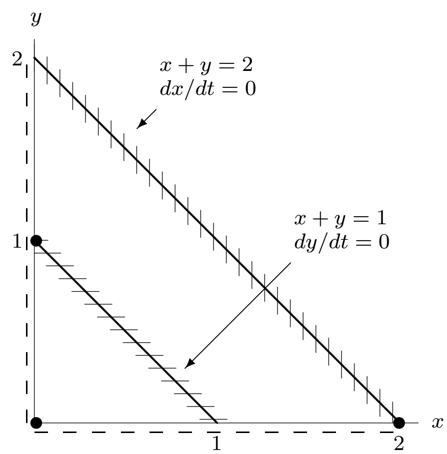


Figure 11.35: Nullclines and equilibrium points (dots)

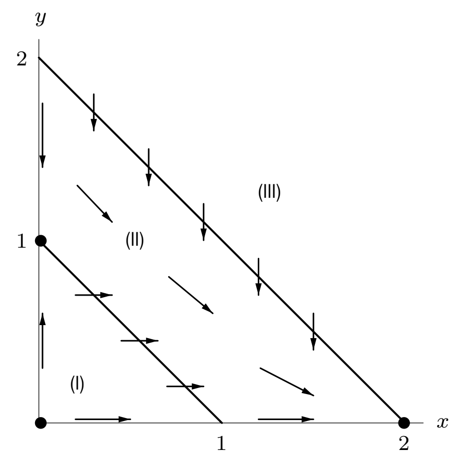


Figure 11.36: General directions of trajectories and equilibrium points (dots)

Looking at sectors in Figure 11.36, we see that no matter in what sector the initial point lies, the trajectory will head toward the equilibrium point $(2,0)$.

5. We first find the nullclines. Vertical nullclines occur where $\frac{dx}{dt} = 0$, which happens when $x = 0$ or $y = \frac{1}{3}(2-x)$. Horizontal nullclines occur where $\frac{dy}{dt} = y(1-2x) = 0$, which happens when $y = 0$ or $x = \frac{1}{2}$. These nullclines are shown in Figure 11.37.

Equilibrium points (also shown in Figure 11.37) occur at the intersections of vertical and horizontal nullclines. There are three such points for this system of equations; $(0,0)$, $(\frac{1}{2},\frac{1}{2})$ and $(2,0)$.

The nullclines divide the positive quadrant into four regions as shown in Figure 11.37. Trajectory directions for these regions are shown in Figure 11.38.

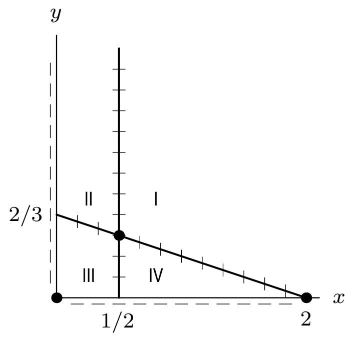


Figure 11.37: Nullclines and equilibrium points (dots)

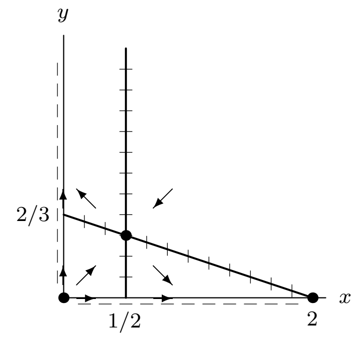


Figure 11.38: General directions of trajectories and equilibrium points (dots)

6. We first find nullclines. Vertical nullclines occur where $\frac{dx}{dt} = x(2 - x - 2y) = 0$, which happens when $x = 0$ or $y = \frac{1}{2}(2 - x)$. Horizontal nullclines occur where $\frac{dy}{dt} = y(1 - 2x - y) = 0$, which happens when $y = 0$ or $y = 1 - 2x$. These nullclines are shown in Figure 11.39.

Equilibrium points (also shown in the figure below) occur at the intersections of vertical and horizontal nullclines. There are three such points for this system; $(0, 0)$, $(0, 1)$, and $(2, 0)$.

The nullclines divide the positive quadrant into three regions as shown in the figure below. Trajectory directions for these regions are shown in Figure 11.40.

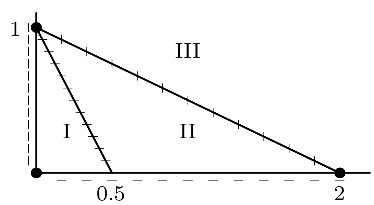


Figure 11.39: Nullclines and equilibrium points (dots)

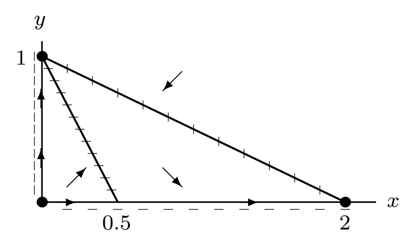


Figure 11.40: General directions of trajectories and equilibrium points (dots)

7. We first find the nullclines. Vertical nullclines occur where $\frac{dx}{dt} = x(1 - y - \frac{x}{3}) = 0$, which happens when $x = 0$ or $y = 1 - \frac{x}{3}$. Horizontal nullclines occur where $\frac{dy}{dt} = y(1 - \frac{y}{2} - x) = 0$, which happens when $y = 0$ or $y = 2(1 - x)$. These nullclines are shown in Figure 11.41.

Equilibrium points (also shown in Figure 11.41) occur at the intersections of vertical and horizontal nullclines. There are four such points for this system: $(0, 0)$, $(0, 2)$, $(3, 0)$, and $(\frac{3}{5}, \frac{4}{5})$.

The nullclines divide the positive quadrant into four regions as shown in Figure 11.41. Trajectory directions for these regions are shown in Figure 11.42.

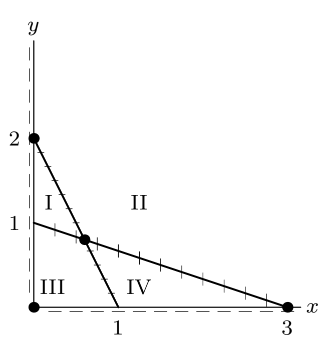


Figure 11.41: Nullclines and equilibrium points (dots)

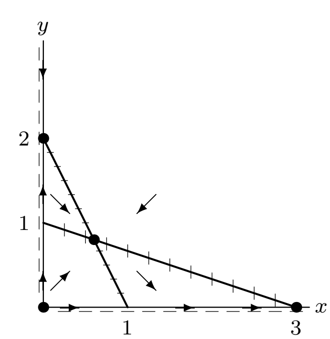


Figure 11.42: General directions of trajectories and equilibrium points (dots)

8. We first find the nullclines. Again, we assume $x,\, y \geq 0$.
$\frac{dx}{dt} = x(1 - x - \frac{y}{3}) = 0$ when $x = 0$ or $x + y/3 = 1$.
$\frac{dy}{dt} = y(1 - y - \frac{x}{2}) = 0$ when $y = 0$ or $y + x/2 = 1$.
These nullclines are shown in Figure 11.43. There are four equilibrium points for these equations. Three of them are the points, $(0,0)$, $(0,1)$, and $(1,0)$. The fourth is the intersection of the two lines $x + y/3 = 1$ and $y + x/2 = 1$. This point is $(\frac{4}{5}, \frac{3}{5})$.

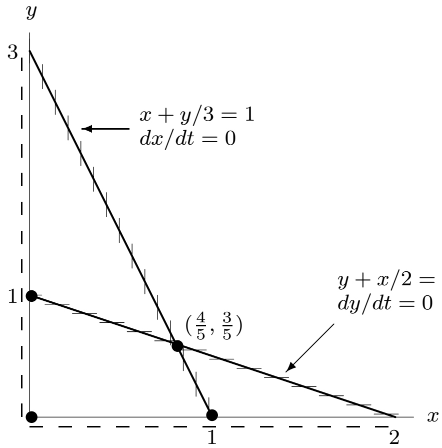


Figure 11.43: Nullclines and equilibrium points (dots)

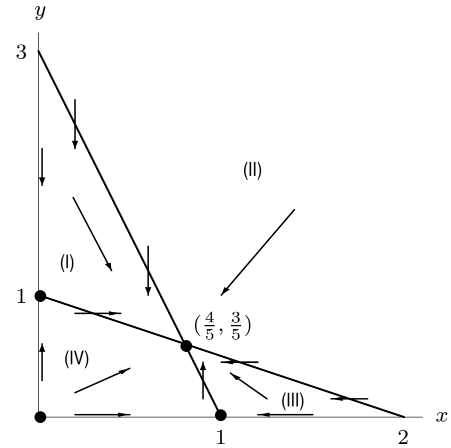


Figure 11.44: General directions of trajectories and equilibrium points (dots)

Looking at sectors in Figure 11.44, we see that no matter in what sector the initial point lies, the trajectory will head toward the equilibrium point $(\frac{4}{5}, \frac{3}{5})$. Only if the initial point lies on the x- or y-axis, will the trajectory head toward the equilibrium points at $(1,0)$, $(0,1)$, or $(0,0)$. In fact, the trajectory will go to $(0,0)$ only if it starts there, in which case $x(t) = y(t) = 0$ for all t. From direction of the trajectories in Figure 11.44, it appears that if the initial point is in sectors (I) or (III), then it will remain in that sector as it heads toward the equilibrium.

9. We assume that $x,\, y \geq 0$ and then find the nullclines. $\frac{dx}{dt} = x(1 - \frac{x}{2} - y) = 0$ when $x = 0$ or $y + \frac{x}{2} = 1$.
$\frac{dy}{dt} = y(1 - \frac{y}{3} - x) = 0$ when $y = 0$ or $x + \frac{y}{3} = 1$.
We find the equilibrium points. They are $(2,0)$, $(0,3)$, $(0,0)$, and $(\frac{4}{5}, \frac{3}{5})$. The nullclines and equilibrium points are shown in Figure 11.45.

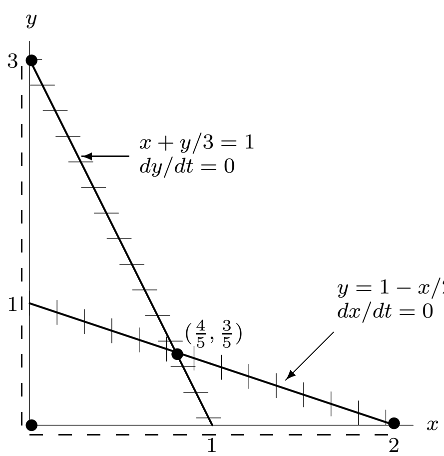


Figure 11.45: Nullclines and equilibrium points (dots)

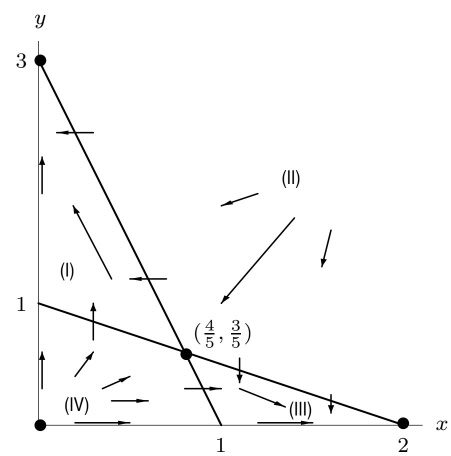


Figure 11.46: General directions of trajectories and equilibrium points (dots)

Figure 11.46 shows that if the initial point is in sector (I), the trajectory heads toward the equilibrium point $(0,3)$. Similarly, if the trajectory begins in sector (III), then it heads toward the equilibrium $(2,0)$ over time. If the trajectory begins in sector (II) or (IV), it can go to any of the three equilibrium points $(2,0)$, $(0,3)$, or $(\frac{4}{5}, \frac{3}{5})$.

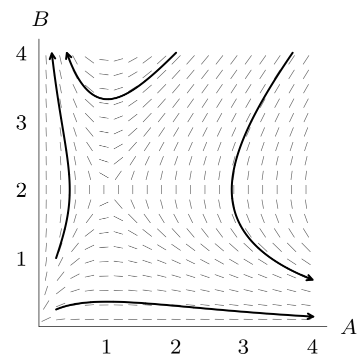
10. (a) If B were not present, then we'd have $A' = 2A$, so company A's net worth would grow exponentially. Similarly, if A were not present, B would grow exponentially. The two companies restrain each other's growth, probably by competing for the market.

(b) To find equilibrium points, find the solutions of the pair of equations

$$A' = 2A - AB = 0$$
$$B' = B - AB = 0$$

The first equation has solutions $A = 0$ or $B = 2$. The second has solutions $B = 0$ or $A = 1$. Thus the equilibrium points are (0,0) and (1,2).

(c) In the long run, one of the companies will go out of business. Two of the trajectories in the figure below go toward the A axis; they represent A surviving and B going out of business. The trajectories going toward the B axis represent A going out of business. Notice both the equilibrium points are unstable.



11. (a) The nullclines are $P = 0$ or $P_1 + 3P_2 = 13$ (where $dP_1/dt = 0$) and $P = 0$ or $P_2 + 0.4P_1 = 6$ (where $dP_2/dt = 0$).

(b) The phase plane in Figure 11.47 shows that P_2 will eventually exclude P_1 regardless of where the experiment starts so long as there were some P_2 originally. Consequently, the data points would have followed a trajectory that starts at the origin, crosses the first nullcline and goes left and upward between the two nullclines to the point $P_1 = 0$, $P_2 = 6$.

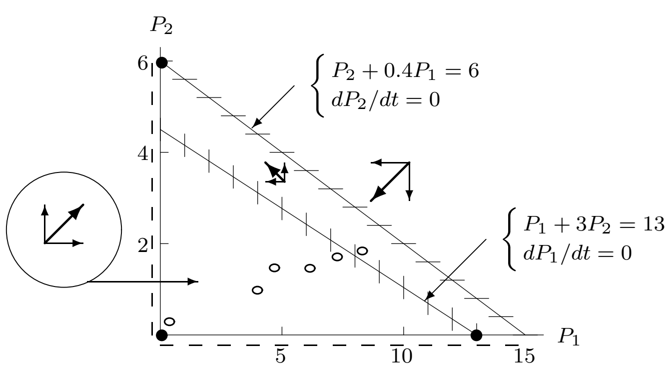


Figure 11.47: Nullclines and equilibrium points (dots) for Gause's yeast data (hollow dots)

12. (a) In the equation for dx/dt, the term involving x, namely $-0.2x$, is negative meaning that as x increases, dx/dt decreases. This corresponds to the statement that the more a country spends on armaments, the less it wants to increase spending.

On the other hand, since $+0.15y$ is positive, as y increases, dx/dt increases, corresponding to the fact that the more a country's opponent arms, the more the country will arm itself.

The constant term, 20, is positive means that if both countries are unarmed initially, (so $x = y = 0$), then dx/dt is positive and so the country will start to arm. In other words, disarmament is not an equilibrium situation in this model.

(b) The nullclines are shown in Figure 11.48. When $dx/dt = 0$, the trajectories are vertical (on the line $-0.2x+0.15y+20 = 0$); when $dy/dt = 0$ the trajectories are horizontal (on $0.1x - 0.2y + 40 = 0$). There is only one equilibrium point, $x = y = 400$.

(c) In region I, try $x = 400$, $y = 0$, giving

$$\frac{dx}{dt} = -0.2(400) + 0.15(0) + 20 < 0$$
$$\frac{dy}{dt} = 0.1(400) - 0.2(0) + 4 - 0 > 0$$

In region II, try $x = 500$, $y = 500$, giving

$$\frac{dx}{dt} = -0.2(500) + 0.15(500) + 20 < 0$$
$$\frac{dy}{dt} = 0.1(500) - 0.2(500) + 40 < 0$$

In region III, try $x = 0$, $y = 400$, giving

$$\frac{dx}{dt} = -0.2(0) + 0.15(400) + 20 > 0$$
$$\frac{dy}{dt} = 0.1(0) - 0.2(400) + 40 < 0$$

In region IV, try $x = 0$, $y = 0$, giving

$$\frac{dx}{dt} = -0.2(0) + 0.15(0) + 20 > 0$$
$$\frac{dy}{dt} = 0.1(0) - 0.2(0) + 40 > 0$$

See Figure 11.48.

(d) The one equilibrium point is stable.

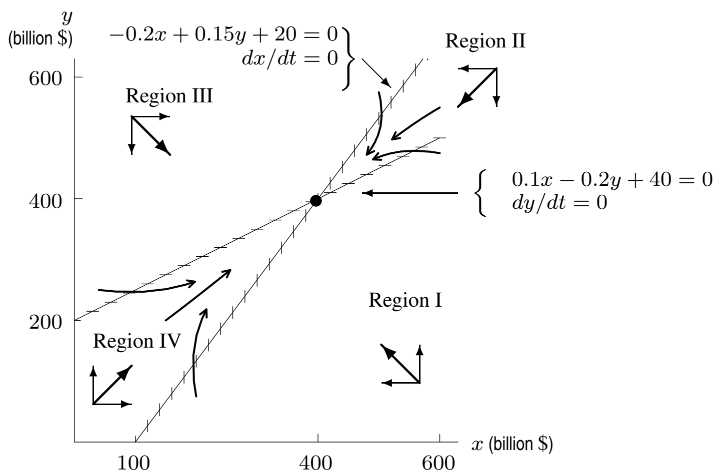


Figure 11.48: Nullclines and equilibrium point(dot) for arms race

(e) If both sides disarm, then both sides spend \$0. Thus initially $x = y = 0$, and $dx/dt = 20$ and $dy/dt = 40$. Since both dx/dt and dy/dt are positive, both sides start arming. Figure 11.48 shows that they will both arm until each is spending about \$400 billion.

