Qualifying Exam in Basic Analysis, January 2013
Duke University, Mathematics Department
Time Allowed: 3 hours

All answers and statements should be proved; no partial credit is given to answers without justification.

1. Given an example of a bounded closed metric space that is not compact. [10pts]

2. Suppose (X, d) is a metric space. Let T be a contraction on X, i.e. $d(Tx, Ty) \leq \alpha \cdot d(x, y)$, for some $\alpha < 1$. Consider the statement:

   for any $x_0 \in X$, the sequence $\{T^n x_0\}_{n=0}^{+\infty}$ converges to a point $x^* \in X$

   (i) Show that this statement is false in general.

   (ii) Define the notion of completeness of a metric space, and prove that if (X, d) is complete, the statement above is true. [20pts]

3. State and prove the fundamental Theorem of calculus, and how it can be applied to compute $\int_0^1 f'(x)dx$, when f' is continuous $[0, 1] \to \mathbb{R}$. Then state and prove a (partial) generalization of this result to calculate $\int_\Omega \nabla f(\mathbf{x})d\mathbf{x}$ for a function $f : \Omega \to \mathbb{R}$, where Ω is a connected domain in $\mathbb{R}^2$ with smooth boundary. [20pts]

4. Is $\frac{(e^{-|x|}-1)\cos x}{x}$ Riemann integrable on $\mathbb{R}$? Prove or disprove. [10pts]

5. Consider the sequence of functions

$$f_n(x) = (x^{2n} + 2^{nx})^{\frac{1}{n}} \qquad n \geq 1\,.$$

   Determine the set E of pointwise convergence to its limit function on E. Prove pointwise convergence on E. Then determine if f_n converges uniformly on E. [20pts]

6. Consider the function

$$f(x, y) = x^2 y^2 e^{-2x^2-3y^2}\,.$$

   (i) Determine if the following limits exist, and calculate those which do:

$$\lim_{(x,y)\to\infty} f(x, y) \quad , \quad \lim_{(x,y)\to\infty} f(x, y)e^{2x^2}\,.$$

   (ii) Determine the stationary points of f in $\mathbb{R}^2$ and determine their nature.

[20pts]