Qualifying Exam in Basic Analysis, August 2011
Duke University, Mathematics Department
Time Allowed: 3 hours

All answers and statements should be proved; no partial credit is given to answers without justification.

Part I: 6 points each, do all 9 questions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ when x is rational and $f(x) = x$ when x is irrational. Prove that f is continuous at $x = 0$ and f is discontinuous for any $x \neq 0$.
2. True or False: If $x_n \in \mathbb{R}$ is a bounded sequence such that $\limsup x_n \leq \liminf x_n$ then x_n converges.
3. Give an example of a sequence of functions $f_n : [0, 1] \to \mathbb{R}$ such that $f_n \to 0$ pointwise on $[0, 1]$ yet
$$\int_0^1 f_n(x)\, dx \not\to 0.$$
4. True or False: If $x_n y_n$ is a convergent sequence of nonzero real numbers and if $\frac{x_n}{y_n}$ converges, then both x_n and y_n are convergent.
5. Let $f(x) = x \tan^2 x$ for $x \in (0, \pi/2)$. Calculate $(f^{-1})'(\pi/4)$.
6. Consider the differential equation
$$f''(x) = a(x)f(x)$$
on $\mathbb{R}$, where $a(x) \geq 0$ and continuous. Suppose f solves the equation, and that $f(0) = f(1) = 0$. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.
7. Show that equation $\arctan(2x - 1) = 5 - 2x$ has at least one real solution.
8. Let $f \in \mathcal{C}^1([a, b])$ and $f'(x) < 0$ for all $x \in [a, b]$. Prove that f is strictly monotonically descreasing, i.e. $f(x) < f(y)$ for all $x > y$, $x, y \in [a, b]$
9. Let $f \in \mathcal{C}([0, 1])$. Prove or disprove:
   (a) if $\int_0^1 f(t)dt = 0$, then $f = 0$.
   (b) if $f(x) \geq 0$ for every $x \in [0, 1]$ and $\int_0^1 f(t)dt = 0$, then $f = 0$.
   (c) if for every $x \in [0, 1]$, $\int_0^x f(t)dt = 0$, then $f = 0$.

Part II: 10 points each. Choose 5 out of 7 questions. Only 5 answers will be counted in your score.

1. Show that
$$\lim_{n\to\infty} \left(\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \cdots + \frac{n}{n^2 + n^2} \right) = \frac{\pi}{4}$$
2. (a) Show that the following is an inner product on $\mathbb{R}^2$;
$$\langle (x_1, y_1), (x_2, y_2) \rangle = 2x_1x_2 + x_1y_2 + y_1x_2 + 2y_1y_2\,.$$
   (b) Explain why
$$d((x_1, y_1), (x_2, y_2)) = 2(x_1 - x_2)^2 + 2(x_1 - x_2)(y_1 - y_2) + 2(y_1 - y_2)^2$$
   gives a metric on $\mathbb{R}^2$.
   (c) Explain why
$$\left\{(x, y) : 2x^2 + 2xy + 2y^2 \leq 1\right\}$$
   is a convex set in $\mathbb{R}^2$.

3. Let $\gamma : [0,1] \to \mathbb{R}^3$ be a continuous path with $\gamma(0) = (0,0,0)$ and $\gamma(1) = (1,1,1)$. Show that γ meets the unit sphere $S = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1\}$.

4. A real-valued function f defined on an interval (a,b) is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

whenever $a < x, y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous.

5. Consider a bounded sequence $\{x_n\}$ in $\mathbb{R}$. Let ω_x be the set of limit points of $\{x_n\}$, i.e. the set of points y in $\mathbb{R}$ for which there exist a subsequence of $\{x_n\}$ converging to y.
   (a) What can you say about the cardinality of ω_x? Can it be 0 (i.e. $\omega_x = \emptyset$)? Finite? Countably infinite? Uncountably infinite?
   (b) Is ω_x bounded?
   (c) Let $\{y_n\}$ be a second bounded sequence in $\mathbb{R}$, and ω_y the corresponding set of limit points. Let $a \in \omega_x$ and $b \in \omega_y$. Is $a \cdot b$ necessarily a limit point of $\{x_n \cdot y_n\}$?
   (d) Prove or disprove: ω_x is closed.

6. Consider the series of functions

$$f(x) = \sum_{n=1}^{+\infty} f_n(x)$$

where

$$f_n(x) = \sum_{n=1}^{+\infty} \frac{\tanh(nx) - nx^4}{n(n^4+x^4)} .$$

Recall that $\tanh(y) = \frac{e^{2y}-1}{e^{2y}+1}$.
   (a) Show that the series converges pointwise on $\mathbb{R}$. [*Hint:* show that $f_n(x)/(-x^4/n^4) \to 1$ as $n \to +\infty$, for every $x \neq 0$.]
   (b) Show that, in general, for a series of functions $\sum_{n=1}^{+\infty} f_n$ to converge uniformly on a subset E of $\mathbb{R}$, it is necessary that the f_n converges uniformly to 0. Is this also a sufficient condition?
   (c) Determine on which intervals (bounded and unbounded) of $\mathbb{R}$ the convergence is uniform. [*Hint*: for unbounded intervals, look at $\lim_{x\to\pm\infty} f_n(x)$, then use (b).]
   (d) Show that

$$\left| \int_0^1 f(x)dx \right| \leq \sum_{n=1}^{+\infty} \left(\frac{1}{n^5} + \frac{1}{5n^4} \right) .$$

7. Let

$$K(f)(x) = \int_0^1 k(x,y) f(y) dy \qquad x \in [0,1]$$

where $k \in \mathcal{C}([0,1] \times [0,1])$.
   (a) Show that if f is square-integrable on $[0,1]$ (i.e. $\int_0^1 |f(x)|^2 dx$), then so is $|Kf|^2$.
   (b) Show that K is a linear operator from square-integrable functions to square-integrable functions, i.e. $K(\alpha f + \beta g) = \alpha K f + \beta K g$ for any $\alpha, \beta \in \mathbb{R}$ and f, g square integrable.
   (c) Show that for any $\epsilon > 0$ there exists a linear operator K_ϵ such that (a) $\int_0^1 |Kf(x) - K_\epsilon f(x)|^2 dx < \epsilon \int_0^1 |f(x)|^2 dx$ for all square integrable f, and (b) K_ϵ has a finite-dimensional range, i.e. show that the subspace spanned by $\{K_\epsilon f\}$ with f ranging over all square integrable functions on $[0,1]$ is a finite-dimensional vector space.