(f) If the country spending $\$y$ billion is unarmed, then $y = 0$ and the corresponding point on the phase plane is on the x-axis. Any trajectory starting on the x-axis tends toward the equilibrium point $x = y = 400$. Similarly, a trajectory starting on the y-axis represents the other country being unarmed; such a trajectory also tends to the same equilibrium point.

Thus, if either side disarms unilaterally, that is, if we start out with one of the countries spending nothing, then over time, they will still both end up spending roughly \$400 billion.

(g) This model predicts that, in the long run, both countries will spend near to \$400 billion, no matter where they start.

13. (a)

$$\frac{dx}{dt} = 0 \text{ when } x = \frac{10.5}{0.45} = 23.3$$
$$\frac{dy}{dt} = 0 \text{ when } 8.2x - 0.8y - 142 = 0$$

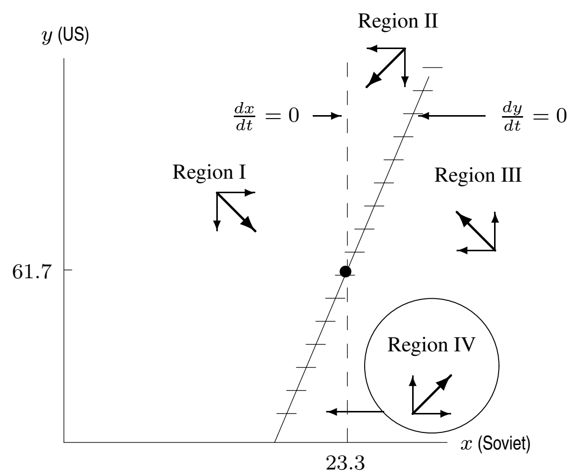


Figure 11.49: Nullclines and equilibrium point (dot) for US-Soviet arms race

There is an equilibrium point where the trajectories cross at $x = 23.3$, $y = 61.7$

In region I, $\frac{dx}{dt} > 0$, $\frac{dy}{dt} < 0$.

In region II, $\frac{dx}{dt} < 0$, $\frac{dy}{dt} < 0$.

In region III, $\frac{dx}{dt} < 0$, $\frac{dy}{dt} > 0$.

In region IV, $\frac{dx}{dt} > 0$, $\frac{dy}{dt} > 0$.

(b)

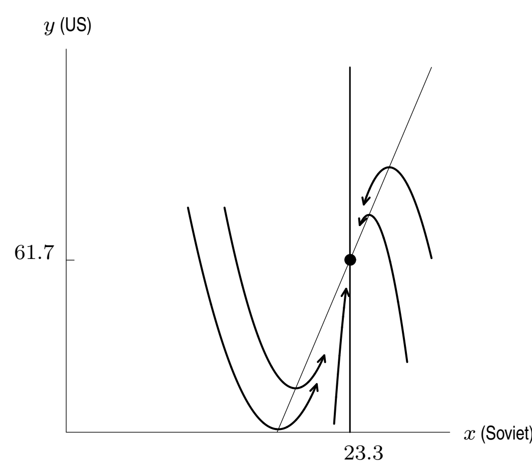


Figure 11.50: Trajectories for US-Soviet arms race.

(c) All the trajectories tend toward the equilibrium point $x = 23.3$, $y = 61.7$. Thus the model predicts that in the long run the arms race will level off with the Soviet Union spending 23.3 billion dollars a year on arms and the US 61.7 billion dollars.

(d) As the model predicts, yearly arms expenditure did tend toward 23 billion for the Soviet Union and 62 billion for the US.

Solutions for Section 11.10

Exercises

1. If $y = 2\cos t + 3\sin t$, then $y' = -2\sin t + 3\cos t$ and $y'' = -2\cos t - 3\sin t$. Thus, $y'' + y = 0$.

2. If $y(t) = 3\sin(2t) + 2\cos(2t)$ then

$$y' = 6\cos(2t) - 4\sin(2t)$$
$$y'' = -12\sin(2t) - 8\cos(2t) = -4(3\sin(2t) + 2\cos(2t)) = -4y$$

as required.

3. If $y = A\cos t + B\sin t$, then $y' = -A\sin t + B\cos t$ and $y'' = -A\cos t - B\sin t$. Thus, $y'' + y = 0$.

4. If $y(t) = A\sin(2t) + B\cos(2t)$ then

$$y' = 2A\cos(2t) - 2B\sin(2t)$$
$$y'' = -4A\sin(2t) - 4B\cos(2t)$$

therefore

$$y'' + 4y = -4A\sin(2t) - 4B\cos(2t) + 4(A\sin(2t) + B\cos(2t)) = 0$$

for all values of A and B, so the given function is a solution.

5. If $y(t) = A\sin(\omega t) + B\cos(\omega t)$ then

$$y' = \omega A\cos(\omega t) - \omega B\sin(\omega t)$$
$$y'' = -\omega^2 A\sin(\omega t) - \omega^2 B\cos(\omega t)$$

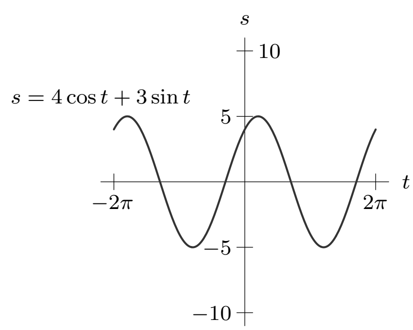
therefore

$$y'' + \omega^2 y = -\omega^2 A\sin(\omega t) - \omega^2 B\cos(2t) + \omega^2(A\sin(\omega t) + B\cos(\omega t)) = 0$$

for all values of A and B, so the given function is a solution.

6. $y = A\cos\alpha t$
$y' = -\alpha A\sin\alpha t$
$y'' = -\alpha^2 A\cos\alpha t$
If $y'' + 5y = 0$, then $-\alpha^2 A\cos\alpha t + 5A\cos\alpha t = 0$, so $A(5-\alpha^2)\cos\alpha t = 0$. This is true for all t if $A = 0$, or if $\alpha = \pm\sqrt{5}$.
We also have the initial condition: $y'(1) = -\alpha A\sin\alpha = 3$. Notice that this equation will not work if $A = 0$. If $\alpha = \sqrt{5}$, then
$A = -\frac{3}{\sqrt{5}\sin\sqrt{5}} \approx -1.705$.
Similarly, if $\alpha = -\sqrt{5}$, we find that $A \approx -1.705$. Thus, the possible values are $A = -\frac{3}{\sqrt{5}\sin\sqrt{5}} \approx -1.705$ and $\alpha = \pm\sqrt{5}$.

7. (a)



(b) Trace along the curve to the highest point; which has coordinates of about $(0.66, 5)$, so $A \approx 5$. If $s = 5\sin(t+\phi)$, then the maximum occurs where $t \approx 0.66$ and $t + \phi = \pi/2$, that is $0.66 + \phi \approx 1.57$, giving $\phi \approx 0.91$.

(c) Analytically

$$A = \sqrt{4^2 + 3^2} = 5$$

and

$$\tan\phi = \frac{4}{3} \quad \text{so} \quad \phi = \arctan\left(\frac{4}{3}\right) = 0.93.$$

8. We want to find A and ϕ such that

$$\cos t - \sin t = A\sin(t+\phi).$$

We know that $A = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Also, $\tan\phi = 1/(-1) = -1$, so $\phi = -\pi/4$ or $\phi = 3\pi/4$. Since $C_1 = 1 > 0$, we take $\phi = 3\pi/4$, giving

$$s(t) = \sqrt{2}\sin\left(t + \frac{3\pi}{4}\right)$$

as our solution. The graph of $s(t)$ is in Figure 11.51.

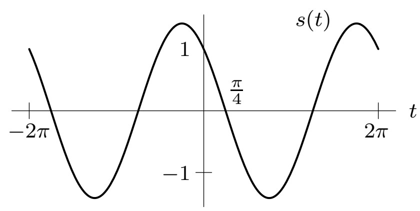


Figure 11.51: Graph of the function $s(t) = \sqrt{2}\sin(t + \frac{3\pi}{4})$

9. The amplitude is $\sqrt{3^2 + 7^2} = \sqrt{58}$.

10. If we write $y = 3\sin 2t + 4\cos 2t$ in the form $y(t) = A\sin(2t+\phi)$, then $A = \sqrt{3^2+4^2} = 5$.

11. Take $\omega = 2$. The amplitude is $A = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$. The phase shift is $\psi = \tan^{-1}\dfrac{12}{5}$.

12. The amplitude is $A = \sqrt{7^2 + 24^2} = \sqrt{625} = 25$.
The phase shift, ϕ, is given by $\tan\phi = \frac{24}{7}$, so $\phi = \arctan\frac{24}{7} \approx 1.287$ or $\phi \approx -1.855$.
Since $C_1 = 24 > 0$, we want $\phi = 1.287$, so the solution is $25\sin(\omega t + 1.287)$.

Problems

13. First, we note that the solutions of:
(a) $x'' + x = 0$ are $x = A\cos t + B\sin t$;
(b) $x'' + 4x = 0$ are $x = A\cos 2t + B\sin 2t$;
(c) $x'' + 16x = 0$ are $x = A\cos 4t + B\sin 4t$.
This follows from what we know about the general solution to $x'' + \omega^2 x = 0$.
The period of the solutions to (a) is 2π, the period of the solutions to (b) is π, and the period of the solutions of (c) is $\frac{\pi}{2}$.
Since the t-scales are the same on all of the graphs, we see that graphs (I) and (IV) have the same period, which is twice the period of graph (III). Graph (II) has twice the period of graphs (I) and (IV). Since each graph represents a solution, we have the following:

- equation (a) goes with graph (II)
  equation (b) goes with graphs (I) and (IV)
  equation (c) goes with graph (III)
- The graph of (I) passes through $(0,0)$, so $0 = A\cos 0 + B\sin 0 = A$. Thus, the equation is $x = B\sin 2t$. Since the amplitude is 2, we see that $x = 2\sin 2t$ is the equation of the graph. Similarly, the equation for (IV) is $x = -3\sin 2t$. The graph of (II) also passes through $(0,0)$, so, similarly, the equation must be $x = B\sin t$. In this case, we see that $B = -1$, so $x = -\sin t$.
  Finally, the graph of (III) passes through $(0,1)$, and 1 is the maximum value. Thus, $1 = A\cos 0 + B\sin 0$, so $A = 1$. Since it reaches a local maximum at $(0,1)$, $x'(0) = 0 = -4A\sin 0 + 4B\cos 0$, so $B = 0$. Thus, the solution is $x = \cos 4t$.

14. (a) Let $y = c_1\sinh wt + c_2\cosh wt$. Then

$$y' = w(c_1\cosh wt + c_2\sinh wt)$$

and

$$y'' = w^2(c_1\sinh wt + c_2\cosh wt) = w^2 y,$$

so $y'' - w^2 y = 0$.

(b) (i) Since $y(0) = 0$, we have $c_1 \sinh 0 + c_2 \cosh 0 = c_2 = 0$. Since $y(1) = 6$ and $c_2 = 0$, we have that

$$y(1) = c_1 \sinh w = 6,$$

so

$$c_1 = \frac{6}{\sinh w}$$

Therefore $y = \dfrac{6 \sinh wt}{\sinh w}$.

(ii) Since $y'(0) = 0$, we have $y'(0) = w(c_1 \cosh 0 + c_2 \sinh 0) = wc_1 = 0$, so $c_1 = 0$. Thus

$$y(1) = c_2 \cosh w = 6$$

implies that

$$c_2 = \frac{6}{\cosh w},$$

giving us that $y = \dfrac{6 \cosh wt}{\cosh w}$.

15. At $t = 0$, we find that $y = 2$, which is clearly the highest point since $-1 \leq \cos 3t \leq 1$. Thus, at $t = 0$ the mass is at its highest point. Since $y' = -6 \sin 3t$, we see $y' = 0$ when $t = 0$. Thus, at $t = 0$ the object is at rest, although it will move down after $t = 0$.

16. At $t = 0$, we find that $y = 0$. Since $-1 \leq \sin 3t \leq 1$, y ranges from -0.5 to 0.5, so at $t = 0$ it is starting in the middle. Since $y' = -1.5 \cos 3t$, we see $y' = -1.5$ when $t = 0$, so the mass is moving downward.

17. At $t = 0$, we find that $y = -1$, which is clearly the lowest point on the path. Since $y' = 3 \sin 3t$, we see that $y' = 0$ when $t = 0$. Thus, at $t = 0$ the object is at rest, although it will move up after $t = 0$.

18. All the differential equations have solutions of the form $s(t) = C_1 \sin \omega t + C_2 \cos \omega t$. Since for all of them, $s'(0) = 0$, we have $s'(0) = 0 = C_1 \omega \cos 0 - C_2 \omega \sin 0 = 0$, giving $C_1 \omega = 0$. Thus, either $C_1 = 0$ or $\omega = 0$. If $\omega = 0$, then $s(t)$ is a constant function, and since the equations represent oscillating springs, we don't want $s(t)$ to be a constant function. Thus, $C_1 = 0$, so all four equations have solutions of the form $s(t) = C \cos \omega t$.

i) $s'' + 4s = 0$, so $\omega = \sqrt{4} = 2$. $s(0) = C \cos 0 = C = 5$. Thus, $s(t) = 5 \cos 2t$.

ii) $s'' + \frac{1}{4}s = 0$, so $\omega = \sqrt{\frac{1}{4}} = \frac{1}{2}$. $s(0) = C \cos 0 = C = 10$. Thus, $s(t) = 10 \cos \frac{1}{2}t$.

iii) $s'' + 6s = 0$, so $\omega = \sqrt{6}$. $s(0) = C = 4$, Thus, $s(t) = 4 \cos \sqrt{6}t$.

iv) $s'' + \frac{1}{6}s = 0$, so $\omega = \sqrt{\frac{1}{6}}$. $s(0) = C = 20$. Thus, $s(t) = 20 \cos \sqrt{\frac{1}{6}}t$.

(a) Spring (iii) has the shortest period, $\frac{2\pi}{\sqrt{6}}$. (Other periods are π, 4π, $2\pi\sqrt{6}$)

(b) Spring (iv) has the largest amplitude, 20.

(c) Spring (iv) has the longest period, $2\pi\sqrt{6}$.

(d) Spring (i) has the largest maximum velocity. We can see this by looking at $v(t) = s'(t) = -C\omega \sin \omega t$. The velocity is just a sine function, so we look for the derivative with the biggest amplitude, which will have the greatest value. The velocity function for Spring i) has amplitude 10, the largest of the four springs. (The other velocity amplitudes are $10 \cdot \frac{1}{2} = 5$, $4\sqrt{6} \approx 9.8$, $\frac{20}{\sqrt{6}} \approx 8.2$)

19. (a) Since $\omega^2 = 9$, $\omega = 3$, and so the general solution is of the form

$$y(t) = A \sin(3t) + B \cos(3t).$$

(b) (i) $y(0) = 0$, gives $A \sin(0) + B \cos(0) = 0$ so that $B = 0$.

$$y'(t) = 3A \cos(3t)$$

$y'(0) = 1$ gives $3A = 1$ and so

$$y(t) = \frac{1}{3} \sin(3t).$$

(ii) $y(0) = 1$, gives $A \sin(0) + B \cos(0) = 1$ so that $B = 1$.

$$y'(t) = 3A \cos(3t) - 3 \sin(3t)$$

$y'(0) = 0$ gives $3A = 0$ and so

$$y(t) = \cos(3t).$$

(iii) $y(0) = 1$, gives $A\sin(0) + B\cos(0) = 1$ so that $B = 1$. $y(1) = 0$ gives $A\sin 3 + \cos 3 = 0$ and so $A = \dfrac{-\cos 3}{\sin 3}$, so

$$y(t) = \frac{-\cos 3}{\sin 3}\sin(3t) + \cos(3t).$$

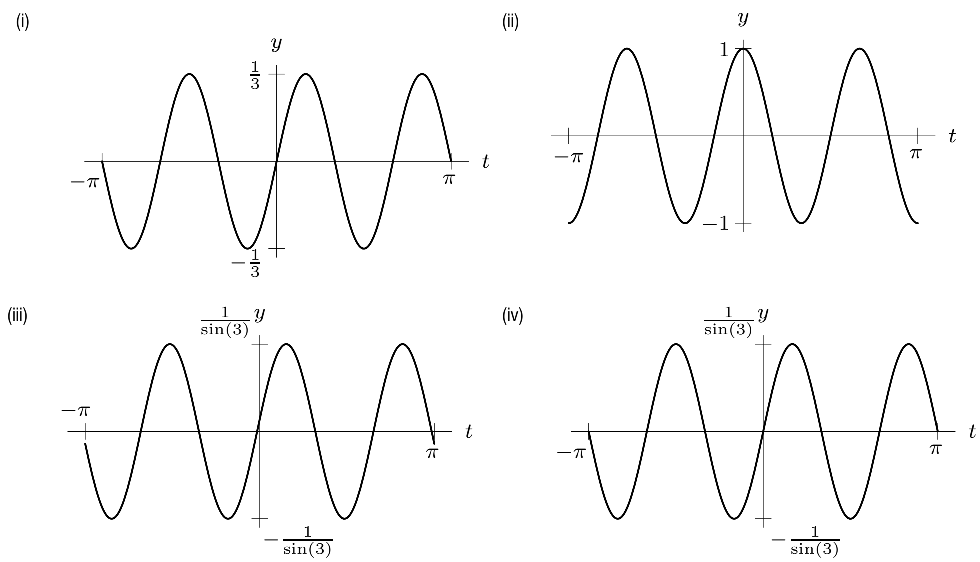
Note that using the trigonometric identities, we can write this as:

$$\begin{aligned} y(t) &= \frac{-\cos 3}{\sin 3}\sin(3t) + \cos(3t) \\ &= \frac{1}{\sin 3}(\sin 3\cos(3t) - \cos 3\sin(3t)) \\ &= \frac{1}{\sin 3}\sin(3 - 3t). \end{aligned}$$

(iv) $y(0) = 0$, gives $A\sin(0) + B\cos(0) = 0$ so that $B = 0$. $y(1) = 1$ gives $A\sin(3) = 1$ and so $A = \dfrac{1}{\sin(3)}$ so

$$y(t) = \frac{1}{\sin(3)}\sin(3t).$$

(c)



20. (a) General solution

$$x(t) = A\cos 4t + B\sin 4t.$$

Thus,

$$5 = A\cos 0 + B\sin 0 \quad \text{so } A = 5.$$

Since $x'(0) = 0$, we have

$$0 = -4A\sin 0 + 4B\cos 0 \quad \text{so } B = 0.$$

Thus,

$$x(t) = 5\cos 4t$$

so amplitude $= 5$, period $= \frac{2\pi}{4} = \frac{\pi}{2}$.

