# Qualifying Exam in Basic Analysis, August 2012
Duke University, Mathematics Department
Time Allowed: 3 hours

All answers and statements should be proved; no partial credit is given to answers without justification.

## Part I: 6 points each, do all 6 questions

1. State and prove the Cauchy-Schwartz inequality.

2. State some reasonable conditions under which a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$
everywere on $\mathbb{R}^2$ and prove this equality under the condition you give.

3. For $a_1, \ldots, a_n > 0$, let the arithmetic mean be
$$\frac{1}{n}\sum_{i=1}^{n} a_i$$
and the geometric mean be
$$\sqrt[n]{\prod_{i=1}^{n} a_i}\,.$$
Is one always larger than the other? State a theorem and prove it, or provide suitable counterexamples.

4. State and prove the contraction mapping principle (also known as the Banach fixed point theorem).

5. Let $f : [0,1] \to \mathbb{R}$. Define what it means for $f$ to be continuous at a point $x_0 \in (0,1)$. Then give an example of a function that is discontinuous everywhere except at $x_0 = \frac{1}{2}$.

6. Let $f$ be the solution of
$$f'(x) = -(x^2 e^{-x^2} + 1) f(x)^3 \quad , \quad f(0) = 2\,.$$
How many solutions does the equation $f(x) = \sin(x)$ have on $\mathbb{R}$?

## Part II: 10 points each. Do all 5 questions.

1. Is
$$f(x) = \frac{1 + x^2}{x\sqrt[4]{1 - \cos\frac{1}{x}}}$$
in $L^1(0,1)$?

2. Define, for $x \in [0,1]$
$$f(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n^3}\cos(2\pi n x)\,.$$
Show that $f$ is well-defined, i.e. the right-hand side converges pointwise for every $x \in [0,1]$. Is $f$ continuous on $[0,1]$? Is $f$ differentiable on $[0,1]$? If so, what is the derivative of $f$?

3. Let $f_n$ be a sequence of continuous functions $[0,1] \to \mathbb{R}$ such that each $f_n$ satisfies $f_n(0) = 1$ and is 1-Lipschitz,, i.e. $|f_n(x) - f_n(y)| \le |x - y|$ for all $x, y \in [0,1]$. Show that $\{f_n\}$ has a subsequence that converges uniformly on $[0,1]$. [*Hint*: Find a subsequence that converges point wise in suitably many points (e.g. a countable dense subset of $[0,1]$), and then show that it (or a subsequence thereof) converges uniformly on all of $[0,1]$ ].

4. Let, for $a > 0$,

$$f(x,y) = \begin{cases} \frac{x^5+y^4}{(x^2+y^2)^a+x^2y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Determine for which values of $a$ this function has each of the following properties:

- $f$ is continuous at $(0,0)$;
- $f$ is differentiable at $(0,0)$.

5. Let for $n \geq 1$

$$g_n(x) = \frac{x^3}{n+x^4} \tanh \frac{1}{nx^2+2},$$

where of course $\tanh(x) = \frac{e^x-e^{-x}}{e^x+e^{-x}}$. Study the point-wise and uniform convergence of the sequence $\{g_n\}$, i.e. determine the subsets of $\mathbb{R}$ where the sequence $\{g_n\}$ converges point-wise, and the subsets of $\mathbb{R}$ where the convergence is uniform.