Qualifying Exam in Basic Analysis, August 2010
Duke University, Mathematics Department
Time Allowed: 3 hours

Part I: 6 points each, do all questions

1. Prove that a metric space (X, ρ) is complete if every Cauchy sequence $\{x_n\}_{n=1}^{\infty} \subset X$ has a convergent *subsequence*.

2. Suppose the function $f : \mathbb{R}^2 \to \mathbb{R}$ is C^1 and $f(x, y) > f(0, 0)$ if $x^2 + y^2 = 1$. Prove there is a point (x_0, y_0) at which $Df(x_0, y_0) = (0, 0)$.

3. Prove or disprove the following statement: If $f : [0, 1] \to \mathbb{R}$ is a continuous function, then for any $y \in \mathbb{R}$, the set
$$f^{-1}(y) = \{x \in [0, 1] \mid f(x) = y\}$$
is a compact set.

4. Show that the equation $2x + \sin x + 1 = 0$ has *exactly one* real root.

5. Show that both first partial derivatives of the function
$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$
exist at the origin, but the function is not differentiable there.

6. Use Green's theorem to compute the line integral
$$\int_{\Gamma} -xy\, dx + xy\, dy$$
where Γ is the circle $x^2 + y^2 = a^2$. State Green's theorem and justify your answer.

7. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function such that $\phi_{xx}(x, y) - \phi_{yy}(x, y) = x$ for any x, y. Let $\psi(x, y) = \phi(x + y, x - y)$. Write $\psi_{xy}(x, y)$ in the simplest possible form. (Subscripts indicate partial derivatives.)

8. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of real-valued, continuously differentiable functions on the interval $(0, 1)$. Suppose that $f_n(x) \to f(x)$ uniformly on $(0, 1)$ and $f_n'(x) \to g(x)$ uniformly on $(0, 1)$ where g is a continuous function. Prove that $f'(x) = g(x)$ for all $x \in (0, 1)$.

9. Prove that $\ln(1.1) \leq 286/3000$.

10. If
$$X_k = \prod_{n=1}^{k} \left(1 + \frac{3}{n^2}\right) = \left(1 + \frac{3}{1}\right)\left(1 + \frac{3}{2^2}\right)\left(1 + \frac{3}{3^2}\right)\cdots\left(1 + \frac{3}{k^2}\right),$$
show that the limit
$$\lim_{k\to\infty} X_k$$
exists and is finite.

Part II: 10 points each. Choose 4 out of 5 questions. Only 4 questions will be counted in your score.

1. Consider the set

$$X = \left\{ \{a_n\}_{n=1}^{\infty} \;\middle|\; a_n \in \mathbb{R}, \quad \sum_{n=1}^{\infty} a_n^2 < \infty \right\}$$

of sequences in $\mathbb{R}$ which are square-summable. Given two sequences $\{a_n\} \in X$ and $\{b_n\} \in X$, define

$$\rho(\{a_n\}, \{b_n\}) = \left(\sum_{n=1}^{\infty} |a_n - b_n|^2 \right)^{1/2}.$$

Prove that (X, ρ) is a complete metric space.

2. Consider the function $f(x,y) = -x^3/3 + xy - y^2$. Find all critical points of f. For each critical point compute the Hessian and decide whether the critical points are (i) local minima, (ii) local maxima, (iii) neither, or (iv) impossible to decide based on the Hessian.

3. Consider a curve in $\mathbb{R}^2$ parametrized by $t \mapsto \big(u(t), v(t)\big)$ where $u, v : \mathbb{R} \to \mathbb{R}$ are C^2 functions. Let $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\boldsymbol{f}(t,s) = \big(u(t) - sv'(t), v(t) + su'(t)\big).$$

   (a) Show that $\boldsymbol{f}_t(t,0) \cdot \boldsymbol{f}_s(t,0) = 0$ for all t. (Subscripts indicate partial derivatives.)

   (b) Show that for any $t_0 \in \mathbb{R}$ where $\big(u'(t_0), v'(t_0)\big) \neq (0,0)$, the inverse function theorem applies to show that for all (x,y) near $\big(u(t_0), v(t_0)\big)$ there is a unique (t,s) near $(t_0, 0)$ such that $(x,y) = \boldsymbol{f}(t,s)$. Why do we need to assume that u and v are C^2?

4. Find the maximum of $f(x,y,z) = x^2y^2z^2$ subject to the constraint $x^2 + y^2 + 2z^2 = 4$.

5. Suppose that $f(x) : \mathbb{R} \to \mathbb{R}$ is a smooth function, $f''(x) > 0$ for all x, and $f(0) = 0$. Prove that the function

$$x \mapsto \frac{f(x)}{x}$$

is increasing in x for $x > 0$. Hint: observe that $f(x)/x$ is the slope of a secant line through the points $(0, f(0))$ and $(x, f(x))$.