(b) General solution

$$x(t) = A\cos\left(\frac{t}{5}\right) + B\sin\left(\frac{t}{5}\right).$$

Since $x(0) = -1$, we have $A = -1$.
Since $x'(0) = 2$, we have

$$2 = -\frac{A}{5}\sin 0 + \frac{B}{5}\cos 0 \quad \text{so } B = 10.$$

Thus,

$$x(t) = -\cos\left(\frac{t}{5}\right) + 10\sin\left(\frac{t}{5}\right).$$

So, amplitude $= \sqrt{(-1)^2 + 10^2} = \sqrt{101}$, period $= \dfrac{2\pi}{1/5} = 10\pi$.

21. (a) Since a mass of 3 kg stretches the spring by 2 cm, the spring constant k is given by

$$3g = 2k \quad \text{so} \quad k = \frac{3g}{2}.$$

See Figure 11.52.

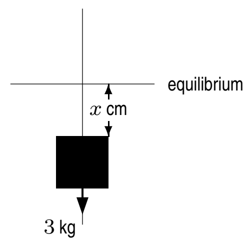


Figure 11.52

Suppose we measure the displacement x from the equilibrium; then, using

$$\text{Mass} \cdot \text{Acceleration} = \text{Force}$$

gives

$$3x'' = -kx = -\frac{3gx}{2}$$
$$x'' + \frac{g}{2}x = 0$$

Since at time $t = 0$, the brick is 5 cm below the equilibrium and not moving, the initial conditions are $x(0) = 5$ and $x'(0) = 0$.

(b) The solution to the differential equation is

$$x = A\cos\left(\sqrt{\frac{g}{2}}t\right) + B\sin\left(\sqrt{\frac{g}{2}}t\right).$$

Since $x(0) = 5$, we have

$$x = A\cos(0) + B\sin(0) = 5 \quad \text{so} \quad A = 5.$$

In addition,

$$x'(t) = -5\sqrt{\frac{g}{2}}\sin\left(\sqrt{\frac{g}{2}}t\right) + B\sqrt{\frac{g}{2}}\cos\left(\sqrt{\frac{g}{2}}t\right)$$

so

$$x'(0) = -5\sqrt{\frac{g}{2}}\sin(0) + B\sqrt{\frac{g}{2}}\cos(0) = 0 \quad \text{so} \quad B = 0.$$

Thus,

$$x = 5\cos\sqrt{\frac{g}{2}}t.$$

22. (a) We are given $\frac{d^2x}{dt^2} = -\frac{g}{l}x$, so $x = C_1\cos\sqrt{\frac{g}{l}}t + C_2\sin\sqrt{\frac{g}{l}}t$. We use the initial conditions to find C_1 and C_2.

$$x(0) = C_1\cos 0 + C_2\sin 0 = C_1 = 0$$
$$x'(0) = -C_1\sqrt{\frac{g}{l}}\sin 0 + C_2\sqrt{\frac{g}{l}}\cos 0 = C_2\sqrt{\frac{g}{l}} = v_0$$

Thus, $C_1 = 0$ and $C_2 = v_0\sqrt{\frac{l}{g}}$, so $x = v_0\sqrt{\frac{l}{g}}\sin\sqrt{\frac{g}{l}}t$.

(b) Again, $x = C_1\cos\sqrt{\frac{g}{l}}t + C_2\sin\sqrt{\frac{g}{l}}t$, but this time, $x(0) = x_0$, and $x'(0) = 0$.
Thus, as before, $x(0) = C_1 = x_0$, and $x'(0) = C_2\sqrt{\frac{g}{l}} = 0$. In this case, $C_1 = x_0$ and $C_2 = 0$. Thus, $x = x_0\cos\sqrt{\frac{g}{l}}t$.

23. (a) If x_0 is increased, the amplitude of the function x is increased, but the period remains the same. In other words, the pendulum will start higher, but the time to swing back and forth will stay the same.

(b) If l is increased, the period of the function x is increased. (Remember, the period of $x_0 \cos \sqrt{\frac{g}{l}}t$ is $\frac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{l/g}$.) In other words, it will take longer for the pendulum to swing back and forth.

24. (a) Let $x = \omega t$ and $y = \phi$. Then

$$\begin{aligned} A\sin(\omega t + \phi) &= A(\sin\omega t\cos\phi + \cos\omega t\sin\phi) \\ &= (A\sin\phi)\cos\omega t + (A\cos\phi)\sin\omega t. \end{aligned}$$

(b) If we want $A\sin(\omega t + \phi) = C_1\cos\omega t + C_2\sin\omega t$ to be true for all t, then by looking at the answer to part (a), we must have $C_1 = A\sin\phi$ and $C_2 = A\cos\phi$. Thus,

$$\frac{C_1}{C_2} = \frac{A\sin\phi}{A\cos\phi} = \tan\phi,$$

and

$$\sqrt{C_1^2 + C_2^2} = \sqrt{A^2\sin^2\phi + A^2\cos^2\phi} = A\sqrt{\sin^2\phi + \cos^2\phi} = A,$$

so our formulas are justified.

25. (a) $36\dfrac{d^2Q}{dt^2} + \dfrac{Q}{9} = 0$ so $\dfrac{d^2Q}{dt^2} = -\dfrac{Q}{324}$.
Thus,

$$\begin{aligned} Q &= C_1\cos\frac{1}{18}t + C_2\sin\frac{1}{18}t. \\ Q(0) &= 0 = C_1\cos 0 + C_2\sin 0 = C_1, \\ \text{so}\quad C_1 &= 0. \end{aligned}$$

So, $Q = C_2\sin\dfrac{1}{18}t$, and

$$\begin{aligned} Q' = I &= \frac{1}{18}C_2\cos\frac{1}{18}t. \\ Q'(0) = I(0) = 2 &= \frac{1}{18}C_2\cos\left(\frac{1}{18}\cdot 0\right) = \frac{1}{18}C_2, \\ \text{so}\quad C_2 &= 36. \end{aligned}$$

Therefore, $Q = 36\sin\dfrac{1}{18}t$.

(b) As in part (a), $Q = C_1\cos\dfrac{1}{18}t + C_2\sin\dfrac{1}{18}t$.
According to the initial conditions:

$$\begin{aligned} Q(0) &= 6 = C_1\cos 0 + C_2\sin 0 = C_1, \\ \text{so}\quad C_1 &= 6. \end{aligned}$$

So $Q = 6\cos\dfrac{1}{18}t + C_2\sin\dfrac{1}{18}t$.
Thus,

$$\begin{aligned} Q' = I &= -\frac{1}{3}\sin\frac{1}{18}t + \frac{1}{18}C_2\cos\frac{1}{18}t. \\ Q'(0) = I(0) = 0 &= -\frac{1}{3}\sin\left(\frac{1}{18}\cdot 0\right) + \frac{1}{18}C_2\cos\left(\frac{1}{18}\cdot 0\right) = \frac{1}{18}C_2, \\ \text{so}\quad C_2 &= 0. \end{aligned}$$

Therefore, $Q = 6\cos\dfrac{1}{18}t$.

26. The equation we have for the charge tells us that:

$$\frac{d^2Q}{dt^2} = -\frac{Q}{LC},$$

where L and C are positive.
If we let $\omega = \sqrt{\frac{1}{LC}}$, we know the solution is of the form:

$$Q = C_1 \cos \omega t + C_2 \sin \omega t.$$

Since $Q(0) = 0$, we find that $C_1 = 0$, so $Q = C_2 \sin \omega t$.
Since $Q'(0) = 4$, and $Q' = \omega C_2 \cos \omega t$, we have $C_2 = \dfrac{4}{\omega}$, so $Q = \dfrac{4}{\omega} \sin \omega t$.
But we want the maximum charge, meaning the amplitude of Q, to be $2\sqrt{2}$ coulombs. Thus, we have $\dfrac{4}{\omega} = 2\sqrt{2}$, which gives us $\omega = \sqrt{2}$.
So we now have: $\sqrt{2} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10C}}$. Thus, $C = \frac{1}{20}$ farads.

27. We know that the general formula for Q will be of the form:

$$Q = C_1 \cos \omega t + C_2 \sin \omega t.$$

and

$$I = Q' = -C_1 \sin \omega t + C_2 \cos \omega t$$

Thus, as $t \to \infty$, neither one approaches a limit. Instead, they vary sinusoidally, with the same frequency but out of phase. We can think of the charge on the capacitor as being analogous to the displacement of a mass on a spring, oscillating from positive to negative. The current is then like the velocity of the mass, also oscillating from positive to negative. When the charge is maximal or minimal, the current is zero (just like when the spring is at the top or bottom of its motion), and when the current is maximal, the charge is zero (just like when the spring is at the middle of its motion).

Solutions for Section 11.11

Exercises

1. The characteristic equation is $r^2 + 4r + 3 = 0$, so $r = -1$ or -3.
Therefore $y(t) = C_1 e^{-t} + C_2 e^{-3t}$.

2. The characteristic equation is $r^2 + 4r + 4 = 0$, so $r = -2$.
Therefore $y(t) = (C_1 t + C_2)e^{-2t}$.

3. The characteristic equation is $r^2 + 4r + 5 = 0$, so $r = -2 \pm i$.
Therefore $y(t) = C_1 e^{-2t} \cos t + C_2 e^{-2t} \sin t$.

4. The characteristic equation is $r^2 - 7 = 0$, so $r = \pm\sqrt{7}$.
Therefore $s(t) = C_1 e^{\sqrt{7}t} + C_2 e^{-\sqrt{7}t}$.

5. The characteristic equation is $r^2 + 7 = 0$, so $r = \pm\sqrt{7}i$.
Therefore $s(t) = C_1 \cos \sqrt{7}t + C_2 \sin \sqrt{7}t$.

6. If we try a solution $y(t) = Ae^{rt}$ then

$$r^2 - 3r + 2 = 0$$

which has the solutions $r = 2$ and $r = 1$ so that the general solution is of the form

$$y(t) = Ae^{2t} + Be^{t}$$

7. The characteristic equation is $4r^2 + 8r + 3 = 0$, so $r = -1/2$ or $-3/2$.
Therefore $z(t) = C_1 e^{-t/2} + C_2 e^{-3t/2}$.

8. The characteristic equation is $r^2 + 4r + 8 = 0$, so $r = -2 \pm 2i$.
Therefore $x(t) = C_1 e^{-2t} \cos 2t + C_2 e^{-2t} \sin 2t$.

9. The characteristic equation is $r^2 + r + 1 = 0$, so $r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.
Therefore $p(t) = C_1 e^{-t/2} \cos \frac{\sqrt{3}}{2}t + C_2 e^{-t/2} \sin \frac{\sqrt{3}}{2}t$.

10. If we try a solution $z(t) = Ae^{rt}$ then

$$r^2 + 2 = 0$$

so that the general solution is of the form:

$$y(t) = A\sin\sqrt{2}t + B\cos\sqrt{2}t$$

11. If we try a solution $z(t) = Ae^{rt}$ then

$$r^2 + 2r = 0$$

which has solutions $r = 0$ and $r = -2$ so that the general solution is of the form

$$y(t) = A + Be^{-2t}$$

12. If we try a solution $P(t) = Ae^{rt}$ then

$$r^2 + 2r + 1 = 0$$

which has the repeated solution $r = -1$ so that the general solution is of the form

$$y(t) = (At + B)e^{-t}$$

13. The characteristic equation is

$$r^2 + 5r + 6 = 0$$

which has the solutions $r = -2$ and $r = -3$ so that

$$y(t) = Ae^{-3t} + Be^{-2t}$$

The initial condition $y(0) = 1$ gives

$$A + B = 1$$

and $y'(0) = 0$ gives

$$-3A - 2B = 0$$

so that $A = -2$ and $B = 3$ and

$$y(t) = -2e^{-3t} + 3e^{-2t}$$

14. The characteristic equation is

$$r^2 + 5r + 6 = 0$$

which has the solutions $r = -2$ and $r = -3$ so that

$$y(t) = Ae^{-3t} + Be^{-2t}$$

The initial condition $y(0) = 5$ gives

$$A + B = 5$$

and $y'(0) = 1$ gives

$$-3A - 2B = 1$$

so that $A = -11$ and $B = 16$ and

$$y(t) = -11e^{-3t} + 16e^{-2t}$$

15. The characteristic equation is

$$r^2 - 3r - 4 = 0$$

which has the solutions $r = 4$ and $r = -1$ so that

$$y(t) = Ae^{4t} + Be^{-t}$$

The initial condition $y(0) = 1$ gives

$$A + B = 1$$

and $y'(0) = 0$ gives

$$4A - B = 0$$

so that $A = \frac{1}{5}$ and $B = \frac{4}{5}$ and

$$y(t) = \frac{1}{5}e^{4t} + \frac{4}{5}e^{-t}$$

16. The characteristic equation is

$$r^2 - 3r - 4 = 0$$

which has the solutions $r = 4$ and $r = -1$ so that

$$y(t) = Ae^{4t} + Be^{-t}$$

The initial condition $y(0) = 0$ gives

$$A + B = 0$$

and $y'(0) = 0.5$ gives

$$4A - B = 0.5$$

so that $A = \frac{1}{10}$ and $B = -\frac{1}{10}$ and

$$y(t) = \frac{1}{10}e^{4t} - \frac{1}{10}e^{-t}$$

17. The characteristic equation is $r^2 + 6r + 5 = 0$, so $r = -1$ or -5.
Therefore $y(t) = C_1e^{-t} + C_2e^{-5t}$.
$y'(t) = -C_1e^{-t} - 5C_2e^{-5t}$
$y'(0) = 0 = -C_1 - 5C_2$
$y(0) = 1 = C_1 + C_2$
Therefore $C_2 = -1/4,\ C_1 = 5/4$ and $y(t) = \frac{5}{4}e^{-t} - \frac{1}{4}e^{-5t}$.

18. The characteristic equation is $r^2 + 6r + 5 = 0$, so $r = -1$ or -5.
Therefore $y(t) = C_1e^{-t} + C_2e^{-5t}$.
$y'(t) = -C_1e^{-t} - 5C_2e^{-5t}$
$y'(0) = 5 = -C_1 - 5C_2$
$y(0) = 5 = C_1 + C_2$
Therefore $C_2 = -5/2,\ C_1 = 15/2$ and $y(t) = \frac{15}{2}e^{-t} - \frac{5}{2}e^{-5t}$.

19. The characteristic equation is $r^2 + 6r + 10 = 0$, so $r = -3 \pm i$.
Therefore $y(t) = C_1e^{-3t}\cos t + C_2e^{-3t}\sin t$.
$y'(t) = C_1[e^{-3t}(-\sin t) + (-3e^{-3t})\cos t] + C_2[e^{-3t}\cos t + (-3e^{-3t})\sin t]$
$y'(0) = 2 = -3C_1 + C_2$
$y(0) = 0 = C_1$
Therefore $C_1 = 0, C_2 = 2$ and $y(t) = 2e^{-3t}\sin t$.

20. The characteristic equation is $r^2 + 6r + 10 = 0$, so $r = -3 \pm i$.
Therefore $y(t) = C_1e^{-3t}\cos t + C_2e^{-3t}\sin t$.
$y'(t) = C_1[e^{-3t}(-\sin t) + (-3e^{-3t})\cos t] + C_2[e^{-3t}\cos t + (-3e^{-3t})\sin t]$
$y'(0) = 0 = -3C_1 + C_2$
$y(0) = 0 = C_1$
Therefore $C_1 = C_2 = 0$ and $y(t) = 0$.

