Qualifying Exam in Basic Analysis, January 2011
Duke University, Mathematics Department
Time Allowed: 3 hours

Part I: 6 points each, answer all 9 questions

1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function and that $g(\boldsymbol{x}) := e^{f(\boldsymbol{x})}$.
   (a) Prove that a local max or min of f is also a local max or min of g.
   (b) Suppose that $\boldsymbol{x}_0$ is a critical point of f. Find a relation between the Hessian of f and the Hessian of g at this critical point.
2. Suppose $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{g} : \mathbb{R}^m \to \mathbb{R}^p$ are continuously differentiable functions. Let $\mathbf{h}(x) = \boldsymbol{g}(\boldsymbol{f}(x))$. State the general chain rule for the gradient matrix $D\mathbf{h}(x)$.
3. Let $A \subset \mathbb{R}^2$ be the bounded region defined as the set of points $(x, y) \in \mathbb{R}^2$ satisfying
$$0 \le x \le 2 \quad \text{and} \quad 0 \le y \le x^2.$$
   Let $V \subset \mathbb{R}^3$ be the solid region
$$V = \left\{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in A, \; 0 \le z \le xy \right\}.$$
   Express the volume of V as an integral and evaluate the integral.
4. Consider the function $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by
$$\boldsymbol{f}(\boldsymbol{x}) = \frac{1}{10} \begin{pmatrix} 2 + x_1 + x_2 + x_1 x_2^2 \\ 3 + x_1 - x_2 - x_1^2 x_2 \end{pmatrix}.$$
   Show that there is a unique $\boldsymbol{x} \in [-1, 1] \times [-1, 1]$ satisfying $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$.
5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Let $B_r = \{\boldsymbol{x} : ||\boldsymbol{x} - \boldsymbol{x}_0|| \le r\}$ be the ball of radius r centered at $\boldsymbol{x}_0$. Show that
$$\lim_{r \to 0} \frac{\int_{B_r} f(\boldsymbol{x})\, d\boldsymbol{x}}{\text{vol}\,(B_r)} = f(\boldsymbol{x}_0).$$
6. Let $f(x) = x^6 - 3\sin(\pi x/2) + 2 + \epsilon x$. Prove that if $|\epsilon|$ is small, there is a point $x_\epsilon \in \mathbb{R}$ near $x_0 = 1$ such that $f(x_\epsilon) = 0$.
7. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a function, and suppose that
$$|f(x) - f(y)| \le (x - y)^2$$
   for all $x, y \in \mathbb{R}$. Prove that f is constant.
8. Let $\boldsymbol{x}_0 = (x_0, y_0)$ and $\boldsymbol{x}_1 = (x_1, y_1)$ be two given points on the plane $\mathbb{R}^2$. Compute the integral
$$\int_C x dy - y dx$$
   where C is the segment (straight line) connecting $\boldsymbol{x}_0$ with $\boldsymbol{x}_1$.

9. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers and that series

$$f(x) = \sum_{n=1}^{\infty} a_n x^n$$

converges for some $x > 0$. Prove that the series converges absolutely for any $y \in \mathbb{R}$ such that $|y| < x$.

Part II: 10 points each, choose 4 out of 5 questions. Only 4 questions will be counted in your score.

10. Find all values of $x \in \mathbb{R}$ for which the following series converges:

$$f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{(x+5)^n}{n\, 3^n\, 2}$$

11. Estimate the following integral:

$$\int_0^{0.1} \frac{s}{(1-s)^3}\, ds.$$

12. Let a, b, c be nonzero real numbers. Find the point on the plane $ax + by + cz = 1$ that is closest to the origin.

13. Consider the set $X = C([0,1];\mathbb{R})$ of continuous real-valued functions on $[0,1]$, and let

$$\rho(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$$

(i) Prove that (X, ρ) is a metric space.

(ii) Prove that the metric space (X, ρ) is complete.

(iii) Consider the set

$$D = \{f \in X \mid |f(x)| \leq 1 \ \ \forall x \in [0,1]\}.$$

Is this set compact in X (under the metric topology induced by ρ)? Explain.

14. Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $f'(0) \neq 0$ and satisfying

$$\left|1 - \frac{f'(x)}{f'(0)}\right| \leq \lambda < 1, \quad \forall\, x \in [-M, M]$$

and $|f(0)| < M(1-\lambda)|f'(0)|$. Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(0)}, \quad n = 0, 1, 2, \ldots$$

with $x_0 = 0$, converges to a point $x_* \in [-M, M]$ that solves $f(x_*) = 0$. Can there be another solution in the interval $[-M, M]$?