Qualifying Exam in Basic Analysis January 2010
Duke University, Mathematics Department
Time Allowed: 3 hours

Part I: 6 points each, do all questions

1. Let $(\mathcal{M}, \rho)$ be a metric space and suppose that $x_n \to x$ and $y_n \to y$ in $\mathcal{M}$. Prove that $\rho(x_n, y_n) \to \rho(x, y)$.

2. State L'Hôpital's rule, and use it to compute the limit $\lim_{n\to\infty}(1 - \frac{2}{n})^n$.

3. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $x_n \to \bar{x} \in \mathbb{R}$ as $n \to \infty$. Suppose that $\{f_n(x)\}_{n=1}^{\infty}$ is a sequence of continuous functions such that $f_n(x) \to f(x)$ uniformly for as $n \to \infty$. Prove that $\lim_{n\to\infty} f_n(x_n) = f(\bar{x})$.

4. Let f be a continuous function on $[0, 1]$. Prove that f achieves its maximum.

5. Determine whether the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{2 + \sqrt{n}}{2n^2 - n + 1}$$

6. Compute the power series expansion $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for the function

$$f(x) = \int_0^x \frac{1}{1 + 2s^4}\, ds$$

and determine the radius of convergence of the series.

7. Estimate $\cos(0.1)$ with error less than 5×10^{-5}.

8. If $f(x, y) = -3x^2y^2 + 7xy + 5e^{xy}$, write the integral for the surface area of the graph of f over the region $A = [0, 1]^2 \subset \mathbb{R}^2$ (but don't try to evaluate it).

9. Suppose that a particle's path in $\mathbb{R}^3$ is described by the parametric curve $\gamma(t) : [0, 2] \to \mathbb{R}^3$. For

$$\gamma(t) = (1 + t^2, 2t - \sqrt{3t + 1}, 2 - t), \quad t \in [0, 2],$$

find an equation for the plane in $\mathbb{R}^3$ that includes that point $\gamma(1)$ and is orthogonal to the direction of motion at time $t = 1$.

10. Evaluate the integral

$$\int_A \sqrt{x^2 + y^2}\, dx\, dy$$

where A is the region $\{x > 0\} \cap \{y > 0\} \cap \{x^2 + y^2 < 1\}$.

Part II: 10 points each. Choose 4 out of 5 questions. Only 4 questions will be counted in your score.

1. Suppose that M is a compact metric space and $f(x) : M \to \mathbb{R}$ is continuous. Prove that $f(x)$ must be uniformly continuous.

2. Prove that for any initial $x_0 \in \mathbb{R}$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = \sin(\frac{x_n}{2}) + 4, \quad n = 0, 1, 2, \ldots$$

   converges to a solution of the equation $x = \sin(\frac{x}{2}) + 4$.

3. Suppose $f(x, y) = 2x^2 - 3y^2 + xy + 1 - e^x$.

   (a) Find an affine function of (x, y) that best approximates $f(x, y)$ in a neighborhood of the point $(x_0, y_0) = (1, 2)$.

   (b) Is it true that for any y sufficiently close to $y_0 = 2$, the equation

$$f(x, y) = f(1, 2)$$

   must have a solution x? Explain.

4. Let $V \subset \mathbb{R}^3$ be the solid region defined by $\{0 \le x \le 1, \ \ 0 \le y \le 3, \ \ 0 \le z \le 2\}$. Evaluate the surface integral

$$\int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, dS$$

   where $\mathbf{F} = 2x\mathbf{i} + x^2 z\mathbf{j} - 3y^2\mathbf{k}$, ∂V is the boundary of V, and $\mathbf{n}$ is unit outward normal.

5. Suppose that $\phi(x)$ is continuously differentiable over $[0, 1]$ and that f is continuous on $[0, 1]$. Prove that

$$\lim_{N \to \infty} \sum_{i=1}^{N} f(x_i^*) \left(\phi(x_{i+1}) - \phi(x_i)\right) = \int_0^1 f(x)\phi'(x) \, dx$$

   where $0 = x_0 < x_1 < \cdots < x_N = 1$, and $x_{i+1} - x_i = 1/N$. The point x_i^* may be any point in the interval $[x_i, x_{i+1}]$. The points x_i and x_i^* depend on N.