21. The characteristic equation is

$$r^2 + 5r + 6 = 0$$

which has the solutions $r = -2$ and $r = -3$ so that

$$y(t) = Ae^{-2t} + Be^{-3t}$$

The initial condition $y(0) = 1$ gives

$$A + B = 1$$

and $y(1) = 0$ gives

$$Ae^{-2} + Be^{-3} = 0$$

so that $A = \dfrac{1}{1-e}$ and $B = -\dfrac{e}{1-e}$ and

$$y(t) = \frac{1}{1-e}e^{-2t} + \frac{-e}{1-e}e^{-3t}$$

22. The characteristic equation is

$$r^2 + 5r + 6 = 0$$

which has the solutions $r = -2$ and $r = -3$ so that

$$y(t) = Ae^{-2t} + Be^{-3t}$$

The initial condition $y(-2) = 0$ gives

$$Ae^4 + Be^6 = 0$$

and $y(2) = 3$ gives

$$Ae^{-4} + Be^{-6} = 3$$

so that $A = \frac{3e^8}{e^4-1}$ and $B = -\frac{3e^6}{e^4-1}$ and

$$y(t) = \frac{3e^8}{e^4-1}e^{-2t} - \frac{3e^6}{e^4-1}e^{-3t}$$

23. The characteristic equation is $r^2 + 2r + 2 = 0$, so $r = -1 \pm i$.
Therefore $p(t) = C_1e^{-t}\cos t + C_2e^{-t}\sin t$.
$p(0) = 0 = C_1$ so $p(t) = C_2e^{-t}\sin t$
$p(\pi/2) = 20 = C_2e^{-\pi/2}\sin\frac{\pi}{2}$ so $C_2 = 20e^{\pi/2}$
Therefore $p(t) = 20e^{\frac{\pi}{2}}e^{-t}\sin t = 20e^{\frac{\pi}{2}-t}\sin t$.

24. The characteristic equation is $r^2 + 4r + 5 = 0$, so $r = -2 \pm i$.
Therefore $p(t) = C_1e^{-2t}\cos t + C_2e^{-2t}\sin t$.
$p(0) = 1 = C_1$ so $p(t) = e^{-2t}\cos t + C_2e^{-2t}\sin t$
$p(\pi/2) = 5 = C_2e^{-\pi}$ so $C_2 = 5e^{\pi}$.
Therefore $p(t) = e^{-2t}\cos t + 5e^{\pi}e^{-2t}\sin t = e^{-2t}\cos t + 5e^{\pi-2t}\sin t$.

Problems

25. (a) $x'' + 4x = 0$ represents an undamped oscillator, and so goes with (IV).
(b) $x'' - 4x = 0$ has characteristic equation $r^2 - 4 = 0$ and so $r = \pm 2$. The solution is $C_1e^{-2t} + C_2e^{2t}$. This represents non-oscillating motion, so it goes with (II).
(c) $x'' - 0.2x' + 1.01x = 0$ has characteristic equation $r^2 - 0.2 + 1.01 = 0$ so $b^2 - 4ac = 0.04 - 4.04 = -4$, and $r = 0.1 \pm i$. So the solution is

$$C_1e^{(0.1+i)t} + C_2e^{(0.1-i)t} = e^{0.1t}(A\sin t + B\cos t).$$

The negative coefficient in the x' term represents an amplifying force. This is reflected in the solution by $e^{0.1t}$, which increases as t increases, so this goes with (I).
(d) $x'' + 0.2x' + 1.01x$ has characteristic equation $r^2 + 0.2r + 1.01 = 0$ so $b^2 - 4ac = -4$. This represents a damped oscillator. We have $r = -0.1 \pm i$ and so the solution is $x = e^{-0.1t}(A\sin t + B\cos t)$, which goes with (III).

26. We solve the characteristic equation in each case to obtain solutions to the differential equation.
(a) $r^2 + 5r + 6 = 0$, so $r = -2$ or -3. Then, $y = C_1e^{-2t} + C_2e^{-3t}$.
(b) $r^2 + r - 6 = 0$, so $r = 2$ or -3. Then, $y = C_1e^{2t} + C_2e^{-3t}$.
(c) $r^2 + 4r + 9 = 0$, so $r = -2 \pm \sqrt{5}i$. Then, $y = C_1e^{-2t}\cos(\sqrt{5}t) + C_2e^{-2t}\sin(\sqrt{5}t)$.
(d) $r^2 = -9$, so $r = \pm 3i$. Then, $y = C_1\cos(3t) + C_2\sin(3t)$.

Since (d) is undamped oscillations, it must be graph (I). Similarly, (c) is damped oscillations and so must be graph (II). Equation (a) is exponential decay, and so must be (IV). This leaves (III) to match with (b), which could be exponential growth or decay.

27. $0 = \frac{d^2}{dt^2}(e^{2t}) - 5\frac{d}{dt}(e^{2t}) + ke^{2t} = 4e^{2t} - 10e^{2t} + ke^{2t} = e^{2t}(k-6)$. Since $e^{2t} \neq 0$, we must have $k - 6 = 0$. Therefore $k = 6$.
The characteristic equation is $r^2 - 5r + 6 = 0$, so $r = 2$ or 3. Therefore $y(t) = C_1e^{2t} + C_2e^{3t}$.

28. In the underdamped case, $b^2 - 4c < 0$ so $4c - b^2 > 0$. Since the roots of the characteristic equation are

$$\alpha \pm i\beta = \frac{-b \pm \sqrt{b^2-4c}}{2} = \frac{-b \pm i\sqrt{4c-b^2}}{2}$$

we have $\alpha = -b/2$ and $\beta = (\sqrt{4c - b^2})/2$ or $\beta = -(\sqrt{4c - b^2})/2$. Since the general solution is

$$y = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

and since α is negative, $y \to 0$ as $t \to \infty$.

29. Recall that $s'' + bs' + cs = 0$ is overdamped if the discriminant $b^2 - 4c > 0$, critically damped if $b^2 - 4c = 0$, and underdamped if $b^2 - 4c < 0$. Since $b^2 - 4c = b^2 - 20$, the solution is overdamped if $b > 2\sqrt{5}$ or $b < -2\sqrt{5}$, critically damped if $b = \pm 2\sqrt{5}$, and underdamped if $-2\sqrt{5} < b < 2\sqrt{5}$.

30. Recall that $s'' + bs' + cs = 0$ is overdamped if the discriminant $b^2 - 4c > 0$, critically damped if $b^2 - 4c = 0$, and underdamped if $b^2 - 4c < 0$. This has discriminant $b^2 - 4c = b^2 + 64$. Since $b^2 + 64$ is always positive, the solution is always overdamped.

31. Recall that $F_{\text{drag}} = -c\frac{ds}{dt}$, so to find the largest coefficient of damping we look at the coefficient of s'. Thus spring (iii) has the largest coefficient of damping.

32. The restoring force is given by $F_{\text{spring}} = -ks$, so we look for the smallest coefficient of s. Spring (iv) exerts the smallest restoring force.

33. The frictional force is $F_{\text{drag}} = -c\frac{ds}{dt}$. Thus spring (iv) has the smallest frictional force.

34. All of these differential equations have solutions of the form $C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$. The spring with the longest period has the smallest β. Since $i\beta$ is the complex part of the roots of the characteristic equation, $\beta = \frac{1}{2}(\sqrt{4c - b^2})$. Thus spring (iii) has the longest period.

35. The stiffest spring exerts the greatest restoring force for a small displacement. Recall that by Hooke's Law $F_{\text{spring}} = -ks$, so we look for the differential equation with the greatest coefficient of s. This is spring (ii).

36. The characteristic equation is $r^2 + r - 2 = 0$, so $r = 1$ or -2. Therefore $z(t) = C_1 e^t + C_2 e^{-2t}$. Since $e^t \to \infty$ as $t \to \infty$, we must have $C_1 = 0$. Therefore $z(t) = C_2 e^{-2t}$. Furthermore, $z(0) = 3 = C_2$, so $z(t) = 3e^{-2t}$.

37. (a) If $r_1 = \frac{-b-\sqrt{b^2-4c}}{2}$ then $r_1 < 0$ since both b and $\sqrt{b^2 - 4c}$ are positive.
If $r_2 = \frac{-b+\sqrt{b^2-4c}}{2}$, then $r_2 < 0$ because

$$b = \sqrt{b^2} > \sqrt{b^2 - 4c}.$$

(b) The general solution to the differential equation is of the form

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

and since r_1 and r_2 are both negative, y must go to 0 as $t \to \infty$.

38. In the overdamped case, we have a solution of the form

$$s = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where r_1 and r_2 are real. We find a t such that $s = 0$, hence $C_1 e^{r_1 t} = -C_2 e^{r_2 t}$.

If $C_2 = 0$, then $C_1 = 0$, hence $s = 0$ for all t. But this does not match with Figure 11.13, so $C_2 \neq 0$. We divide by $C_2 e^{r_1 t}$ and get:

$$-\frac{C_1}{C_2} = e^{(r_2 - r_1)t}, \quad \text{where } -\frac{C_1}{C_2} > 0,$$

so the exponential is always positive. Therefore

$$(r_2 - r_1)t = \ln(-\frac{C_1}{C_2})$$

and

$$t = \frac{\ln(-\frac{C_1}{C_2})}{(r_2 - r_1)}.$$

So the mass passes through the equilibrium point only once, when $t = \frac{\ln(-\frac{C_1}{C_2})}{(r_2 - r_1)}$.

39. (a) $\dfrac{d^2y}{dt^2} = -\dfrac{dx}{dt} = y \quad \text{so} \quad \dfrac{d^2y}{dt^2} - y = 0.$

(b) Characteristic equation $r^2 - 1 = 0$, so $r = \pm 1$.
The general solution for y is $y = C_1 e^t + C_2 e^{-t}$, so $x = C_2 e^{-t} - C_1 e^t$.

40. The differential equation is $Q'' + 2Q' + \frac{1}{4}Q = 0$, so the characteristic equation is $r^2 + 2r + \frac{1}{4} = 0$. This has roots $\dfrac{-2 \pm \sqrt{3}}{2} = -1 \pm \dfrac{\sqrt{3}}{2}$. Thus, the general solution is

$$\begin{aligned} Q(t) &= C_1 e^{(-1+\frac{\sqrt{3}}{2})t} + C_2 e^{(-1-\frac{\sqrt{3}}{2})t}, \\ Q'(t) &= C_1\left(-1+\frac{\sqrt{3}}{2}\right)e^{(-1+\frac{\sqrt{3}}{2})t} + C_2\left(-1-\frac{\sqrt{3}}{2}\right)e^{(-1-\frac{\sqrt{3}}{2})t}. \end{aligned}$$

We have

(a)

$$\begin{aligned} & Q(0) = C_1 + C_2 = 0 \\ \text{and} \quad & Q'(0) = \left(-1+\frac{\sqrt{3}}{2}\right)C_1 + \left(-1-\frac{\sqrt{3}}{2}\right)C_2 = 2. \end{aligned}$$

Using the formula for $Q(t)$, we have $C_1 = -C_2$. Using the formula for $Q'(t)$, we have:

$$\begin{aligned} 2 &= \left(-1+\frac{\sqrt{3}}{2}\right)(-C_2) + \left(-1-\frac{\sqrt{3}}{2}\right)C_2 = -\sqrt{3}C_2 \\ \text{so,} \quad C_2 &= -\frac{2}{\sqrt{3}}. \end{aligned}$$

Thus, $C_1 = \dfrac{2}{\sqrt{3}}$, and $Q(t) = \dfrac{2}{\sqrt{3}}\left(e^{(-1+\frac{\sqrt{3}}{2})t} - e^{(-1-\frac{\sqrt{3}}{2})t}\right)$.

(b) We have

$$\begin{aligned} & Q(0) = C_1 + C_2 = 2 \\ \text{and} \quad & Q'(0) = \left(-1+\frac{\sqrt{3}}{2}\right)C_1 + \left(-1-\frac{\sqrt{3}}{2}\right)C_2 = 0. \end{aligned}$$

Using the first equation, we have $C_1 = 2 - C_2$. Thus,

$$\begin{aligned} \left(-1+\frac{\sqrt{3}}{2}\right)(2-C_2) + \left(-1-\frac{\sqrt{3}}{2}\right)C_2 &= 0 \\ -\sqrt{3}C_2 &= 2 - \sqrt{3} \\ C_2 &= -\frac{2-\sqrt{3}}{\sqrt{3}} \\ \text{and} \qquad C_1 = 2 - C_2 &= \frac{2+\sqrt{3}}{\sqrt{3}}. \end{aligned}$$

Thus, $Q(t) = \dfrac{1}{\sqrt{3}}\left((2+\sqrt{3})e^{(-1+\frac{\sqrt{3}}{2})t} - (2-\sqrt{3})e^{(-1-\frac{\sqrt{3}}{2})t}\right)$.

41. In this case, the differential equation describing the charge is $Q'' + Q' + \frac{1}{4}Q = 0$, so the characteristic equation is $r^2 + r + \frac{1}{4} = 0$. This equation has one root, $r = -\frac{1}{2}$, so the equation for charge is

$$\begin{aligned} Q(t) &= (C_1 + C_2 t)e^{-\frac{1}{2}t}, \\ Q'(t) &= -\frac{1}{2}(C_1 + C_2 t)e^{-\frac{1}{2}t} + C_2 e^{-\frac{1}{2}t} \\ &= \left(C_2 - \frac{C_1}{2} - \frac{C_2 t}{2}\right)e^{-\frac{1}{2}t}. \end{aligned}$$

(a) We have

$$\begin{aligned} Q(0) &= C_1 = 0, \\ Q'(0) &= C_2 - \frac{C_1}{2} = 2. \end{aligned}$$

Thus, $C_1 = 0$, $C_2 = 2$, and

$$Q(t) = 2te^{-\frac{1}{2}t}.$$

(b) We have

$$Q(0) = C_1 = 2\,,$$
$$Q'(0) = C_2 - \frac{C_1}{2} = 0.$$

Thus, $C_1 = 2$, $C_2 = 1$, and

$$Q(t) = (2+t)e^{-\frac{1}{2}t}.$$

(c) The resistance was decreased by exactly the amount to switch the circuit from the overdamped case to the critically damped case. Comparing the solutions of parts (a) and (b) in Problems 40, we find that in the critically damped case the net charge goes to 0 much faster as $t \to \infty$.

42. In this case, the differential equation describing charge is $8Q'' + 2Q' + \frac{1}{4}Q = 0$, so the characteristic equation is $8r^2 + 2r + \frac{1}{4} = 0$. This quadratic equation has solutions

$$r = \frac{-2 \pm \sqrt{4 - 4\cdot 8 \cdot \frac{1}{4}}}{16} = -\frac{1}{8} \pm \frac{1}{8}i.$$

Thus, the equation for charge is

$$Q(t) = e^{-\frac{1}{8}t}\left(A \sin\frac{t}{8} + B\cos\frac{t}{8}\right).$$
$$Q'(t) = -\frac{1}{8}e^{-\frac{1}{8}t}\left(A\sin\frac{t}{8} + B\cos\frac{t}{8}\right) + e^{-\frac{1}{8}t}\left(\frac{1}{8}A\cos\frac{t}{8} - \frac{1}{8}B\sin\frac{t}{8}\right)$$
$$= \frac{1}{8}e^{-\frac{1}{8}t}\left((A-B)\cos\frac{t}{8} + (-A-B)\sin\frac{t}{8}\right).$$

(a) We have

$$Q(0) = B = 0,$$
$$Q'(0) = \frac{1}{8}(A-B) = 2.$$

Thus, $B = 0$, $A = 16$, and

$$Q(t) = 16e^{-\frac{1}{8}t}\sin\frac{t}{8}.$$

(b) We have

$$Q(0) = B = 2,$$
$$Q'(0) = \frac{1}{8}(A-B) = 0.$$

Thus, $B = 2$, $A = 2$, and

$$Q(t) = 2e^{-\frac{1}{8}t}\left(\sin\frac{t}{8} + \cos\frac{t}{8}\right).$$

(c) By increasing the inductance, we have gone from the overdamped case to the underdamped case. We find that while the charge still tends to 0 as $t \to \infty$, the charge in the underdamped case oscillates between positive and negative values. In the over-damped case of Problem 40, the charge starts nonnegative and remains positive.

43. The differential equation for the charge on the capacitor, given a resistance R, a capacitance C, and and inductance L, is

$$LQ'' + RQ' + \frac{Q}{C} = 0.$$

The corresponding characteristic equation is $Lr^2 + Rr + \dfrac{1}{C} = 0$. This equation has roots

$$r = -\frac{R}{2L} \pm \frac{\sqrt{R^2 - \frac{4L}{C}}}{2L}.$$

(a) If $R^2 - \dfrac{4L}{C} < 0$, the solution is

$$Q(t) = e^{-\frac{R}{2L}t}(A\sin\omega t + B\cos\omega t) \text{ for some } A \text{ and } B,$$

where $\omega = \dfrac{\sqrt{R^2 - \frac{4L}{C}}}{2L}$. As $t \to \infty$, $Q(t)$ clearly goes to 0.

(b) If $R^2 - \dfrac{4L}{C} = 0$, the solution is

$$Q(t) = e^{-\frac{R}{t}}(A + Bt) \text{ for some } A \text{ and } B.$$

Again, as $t \to \infty$, the charge goes to 0.

(c) If $R^2 - \dfrac{4L}{C} > 0$, the solution is

$$Q(t) = Ae^{r_1 t} + Be^{r_2 t} \text{ for some } A \text{ and } B,$$

where

$$r_1 = -\frac{R}{2L} + \frac{\sqrt{R^2 - \frac{4L}{C}}}{2L}, \quad \text{and} \quad r_2 = -\frac{R}{2L} - \frac{\sqrt{R^2 - \frac{4L}{C}}}{2L}.$$

Notice that r_2 is clearly negative. r_1 is also negative since

$$\begin{aligned}\frac{\sqrt{R^2 - \frac{4L}{C}}}{2L} &< \frac{\sqrt{R^2}}{2L} \quad (L \text{ and } C \text{ are positive})\\ &= \frac{R}{2L}.\end{aligned}$$

Since r_1 and r_2 are negative, again $Q(t) \to 0$, as $t \to \infty$.

