Qualifying Exam in Basic Analysis August 2009
Duke University, Mathematics Department
Time Allowed: 3 hours

Part I: 6 points each, do all questions

1. Let $f_n(x)$ be a sequence of continuous functions that converges uniformly to a function $f(x)$ on $[0,1]$. Suppose that $g(x)$ is continuous on $[0,1]$. Prove that $f_n(x)g(x) \to f(x)g(x)$ uniformly on $[0,1]$.

2. Let $a_n \geq 0$ and suppose that $\sum_{n=0}^{\infty} a_n$ converges. Prove that $\sum_{n=0}^{\infty} a_n^2$ converges.

3. Consider the mapping, $\mathbf{F}(x,y)$, from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $(x,y) \longrightarrow (x(y+1), 3x-2y)$. Prove that $\mathbf{F}$ is invertible near $(0,0)$.

4. Suppose that $f(x)$ is a twice continuously differentiable function on $[0,1]$ and that $f(0)=3$, $f'(0)=2$ and $|f''(x)| \leq 0.5$ for all $x \in [0,1]$. Estimate $\int_0^1 e^{f(x)}\,dx$.

5. Prove that $\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$ converges.

6. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. If a is an accumulation point of $\{a_n\}$ and b is an accumulation point of $\{b_n\}$, is ab necessarily an accumulation point of $\{a_n b_n\}$? Prove it or give a counterexample.

7. Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences of real numbers. Prove that $\sup_n (a_n + b_n) \leq \sup_n a_n + \sup_n b_n$ and give an example to show that strict inequality can hold.

8. Let $C(t) = (x(t), y(t), z(t))$ be a differentiable curve in $\mathbb{R}^3$, where $x(t) = t^3$, $y(t) = \sin(\pi t)$, and $z(t) = t^t$. Give the scalar equation for the plane (e.g. $ax+by+cz=d$) passing through $C(2)$ that is perpendicular to the tangent to the curve at $t=2$.

9. Either prove or give a counterexample to the following statement: If $f(x)$ is a continuous function on $\mathbb{R}$ that is periodic, then it is uniformly continuous.

10. Let f be a continuous function from a metric space, $\langle X, \rho \rangle$, to $\mathbb{R}$. If $\{x_n\}$ is a Cauchy sequence in $\langle X, \rho \rangle$, is $\{f(x_n)\}$ necessarily a Cauchy sequence in $\mathbb{R}$? Prove it or give a counterexample.

Part II: 10 points each. Choose 4 out of 5 questions. Only 4 questions will be counted in your score.

1. Prove that
$$\int_2^4 \left(\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n} \right) dx = \sum_{n=0}^{\infty} \left(\int_2^4 \frac{(x-3)^n}{2^n}\, dx \right).$$

2. Suppose that C is a simple closed curve in $\mathbb{R}^2$ parameterized by $\mathbf{r}(s)$:

   (a) For what general condition on the vector field $\mathbf{V}(x) : \mathbb{R}^2 \to \mathbb{R}^2$ must
$$\int_C \mathbf{V} \cdot \mathbf{n}\, ds = 0$$
   be true? Here $\mathbf{n}$ is the exterior normal vector to C at a point $\mathbf{r}(s) \in C$. Give an example of a non-constant vector field $\mathbf{V}$ satisfying this condition.

(b) For what general condition on the vector field $\mathbf{V}$ must

$$\int_C \mathbf{V} \cdot \mathbf{dr} = 0$$

be true? Give an example of a non-constant vector field $\mathbf{V}$ satisfying this condition.

3. Let $Q \subset \mathbb{R}^2$ be the square

$$Q = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, \;\; 0 \leq y \leq 1\right\}.$$

Define the map $\mathbf{F}(x, y) = (f_1(x, y), f_2(x, y)) : Q \to \mathbb{R}^2$ by

$$f_1(x, y) = 2x - y + 3, \qquad f_2(x, y) = 4 + x^2 + 2y^2$$

Let A denote the image of Q under the map $\mathbf{F}$. Compute the area of A. It may help to express the area as an integral.

4. Suppose $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ and $g(x, y) : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable functions. Suppose that the set of points where $g(x, y) = 0$ is a simple closed curve in R^2.

(a) Prove that the minimum of $f(x, y)$ on the curve $g(x, y) = 0$ must be attained at some point $(x_0, y_0) \in \mathbb{R}^2$.

(b) Prove that if the point (x_0, y_0) is such a minimizer, then ∇f and ∇g must be linearly dependent at $x(x_0, y_0)$: there are $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\lambda_1 \nabla f(x_0, y_0) + \lambda_2 \nabla g(x_0, y_0) = 0$$

and $\lambda_1^2 + \lambda_2^2 > 0$.

5. Define $\ln x = \int_1^x \frac{1}{t}\, dt$. (This definition is the only property of $\ln x$ you may use.)

(a) Prove that $\ln x$ is strictly monotone increasing.

(b) Prove that $\ln x \to \infty$ as $x \to \infty$.

(c) Prove that there is a unique number e such that $\ln e = 1$.