Qualifying Exam in Basic Analysis, August 2013
Duke University, Mathematics Department
Time Allowed: 3 hours

All answers and statements should be proved; no partial credit is given to answers without justification.

Part I: 6 points each, do all 6 questions

1. State and prove the ratio test for convergence of series. You may assume the comparison test.
2. Let f be a continuous function from $[0,1]$ to $\mathbb{R}$. Prove that f is uniformly continuous.
3. Let f be a bounded monotone function on $[0,1]$. Prove that f is Riemann integrable on $[0,1]$.
4. Let M be a set equipped with 2 metrics ρ_1 and ρ_2. Prove $\sigma := \max\{\rho_1, \rho_2\}$ defines a metric on M. Show that a sequence of points in (M, ρ_1) cannot converge to two distinct points in M.
5. If $\lim_{n\to\infty} a_n = L$, prove $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} a_j = L$.
6. Let $f : [0,1] \to [0,1]$ satisfy $|f'| \leq \frac{\sqrt{2}}{2}$. Prove there exists $a \in [0,1]$ with $f(a) = a$.

Part II: 10 points each. Do all 6 questions.

1. Let $f(x) := \int_1^{10} \frac{e^{tx}}{t+1} dt$. Prove that f is differentiable with derivative $f'(x) = \int_1^{10} \frac{te^{tx}}{t+1} dt$.
2. Let $f(x) := \sum_{k=0}^{\infty} \frac{cos(kx)}{k^2+1}$. Prove that f is continuous. Compute (justify!) $\int_0^{\pi} f(s)ds$.
3. Consider the system of equations

$$\begin{aligned} xy^2 + xzu + Ayv^2 &= \quad 2 + A \\ u^3yz + 2xv - u^2v^2 &= \qquad 2. \end{aligned}$$

   When $A = 1$ show that this equation defines $u(x,y,z), v(x,y,z)$ near $(x,y,z) = (1,1,1)$ and $(u,v) = (1,1)$. Compute $\frac{\partial v}{\partial y}(1,1,1)$. For which values of A does this equation **not** determine u and v as C^1 functions of (x,y,z) near $(1,1,1)$?

4. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of differentiable functions satisfying

$$f_n' = \frac{1}{1 + a_n f_n^2}, \text{ and } f_n(0) = 1.$$

   Show that there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ converging uniformly to a continuous function on $[0,10]$.

5. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Suppose that for all $\epsilon > 0$, there is some $n > \frac{1}{\epsilon}$ such that $|x_n| < \epsilon$. Prove the sequence converges to 0.
6. Let $f : \mathbb{R} \to \mathbb{R}$ be C^2. Suppose that $f'(0) = 0$ and $f''(0) < 0$. Prove that there is $\delta > 0$ such that $f(x) < f(0)$ for all $x \in [-\delta, \delta] \setminus \{0\}$.