Thus, for any circuit with a resistor, a capacitor and an inductor, $Q(t) \to 0$ as $t \to \infty$. Compare this with Problem 27 in Section 11.10, where we showed that in a circuit with just a capacitor and inductor, the charge varied along a sine curve.

44. (a) Since $y = \dfrac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2}$, we have $y' = \dfrac{r_1 e^{r_1 t} - r_2 e^{r_2 t}}{r_1 - r_2}$ and $y'' = \dfrac{r_1^2 e^{r_1 t} - r_2^2 e^{r_2 t}}{r_1 - r_2}$. Thus

$$\begin{aligned}y'' + by' + cy &= \frac{r_1^2 e^{r_1 t} - r_2^2 e^{r_2 t}}{r_1 - r_2} + b\frac{r_1 e^{r_1 t} - r_2 e^{r_2 t}}{r_1 - r_2} + c\frac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2}\\ &= \frac{(r_1^2 + br_1 + c)e^{r_1 t}}{r_1 - r_2} - \frac{(r_2^2 + br_2 + c)e^{r_2 t}}{r_1 - r_2}\\ &= 0,\end{aligned}$$

since both r_1 and r_2 satisfy $r^2 + br + c = 0$.

(b) We have a solution $y = (e^{r_1 t} - e^{r_2 t})/(r_1 - r_2)$ and $r_1 = r_2 + h$. Thus

$$\begin{aligned}y &= \frac{e^{(r_2+h)t} - e^{r_2 t}}{h}\\ &= \frac{e^{r_2 t}e^{ht} - e^{r_2 t}}{h}\\ &= e^{r_2 t}\frac{e^{ht} - 1}{h}.\end{aligned}$$

(c) The Taylor series for e^x is $1 + x + x^2/2! + x^3/3! + \cdots$, so

$$\begin{aligned}\frac{e^{ht} - 1}{h} &= \frac{1 + ht + h^2t^2/2! + \cdots - 1}{h}\\ &= t + \frac{ht^2}{2!} + \frac{h^2t^3}{3!} + \cdots.\end{aligned}$$

(d) $\displaystyle\lim_{h\to 0} e^{r_2 t}\frac{e^{ht} - 1}{h} = \lim_{h\to 0} e^{r_2 t}\left(t + \frac{ht^2}{2!} + \frac{h^2t^3}{3!} + \cdots\right) = te^{r_2 t}.$

(e) The solutions to $r^2 + br + c = 0$ are $\dfrac{-b \pm \sqrt{b^2 - 4c}}{2} = -b/2$, since $b^2 - 4c = 0$ because we have a double root. Thus $r_1 = r_2 = -b/2$, so our solution is $y = te^{r_2 t} = te^{-bt/2}$. Thus

$$y' = (1 - \frac{bt}{2})e^{-\frac{bt}{2}}$$

and

$$y'' = \left(-\frac{b}{2} + \frac{b^2t}{4} - \frac{b}{2}\right)e^{-\frac{bt}{2}} = \left(-b + \frac{b^2t}{4}\right)e^{-\frac{bt}{2}}.$$

So

$$y'' + by' + cy = \left((-b + \frac{b^2t}{4}) + b(1 - \frac{bt}{2}) + ct\right)e^{-\frac{bt}{2}} = (-\frac{b^2}{4} + c)te^{-\frac{bt}{2}} = 0,$$

since $b^2 - 4c = 0$.

Solutions for Chapter 11 Review

Exercises

1. This equation is separable, so we integrate, giving

$$\int dP = \int t\,dt$$

so

$$P(t) = \frac{t^2}{2} + C.$$

2. This equation is separable, so we integrate, giving

$$\int \frac{1}{0.2y - 8}\,dy = \int dx$$

so

$$\frac{1}{0.2}\ln|0.2y - 8| = x + C.$$

Thus

$$y(x) = 40 + Ae^{0.2x}.$$

3. This equation is separable, so we integrate, giving

$$\int \frac{1}{10 - 2P}\,dP = \int dt$$

so

$$\frac{1}{-2}\ln|10 - 2P| = t + C.$$

Thus

$$P = 5 + Ae^{-2t}.$$

4. This equation is separable, so we integrate, giving

$$\int \frac{1}{10 + 0.5H}\,dH = \int dt$$

so

$$\frac{1}{0.5}\ln|10 + 0.5H| = t + C.$$

Thus

$$H = Ae^{0.5t} - 20.$$

5. This equation is separable, so we integrate, using the table of integrals or partial fractions, to get

$$\int \frac{1}{R-3R^2}\,dR = 2\int dt$$

$$\int \frac{1}{R}\,dR + \int \frac{3}{1-3R}\,dR = 2\int dt$$

so

$$\ln|R| - \ln|1-3R| = 2t + C$$

$$\ln\left|\frac{R}{1-3R}\right| = 2t + C$$

$$\frac{R}{1-3R} = Ae^{2t}$$

$$R = \frac{Ae^{2t}}{1+3Ae^{2t}}.$$

6. This equation is separable, so we integrate, using the table of integrals or partial fractions, to get:

$$\int \frac{250}{100P-P^2}\,dP = \int dt$$

$$\frac{250}{100}\left(\int \frac{1}{P}\,dP + \int \frac{1}{100-P}\,dP\right) = \int dt$$

so

$$2.5(\ln|P| - \ln|100-P|) = t + C$$

$$2.5\ln\left|\frac{P}{100-P}\right| = t + C$$

$$\frac{P}{100-P} = Ae^{0.4t}$$

$$P = \frac{100Ae^{0.4t}}{1+Ae^{0.4t}}$$

7. $\frac{dy}{dx} + xy^2 = 0$ means $\frac{dy}{dx} = -xy^2$, so $\int \frac{dy}{y^2} = \int -x\,dx$ giving $-\frac{1}{y} = -\frac{x^2}{2} + C$. Since $y(1) = 1$ we have $-1 = -\frac{1}{2} + C$ so $C = -\frac{1}{2}$. Thus, $-\frac{1}{y} = -\frac{x^2}{2} - \frac{1}{2}$ giving $y = \frac{2}{x^2+1}$.

8. $\frac{dP}{dt} = 0.03P + 400$ so $\int \frac{dP}{P+\frac{40000}{3}} = \int 0.03dt$.
$\ln|P + \frac{40000}{3}| = 0.03t + C$ giving $P = Ae^{0.03t} - \frac{40000}{3}$. Since $P(0) = 0$, $A = \frac{40000}{3}$, therefore $P = \frac{40000}{3}(e^{0.03t} - 1)$.

9. $1 + y^2 - \frac{dy}{dx} = 0$ gives $\frac{dy}{dx} = y^2 + 1$, so $\int \frac{dy}{1+y^2} = \int\, dx$ and $\arctan y = x + C$. Since $y(0) = 0$ we have $C = 0$, giving $y = \tan x$.

10. $2\sin x - y^2\frac{dy}{dx} = 0$ giving $2\sin x = y^2\frac{dy}{dx}$. $\int 2\sin x\,dx = \int y^2\,dy$ so $-2\cos x = \frac{y^3}{3} + C$. Since $y(0) = 3$ we have $-2 = 9 + C$, so $C = -11$. Thus, $-2\cos x = \frac{y^3}{3} - 11$ giving $y = \sqrt[3]{33 - 6\cos x}$.

11. $\frac{dk}{dt} = (1 + \ln t)k$ gives $\int \frac{dk}{k} = \int (1 + \ln t)dt$ so $\ln|k| = t\ln t + C$. $k(1) = 1$, so $0 = 0 + C$, or $C = 0$. Thus, $\ln|k| = t\ln t$ and $|k| = e^{t\ln t} = t^t$, giving $k = \pm t^t$.
But recall $k(1) = 1$, so $k = t^t$ is the solution.

12. $\frac{dy}{dx} = \frac{y(3-x)}{x(\frac{1}{2}y-4)}$ gives $\int \frac{(\frac{1}{2}y-4)}{y}dy = \int \frac{(3-x)}{x}dx$ so $\int (\frac{1}{2} - \frac{4}{y})dy = \int (\frac{3}{x} - 1)dx$. Thus $\frac{1}{2}y - 4\ln|y| = 3\ln|x| - x + C$. Since $y(1) = 5$, we have $\frac{5}{2} - 4\ln 5 = \ln|1| - 1 + C$ so $C = \frac{7}{2} - 4\ln 5$. Thus,

$$\frac{1}{2}y - 4\ln|y| = 3\ln|x| - x + \frac{7}{2} - 4\ln 5.$$

We cannot solve for y in terms of x, so we leave the equation in this form.

13. $\frac{dy}{dx} = \frac{0.2y(18+0.1x)}{x(100+0.5y)}$ giving $\int \frac{(100+0.5y)}{0.2y}\, dy = \int \frac{18+0.1x}{x}\, dx$, so

$$\int \left(\frac{500}{y} + \frac{5}{2}\right) dy = \int \left(\frac{18}{x} + \frac{1}{10}\right) dx.$$

Therefore, $500 \ln |y| + \frac{5}{2}y = 18 \ln |x| + \frac{1}{10}x + C$. Since the curve passes through (10,10), $500 \ln 10 + 25 = 18 \ln 10 + 1 + C$, so $C = 482 \ln 10 + 24$. Thus, the solution is

$$500 \ln |y| + \frac{5}{2}y = 18 \ln |x| + \frac{1}{10}x + 482 \ln 10 + 24.$$

We cannot solve for y in terms of x, so we leave the answer in this form.

14. This equation is separable and so we write it as

$$\frac{1}{z(z-1)} \frac{dz}{dt} = 1.$$

We integrate with respect to t, giving

$$\begin{aligned} \int \frac{1}{z(z-1)} dz &= \int dt \\ \int \frac{1}{z-1} dz - \int \frac{1}{z} dz &= \int dt \\ \ln |z-1| - \ln |z| &= t + C \\ \ln |\frac{z-1}{z}| &= t + C, \end{aligned}$$

so that

$$\frac{z-1}{z} = e^{t+C} = ke^t.$$

Solving for z gives

$$z(t) = \frac{1}{1-ke^t}.$$

The initial condition $z(0) = 10$ gives

$$\frac{1}{1-k} = 10$$

or $k = 0.9$. The solution is therefore

$$z(t) = \frac{1}{1-0.9e^t}.$$

15. Using the solution of the logistic equation given on page 563 in Section 11.7, and using $y(0) = 1$, we get $y = \frac{10}{1+9e^{-10t}}$.

16. $\frac{dy}{dx} = \frac{y(100-x)}{x(20-y)}$ gives $\int (\frac{20-y}{y})\, dy = \int \left(\frac{100-x}{x}\right) dx$. Thus, $20 \ln |y| - y = 100 \ln |x| - x + C$. The curve passes through $(1, 20)$, so $20 \ln 20 - 20 = -1 + C$ giving $C = 20 \ln 20 - 19$. Therefore, $20 \ln |y| - y = 100 \ln |x| - x + 20 \ln 20 - 19$. We cannot solve for y in terms of x, so we leave the equation in this form.

17. $\frac{df}{dx} = \sqrt{xf(x)}$ gives $\int \frac{df}{\sqrt{f(x)}} = \int \sqrt{x}\, dx$, so $2\sqrt{f(x)} = \frac{2}{3}x^{\frac{3}{2}} + C$. Since $f(1) = 1$, we have $2 = \frac{2}{3} + C$ so $C = \frac{4}{3}$. Thus, $2\sqrt{f(x)} = \frac{2}{3}x^{\frac{3}{2}} + \frac{4}{3}$, so $f(x) = (\frac{1}{3}x^{\frac{3}{2}} + \frac{2}{3})^2$.
(Note: this is only defined for $x \geq 0$.)

18. $\frac{dy}{dx} = e^{x-y}$ giving $\int e^y\, dy = \int e^x\, dx$ so $e^y = e^x + C$. Since $y(0) = 1$, we have $e^1 = e^0 + C$ so $C = e - 1$. Thus, $e^y = e^x + e - 1$, so $y = \ln(e^x + e - 1)$.
[Note: $e^x + e - 1 > 0$ always.]

19. $\frac{dy}{dx} = e^{x+y} = e^x e^y$ implies $\int e^{-y}\, dy = \int e^x\, dx$ implies $-e^{-y} = e^x + C$. Since $y = 0$ when $x = 1$, we have $-1 = e + C$, giving $C = -1 - e$. Therefore $-e^{-y} = e^x - 1 - e$ and $y = -\ln(1 + e - e^x)$.

20. $e^{-\cos\theta} \frac{dz}{d\theta} = \sqrt{1-z^2} \sin\theta$ implies $\int \frac{dz}{\sqrt{1-z^2}} = \int e^{\cos\theta} \sin\theta\, d\theta$ implies $\arcsin z = -e^{\cos\theta} + C$. According to the initial conditions: $z(0) = \frac{1}{2}$, so $\arcsin \frac{1}{2} = -e^{\cos 0} + C$, therefore $\frac{\pi}{6} = -e + C$, and $C = \frac{\pi}{6} + e$. Thus $z = \sin(-e^{\cos\theta} + \frac{\pi}{6} + e)$.

21. $(1+t^2)y\frac{dy}{dt} = 1-y$ implies that $\int \frac{y\,dy}{1-y} = \int \frac{dt}{1+t^2}$ implies that $\int \left(-1+\frac{1}{1-y}\right)\,dy = \int \frac{dt}{1+t^2}$. Therefore $-y-\ln|1-y| = \arctan t + C$. $y(1)=0$, so $0 = \arctan 1 + C$, and $C = -\frac{\pi}{4}$, so $-y-\ln|1-y| = \arctan t - \frac{\pi}{4}$. We cannot solve for y in terms of t.

22. We have

$$\frac{dy}{dt} = 2^y \sin^3 t,$$

so

$$\int 2^{-y}\,dy = \int \sin^3 t\,dt.$$

Using Integral Table Formula 17, gives

$$-\frac{1}{\ln 2}2^{-y} = -\frac{1}{3}\sin^2 t\cos t - \frac{2}{3}\cos t + C.$$

According to the initial conditions: $y(0)=0$ so

$$-\frac{1}{\ln 2} = -\frac{2}{3} + C, \quad \text{and} \quad C = \frac{2}{3} - \frac{1}{\ln 2}.$$

Thus,

$$-\frac{1}{\ln 2}2^{-y} = -\frac{1}{3}\sin^2 t\cos t - \frac{2}{3}\cos t + \frac{2}{3} - \frac{1}{\ln 2}.$$

Solving for y gives:

$$2^{-y} = \frac{\ln 2}{3}\sin^2 t\cos t + \frac{2\ln 2}{3}\cos t - \frac{2\ln 2}{3} + 1.$$

It can be shown that the right side is always > 0, so we can take natural logs.

$$y\ln 2 = -\ln\left(\frac{\ln 2}{3}\sin^2 t\cos t + \frac{2\ln 2}{3}\cos t - \frac{2\ln 2}{3} + 1\right),$$

so

$$y = \frac{-\ln\left(\frac{\ln 2}{3}\sin^2 t\cos t + \frac{2\ln 2}{3}\cos t - \frac{2\ln 2}{3} + 1\right)}{\ln 2}.$$

23. The characteristic equation is

$$r^2 + \pi^2 = 0$$

so that $r = \pm i\pi$ and

$$z(t) = A\cos \pi t + B\sin \pi t$$

24. The characteristic equation of $9z'' - z = 0$ is

$$9r^2 - 1 = 0.$$

If this is written in the form $r^2 + br + c = 0$, we have that $r^2 - 1/9 = 0$ and

$$b^2 - 4c = 0 - (4)(-1/9) = 4/9 > 0$$

This indicates overdamped motion and since the roots of the characteristic equation are $r = \pm 1/3$, the general solution is

$$y(t) = C_1 e^{\frac{1}{3}t} + C_2 e^{-\frac{1}{3}t}.$$

25. The characteristic equation of $9z'' + z = 0$ is

$$9r^2 + 1 = 0$$

If we write this in the form $r^2 + br + c = 0$, we have that $r^2 + 1/9 = 0$ and

$$b^2 - 4c = 0 - (4)(1/9) = -4/9 < 0$$

This indicates underdamped motion and since the roots of the characteristic equation are $r = \pm\frac{1}{3}i$, the general equation is

$$y(t) = C_1\cos\left(\frac{1}{3}t\right) + C_2\sin\left(\frac{1}{3}t\right)$$

26. The characteristic equation of $y'' + 6y' + 8y = 0$ is

$$r^2 + 6r + 8 = 0.$$

We have that

$$b^2 - 4c = 6^2 - 4(8) = 4 > 0.$$

This indicates overdamped motion. Since the roots of the characteristic equation are $r_1 = -2$ and $r_2 = -4$, the general solution is

$$y(t) = C_1 e^{-2t} + C_2 e^{-4t}.$$

27. The characteristic equation is

$$r^2 + 2r + 3 = 0$$

which has the solution

$$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 3}}{2} = -1 \pm \sqrt{-2}$$

so that the general solution is

$$y(t) = e^{-t}(A \sin \sqrt{2}t + B \cos \sqrt{2}t)$$

28. The characteristic equation of $x'' + 2x^{'} + 10x = 0$ is

$$r^2 + 2r + 10 = 0$$

We have that

$$b^2 - 4c = 2^2 - 4(10) = -36 < 0$$

This indicates underdamped motion and since the roots of the characteristic equation are $r = -1 \pm 3i$, the general solution is

$$y(t) = C_1 e^{-t} \cos 3t + C_2 e^{-t} \sin 3t$$

Problems

29. Figure (I) shows a line segment at $(4, 0)$ with positive slope. The only possible differential equation is (b), where $y'(4, 0) = \cos 0 = 1$. Note that (a) is not possible as $y'(4, 0) = e^{-16} = 0.0000001$, a much smaller positive slope than that shown.

