Qualifying Exam in Basic Analysis, January 2012
Duke University, Mathematics Department
Time Allowed: 3 hours

All answers and statements should be proved; no partial credit is given to answers without justification.

Part I: 6 points each, do all 6 questions

1. Find $\int_0^\infty e^{-x^2}\,dx$

2. Let $K \subset \mathbb{R}$ be a compact set and $f(x)$ be a continuous functions on K. Prove there exists $x_0 \in K$ such that $f(x) \le f(x_0)$ for all $x \in K$.

3. By integrating the series
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \cdots$$
prove that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots$. Justify carefully all the steps (especially taking the limit as $x \to 1$ from below).

4. John and Jane are good friends, and avid travelers. John lives in Australia, and Jane in the U.S., and they agree to exchange homes during the spring. The leave their homes at exactly the same (universal, not local) time, and they arrive at each other's home at exactly the same (universal) time. The total time of the journey for both of them is exactly 5 days. John travels to Jane's house through China and Europe, Jane travels to John's house via South America and India. During their travels they never cross the arctic nor the antarctic circles. Does there exist a (universal) time, during their travel, at which John and Jane were at exactly the same latitude? If yes, state and prove a theorem you are using when proving your answer; if no, provide a counterexample. Was all the information above about John's and Jane's travels necessary to draw the same conclusion? Explain.

5. State some reasonable conditions under which a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$
everywere on $\mathbb{R}^2$ and prove this equality under the condition you give.

6. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that all partial derivatives of order 3 exist and are continuous. Write down (explicitly in terms of partial derivatives of f) a quadratic polynomial $P(x, y)$ in x and y such that
$$|f(x, y) - P(x, y)| \le C(x^2 + y^2)^{3/2}$$
for all (x, y) in some small neighborhood of $(0, 0)$, where C is a number that may depend on f but not on x and y. Then prove the above estimate.

Part II: 10 points each. Choose 5 out of 7 questions. Only 5 answers will be counted in your score.

1. Suppose f is a positive bounded measurable function on $[0, 1]$ and $F(x) = \int_0^x f(t)dt$ for $0 \le x \le 1$.
   (a) Show that F is continuous.
   (b) Show that F is differentiable at almost every point in $[0, 1]$
   (c) Show that $F'(x) = f(x)$.

2. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuously differentiable functions with $F(0, 0) = (0, 0)$ and the Jacobian of F at $(0, 0)$ equals the identity matrix. Show that there is a $\delta > 0$ such that for any (a, b) satisfying $a^2 + b^2 < \delta$, then there is a point $(x, y) \in \mathbb{R}^2$ such that $F(x, y) = (a, b)$.

3. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^2$ is continuously differentiable. Suppose for some $\vec{v}_0 \in \mathbb{R}^3$ and $\vec{x}_0 \in \mathbb{R}^2$ that $F(\vec{v}_0) = \vec{x}_0$ and $F'(\vec{v}_0) : \mathbb{R}^3 \to \mathbb{R}^2$ is onto. Show that there is a continuously differentiable function γ, $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^3$ for some $\epsilon > 0$, such that

(a) $\gamma'(0) \neq \vec{0} \in \mathbb{R}^3$, and

(b) $F(\gamma(t)) = \vec{x}_0$ for all $t \in (-\epsilon, \epsilon)$.

4. Consider a function $f : [a, b] \to \mathbb{R}$ which is twice continuously differentiable (including the endpoints). Let $x_i = a + ih, i = 0, 1, \cdots, n, h = (b-a)/n$. Show that there exists M such that for all $n > 1$,

$$\left| \frac{1}{n} \left(\frac{1}{2} f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2} f(x_n) \right) - \int_a^b f(x)\, dx \right| \leq \frac{M}{n^2}$$

5. Define, for $x \in [0, 1]$

$$f(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n^3} \sin(2\pi n x)\,.$$

Show that f is well-defined, i.e. the right-hand side converges pointwise for every $x \in [0, 1]$. If f continuous on $[0, 1]$? Is f differentiable on $[0, 1]$? If so, what is the derivative of f?

6. Consider the weight function $w(x) = \sqrt{1 - x^2}$ on $[0, 1]$. Define the function $\langle \cdot, \cdot, \rangle_w$ on $\mathcal{C}([0, 1])$ defined by

$$\langle f, g \rangle_w := \int_0^1 f(x) g(x) w(x) dx \qquad f, g \in \mathcal{C}([0, 1])\,.$$

(i) Show that $\langle \cdot, \cdot, \rangle_w$ is an inner product on $\mathcal{C}([0, 1])$.

(ii) Can you find a large class of functions w such that the similarly defined $\langle \cdot, \cdot \rangle_w$ is an inner product?

(iii) The norm associated to this inner product is $||f||_w := \sqrt{\langle f, f \rangle} = \sqrt{\int_0^1 |f(x)|^2 w(x) dx}$, for $f \in \mathcal{C}([0, 1])$; this of course also induces a distance

$$d(f, g) := ||f - g||_w = \sqrt{\int_{0,1} |f(x) - g(x)|^2 w(x) dx} \qquad f, g \in \mathcal{C}([0, 1])\,.$$

Define what it means for a sequence $\{f_n\} \subseteq \mathcal{C}([0, 1])$ to converge with respect to this distance, and what it means for such a sequence to be Cauchy.

7. With the same notation as in the previous question:

(i) Define what it would mean that the space $\mathcal{C}([0, 1])$ complete with respect to this distance. Then prove that it is not complete.

(ii) Show that if $\{f_n\} \subseteq \mathcal{C}([0, 1])$ converges uniformly, then it also converges with respect to $|| \cdot ||_w$. Does the converse hold?