Qualifying Exam in Basic Analysis, Fall 2014
Duke University, Mathematics Department
Time Allowed: 3 hours

All answers and statements should be proved; no partial credit is given to answers without justification.

Do all 10 questions

1. a) Prove a uniformly continuous function on $(0,1)$ maps Cauchy sequences to Cauchy sequences. Is this true for all continuous functions? If so prove, if not give a counter example.
   b) Prove a continous function on $(0,1)$ is uniformly continous if and only if it is the restriction to $(0,1)$ of a continuous function on $[0,1]$.

2. Let y be a nonconstant real valued continuous function on the closed unit disk $\overline{D}(0,1)$ in $\mathbb{R}^2$. Prove that the image of y is a closed interval.

3. Let $\{a_n\}_{n=1}^{\infty}$ be a monotonically decreasing sequence of positive numbers. Prove $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

4. Show that there is a disk D around the origin such that the graph of $f(x,y) = x - y + e^{-4x^2+xy-y^2} - \sin(y^3)$ is below the plane $-x+y+z=1$.

5. Show $\operatorname{div} \nabla \ln(\sqrt{x^2+y^2}) = 0$ on $\mathbb{R}^2 \setminus \{(0,0)\}$, and compute
$$\int_{\mathbb{R}^2} \ln(\sqrt{x^2+y^2}) \operatorname{div} \nabla e^{-3x^2-4y^2} dxdy.$$

6. Let f be bounded and Riemann integrable on $[-10,10]$. Show that $F(t) := \int_0^1 f(x+t)dx$ is continuous at $t=0$.

7. Let M_n denote the set of $n \times n$ real matrices. Let $\|A\| := \sup_{\hat{v}\neq 0} \frac{|A\hat{v}|}{|\hat{v}|} = \sup_{|\hat{v}|=1} |A\hat{v}|$, where $|v|$ is the usual norm on $\mathbb{R}^n$. Let $d(A,B) = \|A-B\|$. Prove (M_n, d) is a metric space. Prove if $A, B \in M$ then $\|AB\| \leq \|A\|\|B\|$.

8. Let $A : [0,1] \to M_n$ be a continuous function, with M_n the metric space of Problem 7. Suppose that $A(\frac{1}{2}) = I_n - B$ where $\|B\| < \frac{1}{2}$ and I_n denotes the $n \times n$ identity matrix. Show that
$$A(\tfrac{1}{2})^{-1} = \sum_{k=0}^{\infty} B^k,$$
and $A(t)$ is invertible for t in some neighborhood of $\frac{1}{2}$, and $A(t)^{-1}$ is continuous in a neighborhood of $\frac{1}{2}$.

9. Let $f : \bar{B}(0,2) \to \bar{B}(0,2)$ be a C^1 function from the closed ball of radius 2 into itself. Suppose $|\nabla f| \leq \frac{1}{10}$. Prove there exists $y \in \bar{B}(0,2)$ such that $f(y) = y$.

10. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x,y) = (e^x + y^2 - x^4, \sin(xy), y - y^4 + x^4)$. Prove there exists a neighborhood of $(0,0)$ restricted to which this function is injective.