Figure (II) shows a line segment at $(0, 4)$ with zero slope. The possible differential equations are (d), where $y'(0, 4) = 4(4 - 4) = 0$, and (f), where $y'(0, 4) = 0(3 - 0) = 0$.

Figure (III) shows a line segment at $(4, 0)$ with negative slope of large magnitude. The only possible differential equation is (f), where $y'(4, 0) = 4(3 - 4) = -4$. Note that (c) is not possible as $y'(4, 0) = \cos(4 - 0) = -0.65$, a negative slope of smaller magnitude than that shown.

Figure (IV) shows a line segment at $(4, 0)$ with a negative slope of small magnitude. The only possible differential equation is (c), where $y'(4, 0) = \cos(4 - 0) = -0.65$. Note that (f) is not possible as $y'(4, 0) = 4(3 - 4) = -4$, a negative slope of larger magnitude than that shown.

Figure (V) shows a line segment at $(0, 4)$ with positive slope. Possible differential equations are (a), where $y'(0, 4) = e^{0^2} = 1$, and (c), where $y'(0, 4) = \cos(4 - 4) = 1$.

Figure (VI) shows a line segment at $(0, 4)$ with a negative slope of large magnitude. The only possible differential equation is (e), where $y'(0, 4) = 4(3 - 4) = -4$. Note that (b) is not possible as $y'(0, 4) = \cos 4 = -0.65$, a negative slope of smaller magnitude than that shown.

30. (a) The slope field for $dy/dx = y/x$ is in Figure 11.53.

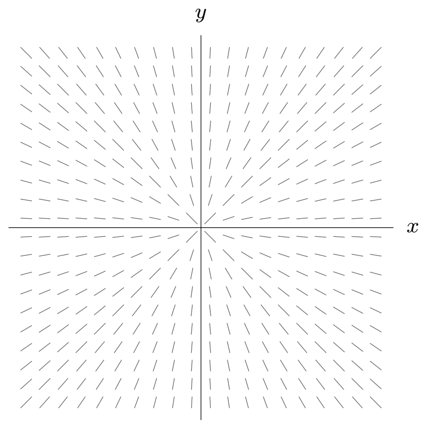


Figure 11.53

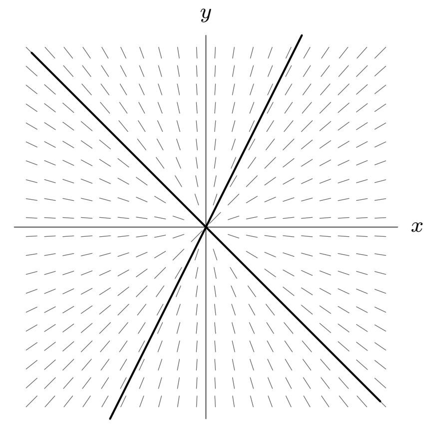


Figure 11.54

(b) See Figure 11.54.

(c) Separating variables gives

$$\int \frac{1}{y}dy = \int \frac{1}{x}dx$$

or

$$\ln|y| = \ln|x| + C$$

which can be written as

$$\ln|y| = \ln|x| + \ln|D|$$

so that

$$y = Dx.$$

Thus, the solutions are lines through the origin, as shown in part (b).

31. (a) We know that the equilibrium solutions are the functions satisfying the differential equation whose derivative everywhere is 0. Thus we have

$$\begin{aligned}\frac{dy}{dt} &= 0\\ 0.2(y-3)(y+2) &= 0\\ (y-3)(y+2) &= 0.\end{aligned}$$

The solutions are $y = 3$ and $y = -2$.

(b)

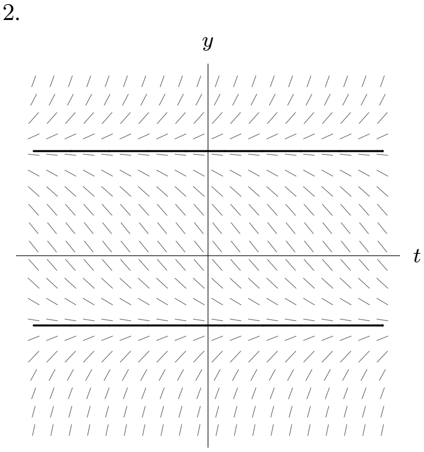
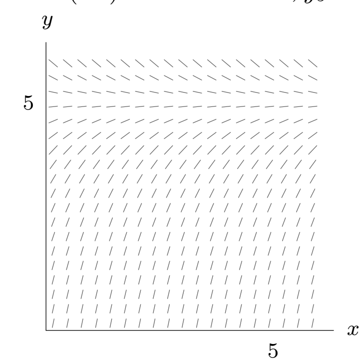


Figure 11.55

Looking at Figure 11.55, we see that the line $y = 3$ is an unstable solution, while the line $y = -2$ is a stable solution.

32. (a) $\Delta x = \frac{1}{5} = 0.2$.
At $x = 0$:
$y_0 = 1, y' = 4$; so $\Delta y = 4(0.2) = 0.8$. Thus, $y_1 = 1 + 0.8 = 1.8$.
At $x = 0.2$:
$y_1 = 1.8, y' = 3.2$; so $\Delta y = 3.2(0.2) = 0.64$. Thus, $y_2 = 1.8 + 0.64 = 2.44$.
At $x = 0.4$:
$y_2 = 2.44, y' = 2.56$; so $\Delta y = 2.56(0.2) = 0.512$. Thus, $y_3 = 2.44 + 0.512 = 2.952$.
At $x = 0.6$:
$y_3 = 2.952, y' = 2.048$; so $\Delta y = 2.048(0.2) = 0.4096$. Thus, $y_4 = 3.3616$.
At $x = 0.8$:
$y_4 = 3.3616, y' = 1.6384$; so $\Delta y = 1.6384(0.2) = 0.32768$. Thus, $y_5 = 3.68928$. So $y(1) \approx 3.689$.

(b)


Since solution curves are concave down for $0 \le y \le 5$, and $y(0) = 1 < 5$, the estimate from Euler's method will be an overestimate.

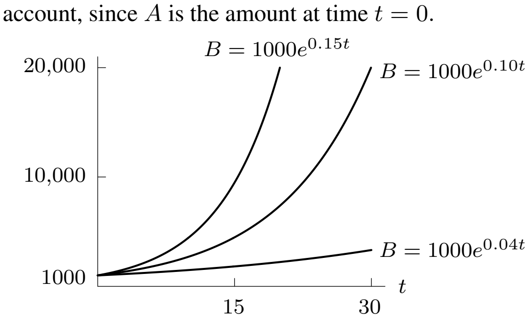
(c) Solving by separation:
$\int \frac{dy}{5-y} = \int dx$, so $-\ln|5-y| = x + C$.
Then $5 - y = Ae^{-x}$ where $A = \pm e^{-C}$. Since $y(0) = 1$, we have $5 - 1 = Ae^0$, so $A = 4$.
Therefore, $y = 5 - 4e^{-x}$, and $y(1) = 5 - 4e^{-1} \approx 3.528$.
(Note: as predicted, the estimate in (a) is too large.)

(d) Doubling the value of n will probably halve the error and, therefore, give a value half way between 3.528 and 3.689, which is approximately 3.61.

33. (a) $\frac{dB}{dt} = \frac{r}{100}B$. The constant of proportionality is $\frac{r}{100}$.
(b) Solving, we have

$$\begin{aligned}
\frac{dB}{B} &= \frac{r\,dt}{100} \\
\int \frac{dB}{B} &= \int \frac{r}{100}\,dt \\
\ln|B| &= \frac{r}{100}t + C \\
B &= e^{(r/100)t+C} = Ae^{(r/100)t}, \qquad A = e^C.
\end{aligned}$$

A is the initial amount in the account, since A is the amount at time $t = 0$.

(c)


34. The rate of disintegration is proportional to the quantity of carbon-14 present. Let Q be the quantity of carbon-14 present at time t, with $t = 0$ in 1977. Then

$$Q = Q_0 e^{-kt},$$

where Q_0 is the quantity of carbon-14 present in 1977 when $t = 0$. Then we know that

$$\frac{Q_0}{2} = Q_0 e^{-k(5730)}$$

so that

$$k = -\frac{\ln(1/2)}{5730} = 0.000121.$$

Thus

$$Q = Q_0 e^{-0.000121t}.$$

The quantity present at any time is proportional to the rate of disintegration at that time so

$$Q_0 = c8.2 \qquad \text{and} \qquad Q = c13.5$$

where c is a constant of proportionality. Thus substituting for Q and Q_0 in

$$Q = Q_0 e^{-0.000121t}$$

gives

$$c13.5 = c8.2e^{-0.000121t}$$

so

$$t = -\frac{\ln(13.5/8.2)}{0.000121} \approx -4120.$$

Thus Stonehenge was built about 4120 years before 1977, in about 2150 B.C.

35. (a) If A is surface area, we know that for some constant K

$$\frac{dV}{dt} = -KA.$$

If r is the radius of the sphere, $V = 4\pi r^3/3$ and $A = 4\pi r^2$. Solving for r in terms of V gives $r = (3V/4\pi)^{1/3}$, so

$$\frac{dV}{dt} = -K(4\pi r^2) = -K4\pi\left(\frac{3V}{4\pi}\right)^{2/3} \qquad \text{so} \qquad \frac{dV}{dt} = -kV^{2/3}$$

where k is another constant, $k = K(4\pi)^{1/3}3^{2/3}$.

(b) Separating variables gives

$$\int \frac{dV}{V^{2/3}} = -\int k\,dt$$
$$3V^{1/3} = -kt + C.$$

Since $V = V_0$ when $t = 0$, we have $3V_0^{1/3} = C$, so

$$3V^{1/3} = -kt + 3V_0^{1/3}.$$

Solving for V gives

$$V = \left(-\frac{k}{3}t + V_0^{1/3}\right)^3.$$

This function is graphed in Figure 11.56.

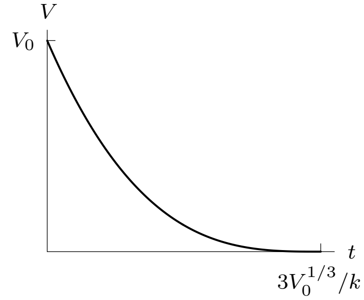


Figure 11.56

(c) The snowball disappears when $V = 0$, that is when

$$-\frac{k}{3}t + V_0^{1/3} = 0$$

giving

$$t = \frac{3V_0^{1/3}}{k}.$$

36. (a) Quantity of A present at time t equals $(a - x)$.
Quantity of B present at time t equals $(b - x)$.
So

$$\text{Rate of formation of } C = k(\text{Quantity of } A)(\text{Quantity of } B)$$

gives

$$\frac{dx}{dt} = k(a - x)(b - x)$$

(b) Separating gives

$$\int \frac{dx}{(a-x)(b-x)} = \int k\,dt.$$

Rewriting the denominator as $(a-x)(b-x) = (x-a)(x-b)$ enables us to use Formula 26 in the Table of Integrals provided $a \neq b$. For some constant K, this gives

$$\frac{1}{a-b}\left(\ln|x-a| - \ln|x-b|\right) = kt + K.$$

Thus

$$\begin{aligned}
\ln\left|\frac{x-a}{x-b}\right| &= (a-b)kt + K(a-b) \\
\left|\frac{x-a}{x-b}\right| &= e^{K(a-b)}e^{(a-b)kt} \\
\frac{x-a}{x-b} &= Me^{(a-b)kt} \quad \text{where } M = \pm e^{K(a-b)}.
\end{aligned}$$

Since $x = 0$ when $t = 0$, we have $M = \frac{a}{b}$. Thus

$$\frac{x-a}{x-b} = \frac{a}{b}e^{(a-b)kt}.$$

Solving for x, we have

$$\begin{aligned}
bx - ba &= ae^{(a-b)kt}(x-b) \\
x(b - ae^{(a-b)kt}) &= ab - abe^{(a-b)kt} \\
x &= \frac{ab(1-e^{(a-b)kt})}{b - ae^{(a-b)kt}} = \frac{ab(e^{bkt} - e^{akt})}{be^{bkt} - ae^{akt}}.
\end{aligned}$$

37. Quantity of A left at time t = Quantity of B left at time t equals $(a - x)$.
Thus

$$\text{Rate of formation of } C = k(\text{Quantity of } A)(\text{Quantity of } B)$$

gives

$$\frac{dx}{dt} = k(a-x)(a-x) = k(a-x)^2.$$

Separating gives

$$\int \frac{dx}{(x-a)^2} = \int k\,dt$$

Integrating gives, for some constant K,

$$-(x-a)^{-1} = kt + K.$$

When $t = 0$, $x = 0$ so $K = a^{-1}$. Solving for x:

$$-(x-a)^{-1} = kt + a^{-1}$$
$$x - a = -\frac{1}{kt + a^{-1}}$$
$$x = a - \frac{a}{akt+1} = \frac{a^2kt}{akt+1}$$

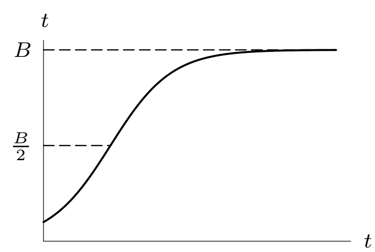
38. Recall that $s'' + bs' + c = 0$ is overdamped if the discriminant $b^2 - 4c > 0$, critically damped if $b^2 - 4c = 0$, and underdamped if $b^2 - 4c < 0$. Since $b^2 - 4c = 16 - 4c$, the circuit is overdamped if $c < 4$, critically damped if $c = 4$, and underdamped if $c > 4$.

39. Recall that $s'' + bs' + cs = 0$ is overdamped if the discriminant $b^2 - 4c > 0$, critically damped if $b^2 - 4c = 0$, and underdamped if $b^2 - 4c < 0$. Since $b^2 - 4c = 8 - 4c$, the solution is overdamped if $c < 2$, critically damped if $c = 2$, and underdamped if $c > 2$.

40. Recall that $s'' + bs' + cs = 0$ is overdamped if the discriminant $b^2 - 4c > 0$, critically damped if $b^2 - 4c = 0$, and underdamped if $b^2 - 4c < 0$. Since $b^2 - 4c = 36 - 4c$, the solution is overdamped if $c < 9$, critically damped if $c = 9$, and underdamped if $c > 9$.

41. (a) $\frac{dp}{dt} = kp(B-p)$, where $k > 0$.

(b) To find when $\frac{dp}{dt}$ is largest, we notice that $\frac{dp}{dt} = kp(B-p)$, as a function of p, is a parabola opening downward with the maximum at $p = \frac{B}{2}$, i.e. when $\frac{1}{2}$ the tin has turned to powder. This is the time when the tin is crumbling fastest.



(c) If $p = 0$ initially, then $\frac{dp}{dt} = 0$, so we would expect p to remain 0 forever. However, since many organ pipes get tin pest, we must reconcile the model with reality. There are two possible ideas which solve this problem. First, we could assume that p is never 0. In other words, we assume that all tin pipes, no matter how new, must contain some small amount of tin pest. Assuming this means that all organ pipes must deteriorate due to tin pest eventually. Another explanation is that the powder forms at a slow rate even if there was none present to begin with. Since not all organ pipes suffer, it is possible that the conversion is catalyzed by some other impurities not present in all pipes.

42. Let I be the number of infected people. Then, the number of healthy people in the population is $M - I$. The rate of infection is

$$\text{Infection rate} = \frac{0.01}{M}(M-I)I.$$

and the rate of recovery is

$$\text{Recovery rate} = 0.009I.$$

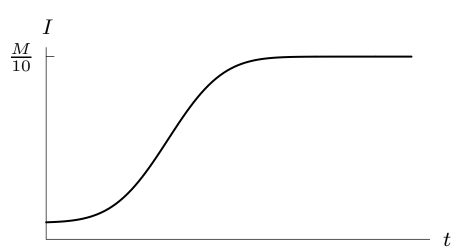
Therefore,

$$\frac{dI}{dt} = \frac{0.01}{M}(M-I)I - 0.009I$$

or

$$\frac{dI}{dt} = 0.001I(1 - 10\frac{I}{M}).$$

This is a logistic differential equation, and so the solution will look like the following graph:



The limiting value for I is $\frac{1}{10}M$, so $1/10$ of the population is infected in the long run.

43. (a) When Juliet loves Romeo (i.e. $j > 0$), Romeo's love for her decreases (i.e. $\frac{dr}{dt} < 0$). When Juliet hates Romeo ($j < 0$), Romeo's love for her grows ($\frac{dr}{dt} > 0$). So j and $\frac{dr}{dt}$ have opposite signs, corresponding to the fact that $-B < 0$. When Romeo loves Juliet ($r > 0$), Juliet's love for him grows ($\frac{dj}{dt} > 0$). When Romeo hates Juliet ($r < 0$), Juliet's love for him decreases ($\frac{dj}{dt} < 0$). Thus r and $\frac{dj}{dt}$ have the same sign, corresponding to the fact that $A > 0$.

(b) Since $\frac{dr}{dt} = -Bj$, we have

$$\frac{d^2r}{dt^2} = \frac{d}{dt}(-Bj) = -B\frac{dj}{dt} = -ABr.$$

Rewriting the above equation as $r'' + ABr = 0$, we see that the characteristic equation is $R^2 + AB = 0$. Therefore $R = \pm\sqrt{AB}i$ and the general solution is

$$r(t) = C_1 \cos \sqrt{AB}t + C_2 \sin \sqrt{AB}t.$$

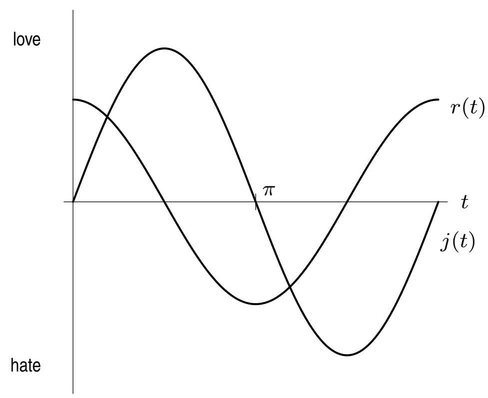
(c) Using $\frac{dr}{dt} = -Bj$, and differentiating r to find j, we obtain

$$j(t) = -\frac{1}{B}\frac{dr}{dt} = -\frac{\sqrt{AB}}{B}(-C_1 \sin \sqrt{AB}t + C_2 \cos \sqrt{AB}t).$$

Now, $j(0) = 0$ gives $C_2 = 0$ and $r(0) = 1$ gives $C_1 = 1$. Therefore, the particular solutions are

$$r(t) = \cos \sqrt{AB}t \quad \text{and} \quad j(t) = \sqrt{\frac{A}{B}} \sin \sqrt{AB}t$$

(d) Consider one period of the graph of $j(t)$ and $r(t)$:



From the graph, we see that they both love each other only a quarter of the time.

44. (a) We have $\Psi = C_1\cos(\omega x) + C_2 \sin(\omega x)$, and we want $\Psi(0) = \Psi(l) = 0$.

$$\Psi(0) = C_1 = 0 \qquad \text{so } C_1 = 0.$$
$$\Psi(l) = C_2 \sin(\omega l) = 0 \quad \text{so } \omega l = n\pi \text{ for some positive integer } n.$$

Thus, $\omega = (n\pi)/l$, so

$$\Psi = C_2 \sin\left(\frac{n\pi x}{l}\right).$$

(b) Using this formula for Ψ, we have

$$\frac{d\Psi}{dx} = \frac{n\pi}{l}C_2 \cos\left(\frac{n\pi x}{l}\right)$$
$$\frac{d^2\Psi}{dx^2} = -\frac{n^2\pi^2}{l^2}C_2 \sin\left(\frac{n\pi x}{l}\right).$$

Thus, substituting for $d^2\Psi/dx^2$ and $\Psi = C_2\sin(n\pi x/l)$, we have

$$\frac{-h^2}{8\pi^2 m}\frac{d^2\Psi}{dx^2} = \frac{h^2}{8\pi^2 m}\cdot\frac{n^2\pi^2}{l^2}C_2 \sin\left(\frac{n\pi x}{l}\right) = \frac{h^2n^2}{8ml^2}\Psi,$$

so

$$E = \frac{h^2n^2}{8ml^2}.$$

(c) Since n must be a positive integer, so $n = 1, 2, 3, 4, \ldots$, the possible values of E are

$$E_1 = \frac{h^2}{8ml^2}, \quad E_2 = \frac{4h^2}{8ml^2}, \quad E_3 = \frac{9h^2}{8ml^2}, \quad E_4 = \frac{16h^2}{8ml^2}, \quad \ldots.$$

The lowest energy level is $E_1 = h^2/(8ml^2)$, and we see that other energy levels are multiples of E_1:

$$E_2 = 4E_1, \quad E_3 = 9E_1, \quad E_4 = 16E_1, \quad \ldots.$$

CAS Challenge Problems

45. (a) We find the equilibrium solutions by setting $dP/dt = 0$, that is, $P(P-1)(2-P) = 0$, which gives three solutions, $P = 0$, $P = 1$, and $P = 2$.

(b) To get your computer algebra system to check that P_1 and P_2 are solutions, substitute one of them into the equation and form an expression consisting of the difference between the right and left hand sides, then ask the CAS to simplify that expression. Do the same for the other function. In order to avoid too much typing, define P_1 and P_2 as functions in your system.

(c) Substituting $t = 0$ gives

$$P_1(0) = 1 - \frac{1}{\sqrt{4}} = 1/2$$
$$P_2(0) = 1 + \frac{1}{\sqrt{4}} = 3/2.$$

We can find the limits using a computer algebra system. Alternatively, setting $u = e^t$, we can use the limit laws to calculate

$$\begin{aligned}
\lim_{t\to\infty} \frac{e^t}{\sqrt{3+e^{2t}}} &= \lim_{u\to\infty} \frac{u}{\sqrt{3+u^2}} = \lim_{u\to\infty} \sqrt{\frac{u^2}{3+u^2}} \\
&= \sqrt{\lim_{u\to\infty} \frac{u^2}{3+u^2}} = \sqrt{\lim_{u\to\infty} \frac{1}{\frac{3}{u^2}+1}} \\
&= \sqrt{\frac{1}{\lim_{u\to\infty} \frac{3}{u^2} + 1}} = \sqrt{\frac{1}{0+1}} = 1.
\end{aligned}$$

Therefore, we have

$$\lim_{t\to\infty} P_1(t) = 1 - 1 = 0$$
$$\lim_{t\to\infty} P_2(t) = 1 + 1 = 2.$$

To predict these limits without having a formula for P, looking at the original differential equation. We see if $0 < P < 1$, then $P(P-1)(2-P) < 0$, so $P' < 0$. Thus, if $0 < P(0) < 1$, then $P'(0) < 0$, so P is initially decreasing, and tends toward the equilibrium solution $P = 0$. On the other hand, if $1 < P < 2$, then $P(P-1)(2-P) > 0$, so $P' > 0$. So, if $1 < P(0) < 2$, then $P'(0) > 0$, so P is initially increasing and tends toward the equilibrium solution $P = 2$.

46. (a) Using the integral equation with $n+1$ replaced by n, we have

$$y_n(a) = b + \int_a^a (y_{n-1}(t)^2 + t^2)\,dt = b + 0 = b.$$

(b) We have $a = 1$ and $b = 0$, so the integral equation tells us that

$$y_{n+1}(s) = \int_1^s (y_n(t)^2 + t^2)\,dt.$$

With $n = 0$, since $y_0(s) = 0$, the CAS gives

$$y_1(s) = \int_1^s 0 + t^2\,dt = -\frac{1}{3} + \frac{s^3}{3}.$$

Then

$$y_2(s) = \int_1^s (y_1(t)^2 + t^2)\,dt = -\frac{17}{42} + \frac{s}{9} + \frac{s^3}{3} - \frac{s^4}{18} + \frac{s^7}{63},$$

and

$$\begin{aligned} y_3(s) &= \int_1^s (y_2(t)^2 + t^2)\,dt \\ &= -\frac{157847}{374220} + \frac{289\,s}{1764} - \frac{17\,s^2}{378} + \frac{82\,s^3}{243} - \frac{17\,s^4}{252} + \frac{s^5}{42} - \frac{s^6}{486} + \frac{s^7}{63} - \frac{11\,s^8}{1764} + \\ &\quad \frac{5\,s^9}{6804} + \frac{2\,s^{11}}{2079} - \frac{s^{12}}{6804} + \frac{s^{15}}{59535}. \end{aligned}$$

(c) The solution y, and the approximations y_1, y_2, y_3 are graphed in Figure 11.57. The approximations appear to be accurate on the range $0.5 \le s \le 1.5$.

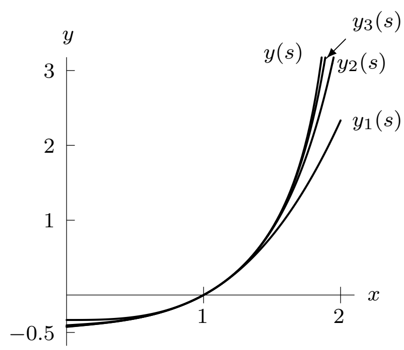


Figure 11.57

47. (a) See Figure 11.58.

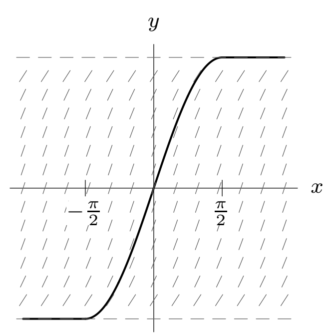


Figure 11.58

(b) Different CASs give different answers, for example they might say $y = \sin x$, or they might say

$$y = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

(c) Both the sample CAS answers in part (b) are wrong. The first one, $y = \sin x$, is wrong because $\sin x$ starts decreasing at $x = \pi/2$, where the slope field clearly shows that y should be increasing at all times. The second answer is better, but it does not give the solution outside the range $-\pi/2 \le x \le \pi/2$. The correct answer is the one sketched in Figure 11.58, which has formula

$$y = \begin{cases} -1 & x \le -\frac{\pi}{2} \\ \sin x & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 1 & x \le \frac{\pi}{2} \le x. \end{cases}$$

CHECK YOUR UNDERSTANDING

1. True. The general solution to $y' = -ky$ is $y = Ce^{-kt}$.
2. False. Suppose $k = -1$. The equation $y'' - y = 0$ or $y'' = y$ has solutions $y = e^t$ and $y = e^{-t}$ and general solution $y = C_1e^t + C_2e^{-t}$.
3. False. The function $y = t^2$ is a solution to $y'' = 2$.
4. False. If $y(0) \le 0$, then $\lim_{x\to\infty} y = -\infty$.
5. True. No matter what initial value you pick, the solution curve has the x-axis as an asymptote.
6. False. There appear to be two equilibrium values dividing the plane into regions with different limiting behavior.
7. False. Euler's method approximates y-values of points on the solution curve.
8. False. In order to be solved using separation of variables, a differential equation must have the form $dy/dx = f(x)g(y)$, so we would need $x + y = f(x)g(y)$. This certainly does not appear to be true. If it were, setting $x = 0$ and $y = 0$, we would have $f(0)g(0) = 0$ so either $f(0) = 0$ or $g(0) = 0$. If $f(0) = 0$, then substituting in $x = 0$ and $y = 1$, we have $0 + 1 = f(0)g(1) = 0$, which is absurd. We get the same contradiction if we assume $g(0) = 0$.
9. True. Rewrite the equation as $dy/dx = xy + x = x(y + 1)$. Since the equation now has the form $dy/dx = f(x)g(y)$, it can be solved by separation of variables.
10. False. It is true that $y = x^3$ is a solution of the differential equation, since $dy/dx = 3x^2 = 3y^{2/3}$, but it is not the only solution passing through $(0, 0)$. Another solution is the constant function $y = 0$. Usually there is only one solution curve to a differential equation passing through a given point, but not always.
11. True. We have $dy/dx > 0$ at every point because $x^2 + y^2 + 1 > 0$, and a positive derivative indicates increasing function.
12. False. We have

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d(x^2+y^2+1)}{dx}\\ &= 2x + 2y\frac{dy}{dx}\\ &= 2x + 2y(x^2+y^2+1)\\ &= 2x + 2y + 2x^2y + 2y^3.\end{aligned}$$

    At the point $(x, y) = (-1, 0)$ we have $d^2y/dx^2 = -2 < 0$. A negative second derivative indicates function concave down. The solution curve of the differential equation that passes through the point $(-1, 0)$ is concave down at $(-1, 0)$.
13. False. This is a logistic equation with equilibrium values $P = 0$ and $P = 2$. Solution curves do not cross the line $P = 2$ and do not go from $(0, 1)$ to $(1, 3)$.
14. True. This is a logistic differential equation. Any solution with $P(0) > 0$ tends toward the carrying capacity, L, as $t \to \infty$.
15. True. Specifying $x(0)$ and $y(0)$ corresponds to picking a starting point in the plane and thereby picking the unique solution curve through that point.
16. False. Competitive exclusion, in which one population drives out another, is modeled by a system of differential equations.
17. True. Since $f'(x) = g(x)$, we have $f''(x) = g'(x)$. Since $g(x)$ is increasing, $g'(x) > 0$ for all x, so $f''(x) > 0$ for all x. Thus the graph of f is concave up for all x.
18. False. We just need an example of a function $f(x)$ which is decreasing for $x > 0$, but whose derivative $f'(x) = g(x)$ is increasing for $x > 0$. An example is $f(x) = 1/x$. Clearly $f(x)$ is decreasing for $x > 0$ but its derivative $f'(x) = -1/x^2$ is clearly increasing for $x > 0$.
19. True. Since $g(x)$ is increasing, $g(x) \ge g(0) = 1$ for all $x \ge 0$. Since $f'(x) = g(x)$, this means that $f'(x) > 0$ for all $x \ge 0$. Therefore $f(x)$ is increasing for all $x \ge 0$.
20. False. If $g(x) > 0$ for all x, then $f(x)$ would have to be increasing for all x so $f(x + p) = f(x)$ would be impossible. For example, let $g(x) = 2 + \cos x$. Then a possibility for f is $f(x) = 2x + \sin x$. Then $g(x)$ is periodic, but $f(x)$ is not.
21. False. Let $g(x) = 0$ for all x and let $f(x) = 17$. Then $f'(x) = g(x)$ and $\lim_{x\to\infty} g(x) = 0$, but $\lim_{x\to\infty} f(x) = 17$.
22. True. Since $\lim_{x\to\infty} g(x) = \infty$, there must be some value $x = a$ such that $g(x) > 1$ for all $x > a$. Then $f'(x) > 1$ for all $x > a$. Thus, for some constant C, we have $f(x) > x + C$ for all $x > a$, which implies that $\lim_{x\to\infty} f(x) = \infty$. More precisely, let $C = f(a) - a$ and let $h(x) = f(x) - x - C$. Then $h(a) = 0$ and $h'(x) = f'(x) - 1 > 0$ for all $x > a$. Thus h is increasing so $h(x) > 0$ for all $x > a$, which means that $f(x) > x + C$ for all $x > a$.

23. False. Let $f(x) = x^3$ and $g(x) = 3x^2$. Then $y = f(x)$ satisfies $dy/dx = g(x)$ and $g(x)$ is even while $f(x)$ is odd.

24. False. The example $f(x) = x^3$ and $g(x) = 3x^2$ shows that you might expect $f(x)$ to be odd. However, the additive constant C can mess things up. For example, still let $g(x) = 3x^2$, but let $f(x) = x^3 + 1$ instead. Then $g(x)$ is still even, but $f(x)$ is not odd (for example, $f(-1) = 0$ but $-f(1) = -2$).

25. True. The slope of the graph of f is $dy/dx = 2x - y$. Thus when $x = a$ and $y = b$, the slope is $2a - b$.

26. True. Saying $y = f(x)$ is a solution for the differential equation $dy/dx = 2x - y$ means that if we substitute $f(x)$ for y, the equation is satisfied. That is, $f'(x) = 2x - f(x)$.

27. False. Since $f'(x) = 2x - f(x)$, we would have $1 = 2x - 5$ so $x = 3$ is the only possibility.

28. True. Differentiate $dy/dx = 2x - y$, to get:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x - y) = 2 - \frac{dy}{dx} = 2 - (2x - y).$$

29. False. Since $f'(1) = 2(1) - 5 = -3$, the point $(1, 5)$ could not be a critical point of f.

30. True. Since $dy/dx = 2x - y$, the slope of the graph of f is negative at any point satisfying $2x < y$, that is any point lying above the line $y = 2x$. The slope of the graph of f is positive at any point satisfying $2x > y$, that is any point lying below the line $y = 2x$.

31. True. When we differentiate $dy/dx = 2x - y$, we get:

$$\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - (2x - y).$$

Thus at any inflection point of $y = f(x)$, we have $d^2y/dx^2 = 2 - (2x - y) = 0$. That is, any inflection point of f must satisfy $y = 2x - 2$.

32. False. Suppose that $g(x) = f(x) + C$, where $C \neq 0$. In order to be a solution of $dy/dx = 2x - y$ we would need $g'(x) = 2x - g(x)$. Instead we have:

$$g'(x) = f'(x) = 2x - f(x) = 2x - (g(x) - C) = 2x - g(x) + C.$$

Since $C \neq 0$, this means $g(x)$ is not a solution of $dy/dx = 2x - y$.

33. True. We will use the hint. Let $w = g(x) - f(x)$. Then:

$$\frac{dw}{dx} = g'(x) - f'(x) = (2x - g(x)) - (2x - f(x)) = f(x) - g(x) = -w.$$

Thus $dw/dx = -w$. This equation is the equation for exponential decay and has the general solution $w = Ce^{-x}$. Thus,

$$\lim_{x\to\infty} (g(x) - f(x)) = \lim_{x\to\infty} Ce^{-x} = 0.$$

34. An example is $dy/dx = e^x$. In fact, if $f(x)$ is any increasing positive function, then the solutions of $dy/dx = f(x)$ are increasing since $f(x) > 0$ and concave up since $d^2y/dx^2 = f'(x) > 0$.

35. We want to have $dy/dx = 0$ when $y - x^2 = 0$, so let $dy/dx = y - x^2$.

36. This family has $f'(x) = 2x$, so let $dy/dx = 2x$.

37. If we differentiate implicitly the equation for the family, we get $2x - 2y\,dy/dx = 0$. When we solve, we get the differential equation we want $dy/dx = x/y$.

PROJECTS FOR CHAPTER ELEVEN

1. (a) (i) Integrating we have

$$\begin{aligned}\frac{dP}{dt} &= 30.2\\ P &= 30.2t + C.\end{aligned}$$

Since $P(0) = 95$, we have $C = 95$, so

$$P = 30.2t + 95.$$

Substituting $t = 87$ and rounding to the nearest person gives

$$P = 30.2 \cdot 87 + 95 = 2722.$$

The linear model predicts that 2722 people would have contracted SARS by June 12, 2003.

(ii) Separating variables, we have

$$\begin{aligned}
\frac{1}{P}\frac{dP}{dt} &= 0.12 \\
\int \frac{dP}{P} &= \int 0.12\, dt \\
\ln|P| &= 0.12t + C \\
P &= Ae^{0.12t}.
\end{aligned}$$

Since $P(0) = 95$, we have $A = 95$, so

$$P = 95e^{0.12t}.$$

Substituting $t = 87$ and rounding to the nearest person gives

$$P = 95e^{0.12 \cdot 87} = 3,249,062.$$

The exponential model predicts that 3.249 million people would have contracted SARS by June 12, 2003.

(iii) Writing the differential equation in the form

$$\begin{aligned}
\frac{dP}{dt} &= 0.19P - 0.0002P^2 \\
\frac{dP}{dt} &= 0.19P\left(1 - \frac{0.0002}{0.19}P\right) \\
\frac{dP}{dt} &= 0.19P\left(1 - \frac{P}{950}\right),
\end{aligned}$$

we use the analytic solution derived on page 563 of the text to obtain

$$P = \frac{950}{1 + Ae^{-0.19t}}, \quad \text{with } A = \frac{950 - 95}{95} = 9$$

so

$$P = \frac{950}{1 + 9e^{-0.19t}}.$$

Substituting $t = 87$ and rounding to the nearest person gives

$$P = \frac{950}{1 + 9e^{-0.19 \cdot 87}} = 950.$$

The logistic model predicts that 950 people will have contracted SARS by June 12, 2003.

(b) (i) The three methods give very different predictions. The linear and logistic are about 3000 and 1000, respectively, while the exponential model is 3 million, nearly half the population of Hong Kong.

(ii) The number of new cases per day is approximated by the derivative, dP/dt. The linear model predicts a constant number of new cases each day; the exponential model predicts an increasing number of new cases each day; the logistic model predicts that the number of new cases per day will first increase and then decrease.

(iii) The general trend in the figure shows that the number of new cases per day first climbed and then fell, suggesting that the logistic model fits best. The high values are largely Mondays, and represent two days of data recorded as one, since no new cases were reported on Sundays.

(c) (i) The formula

$$P = \frac{950}{1 + 9e^{-0.19t}}$$

has limiting value $P = 950$ as $t \to \infty$. Thus, this formula predicts that the maximum number of cases expected is 950.

(ii) The graph allows us to estimate (very roughly) when the daily increase was largest, namely about April 10. Since the maximum rate of change of P (and the maximum daily increase in P) occurs at $L/2$, where L is the maximum value of P, we expect the maximum value of P to be about $2 \cdot 998 \approx 2000$.

(d) See Figures 11.59–11.61. The dots represent the actual data.

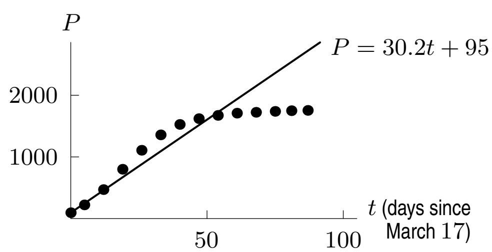


Figure 11.59: Linear predictions and actual data

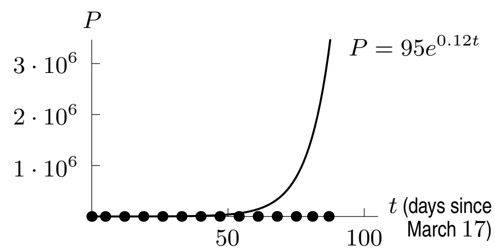


Figure 11.60: Exponential predictions and actual data

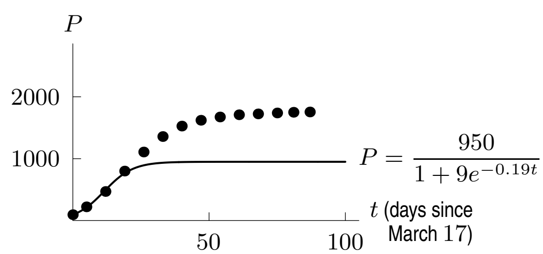


Figure 11.61: Logistic predictions and actual data

2. (a) Since I_0 is the number of infecteds on day $t = 0$, March 17, we have $I_0 = 95$. Since S_0 is the initial number of susceptibles, which is the whole population of Hong Kong, $S_0 \approx 6.8$ million.

(b) For $a = 1.25 \cdot 10^{-8}$ and $b = 0.06$, the system of equations is

$$\frac{dS}{dt} = -1.25 \cdot 10^{-8} SI$$
$$\frac{dI}{dt} = 1.25 \cdot 10^{-8} SI - 0.06I.$$

So, by the chain rule,

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{1.25 \cdot 10^{-8} SI - 0.06I}{-1.25 \cdot 10^{-8} SI} = -1 + \frac{4.8 \cdot 10^6}{S}.$$

The slope field and trajectory are in Figure 11.62.

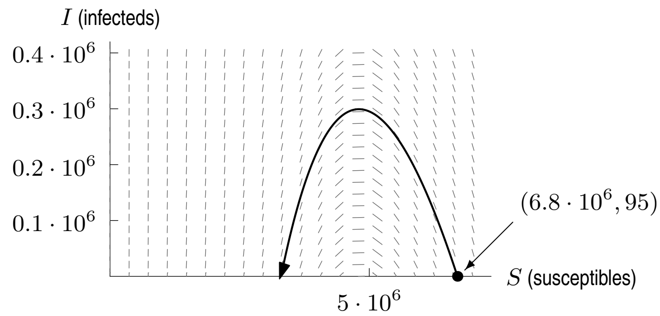


Figure 11.62

(c) The maximum value of I is about $300{,}000$; this gives us the maximum number of infecteds at any one time—the total number of people infected during the course of the disease is much greater than this. The trajectory meets the S-axis at about 3.3 million; this tells us that when the disease dies out, there are still 3.3 million susceptibles who have never had the disease. Therefore $6.8 - 3.3 = 3.5$ million people are predicted to have had the disease.

The threshold value of S occurs where $dI/dt = 0$ and $I \neq 0$, so, for $a = 1.25 \cdot 10^{-8}$ and $b = 0.06$,

$$\frac{dI}{dt} = 1.25 \cdot 10^{-8} SI - 0.06I = 0,$$

giving

$$\text{Threshold value} = S = \frac{0.06}{1.25 \cdot 10^{-8}} = 4.8 \cdot 10^6 \text{ people.}$$

The threshold value tells us that if the initial susceptible population, S_0 is more than 4.8 million, there will be an epidemic. If S_0 is less than 4.8 million, there will not be an epidemic. Since the population of Hong Kong is over 4.8 million, an epidemic is predicted.

(d) The value of b represents the rate at which infecteds are removed from circulation. Quarantine increases the rate people are removed and thus increases b.

(e) For $a = 1.25 \cdot 10^{-8}$ and $b = 0.24$, the system of differential equations is

$$\frac{dS}{dt} = -1.25 \cdot 10^{-8} SI$$
$$\frac{dI}{dt} = 1.25 \cdot 10^{-8} SI - 0.24I.$$

So, by the chain rule,

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{1.25 \cdot 10^{-8} SI - 0.24I}{-1.25 \cdot 10^{-8} SI} = -1 + \frac{19.2 \cdot 10^6}{S}.$$

The slope field is in Figure 11.63. The solution trajectory does not show as the disease dies out right away.

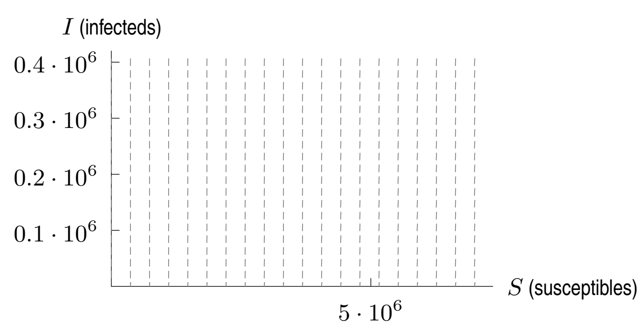


Figure 11.63

(f) The threshold value of S occurs where $dI/dt = 0$ and $I \neq 0$, so, for $b = 0.24$ and the same value of a,

$$\frac{dI}{dt} = 1.25 \cdot 10^{-8} SI - 0.24I = 0,$$

giving

$$\text{Threshold value } = S = \frac{0.24}{1.25 \cdot 10^{-8}} = 19.2 \cdot 10^6 \text{ people.}$$

The threshold value tells us that if S_0 is less than 19.2 million, there will be no epidemic. The population of Hong Kong is 6.8 million, so S_0 is below this value. Thus no epidemic is predicted.

Policies, such as quarantine, which raise the value of b can be effective in preventing an epidemic. In this case, the value of b increased sufficiently that the population of Hong Kong fell below the threshold value, and a potential epidemic was averted. However, we do not have evidence that the quarantine policy was responsible for the increase in b.

(g) Policy I: Closing off the city changes the initial values of S_0 and I_0 but not the values of a and b. If not one infected person enters the city, then $I_0 = 0$ and the solution trajectory is an equilibrium point on the S-axis. However, in practice it is almost impossible to cut off a city completely, so usually $I_0 > 0$. Also, by the time a policy to close off a city is put into effect, there may already be infected people inside the city, so again $I_0 > 0$. Thus, whether or not there is an epidemic depends on whether S_0 is greater than the threshold value, not on the value of I_0 (provided $I_0 > 0$).

For example, in the case of Hong Kong with the March values of a and b, changing the value of I_0 to 1 leaves the solution trajectory much as before; see Figure 11.64. The main difference is that the epidemic occus slightly later. So a policy of isolating a city only works if it keeps the disease out of the city of the city entirely. Thus, Policy I does not help the city except in the exceptional case that *every* infected person is kept out.

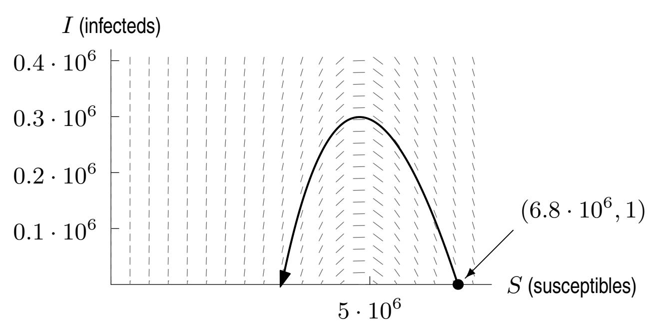


Figure 11.64

Policy II: From the analysis of the Hong Kong data, we see that a quarantine policy can help prevent an epidemic if the value of b is increased enough to bring S_0 below the threshold value. Thus, Policy II can be very effective.

3. (a)

$$p(x) = \text{the number of people with incomes} \geq x.$$

$$p(x + \Delta x) = \text{the number of people with incomes} \geq x + \Delta x.$$

So the number of people with incomes between x and $x + \Delta x$ is

$$p(x) - p(x + \Delta x) = -\Delta p.$$

Since all the people with incomes between x and $x + \Delta x$ have incomes of about x (if Δx is small), the total amount of money earned by people in this income bracket is approximately $x(-\Delta p) = -x\Delta p$.

(b) Pareto's law claims that the average income of all the people with incomes $\geq x$ is kx. Since there are $p(x)$ people with income $\geq x$, the total amount of money earned by people in this group is $kxp(x)$.

The total amount of money earned by people with incomes $\geq (x+\Delta x)$ is therefore $k(x+\Delta x)p(x+\Delta x)$. Then the total amount of money earned by people with incomes between x and $x+\Delta x$ is

$$kxp(x) - k(x+\Delta x)p(x+\Delta x).$$

Since $\Delta p = p(x+\Delta x) - p(x)$, we can substitute $p(x+\Delta x) = p(x) + \Delta p$. Thus the total amount of money earned by people with incomes between x and $x+\Delta x$ is

$$kxp(x) - k(x+\Delta x)(p(x)+\Delta p).$$

Multiplying out, we have

$$kxp(x) - kxp(x) - k(\Delta x)p(x) - kx\Delta p - k\Delta x\Delta p$$

Simplifying and dropping the second order term $\Delta x\Delta p$ gives the total amount of money earned by people with incomes between x and $x+\Delta x$ as

$$-kp\Delta x - kx\Delta p.$$

(c) Setting the answers to parts (a) and (b) equal gives

$$-x\Delta p = -kp\Delta x - kx\Delta p.$$

Dividing by Δx, and letting $\Delta x \to 0$ so that $\frac{\Delta p}{\Delta x} \to p'$, we have

$$\begin{aligned} x\frac{\Delta p}{\Delta x} &= kp + kx\frac{\Delta p}{\Delta x} \\ xp' &= kp + kxp' \end{aligned}$$

so

$$(1-k)xp' = kp.$$

(d) We solve this equation by separating variables

$$\begin{aligned} \int \frac{dp}{p} &= \int \frac{k}{(1-k)}\frac{dx}{x} \\ \ln p &= \frac{k}{(1-k)}\ln x + C \quad \text{(no absolute values needed since } p, x > 0\text{)} \\ \ln p &= \ln x^{k/(1-k)} + \ln A \quad \text{(writing } C = \ln A\text{)} \\ \ln p &= \ln[Ax^{k/(1-k)}] \quad \text{(using } \ln(AB) = \ln A + \ln B\text{)} \\ p &= Ax^{k/(1-k)} \end{aligned}$$

(e) We take $A = 1$. For $k = 10$, $p = x^{-10/9} \approx x^{-1}$. For $k = 1.1$, $p = x^{-11}$. The functions are graphed in Figure 11.65. Notice that the larger the value of k, the less negative the value of $k/(1-k)$ (remember $k > 1$), and the slower $p(x) \to 0$ as $x \to \infty$.

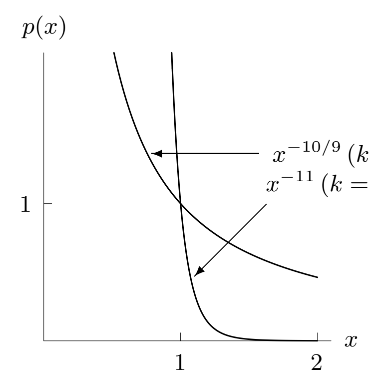


Figure 11.65

4. (a) Writing $F = b\left(\frac{a^2-ar}{r^3}\right) = 0$ shows $F = 0$ when $r = a$, so $r = a$ gives the equilibrium position.

(b) Expanding $1/r^3$ about $r = a$ gives

$$\begin{aligned}
\frac{1}{r^3} = \frac{1}{(a+r-a)^3} &= \frac{1}{a^3}\left(1+\frac{r-a}{a}\right)^{-3} \\
&= \frac{1}{a^3}\left(1 - 3\left(\frac{r-a}{a}\right) + \frac{(-3)(-4)}{2!}\left(\frac{r-a}{a}\right)^2 - \cdots\right) \\
&= \frac{1}{a^3}\left(1 - \frac{3(r-a)}{a} + \frac{6(r-a)^2}{a^2} - \cdots\right).
\end{aligned}$$

Similarly, expanding $1/r^2$ about $r = a$ gives

$$\begin{aligned}
\frac{1}{r^2} = \frac{1}{(a+r-a)^2} &= \frac{1}{a^2}\left(1+\frac{r-a}{a}\right)^{-2} \\
&= \frac{1}{a^2}\left(1 - 2\left(\frac{r-a}{a}\right) + \frac{(-2)(-3)}{2!}\left(\frac{r-a}{a}\right)^2 - \cdots\right) \\
&= \frac{1}{a^2}\left(1 - 2\left(\frac{r-a}{a}\right) + 3\left(\frac{r-a}{a}\right)^2 - \cdots\right).
\end{aligned}$$

Thus, combining gives

$$\begin{aligned}
F &= b\left(\frac{1}{a}\left(1 - \frac{3(r-a)}{a} + \frac{6(r-a)^2}{a^2} - \cdots\right) - \frac{1}{a}\left(1 - \frac{2(r-a)}{a} + \frac{3(r-a)^2}{a^2} - \cdots\right)\right) \\
&= \frac{b}{a}\left(-\frac{(r-a)}{a} + \frac{3(r-a)^2}{a^2} - \cdots\right) \\
&= \frac{b}{a^2}\left(-(r-a) + \frac{3(r-a)^2}{a} - \cdots\right).
\end{aligned}$$

(c) Setting $x = r - a$ gives

$$F \approx \frac{b}{a^2}\left(-x + \frac{3x^2}{a}\right).$$

(d) For small x, we discard the quadratic term in part (c), giving

$$F \approx \frac{-b}{a^2}x.$$

The acceleration is d^2x/dt^2. Thus, using Newton's Second Law:

$$\text{Force} = \text{Mass} \cdot \text{Acceleration}$$

we get

$$\frac{-bx}{a^2} = m\frac{d^2x}{dt^2}.$$

So

$$\frac{d^2x}{dt^2} + \frac{b}{a^2m}x = 0.$$

This differential equation represents an oscillation of the form $x = C_1\cos\omega t + C_2\sin\omega t$, where $\omega^2 = b/(a^2m)$ so $\omega = \sqrt{b/(a^2m)}$. Thus, we have

$$\text{Period} = \frac{2\pi}{\omega} = 2\pi a\sqrt{\frac{m}{b}}.$$