Qualifying Exam in Basic Analysis June 2009

Part I (60 points)

(a) State two equivalent definitions for what it means for a set in $\mathbb{R}^n$ to be compact.

(b) State what it means for a sequence of functions, f_n, to converge uniformly to a limiting function, f, on a set E. Give an example of a sequence of functions that converges pointwise but not uniformly.

(c) Suppose that $\{a_n\}$ is a sequence of real numbers and that $a_n \to a$. Suppose that $a_n > 4$ for all n. Prove that $a \geq 4$.

(d) Give an example of a series of real numbers that converges (say how you know it), but does not converge absolutely.

(e) Suppose that f is continuously differentiable on $[0,1]$, $f(0) = 0$, and $f'(x) \leq 1$ for all $x \epsilon [0,1]$. Prove that $f(x) \leq x$ for all $x \epsilon [0,1]$.

(f) Consider the following function, f, on the interval $[0,2]$. $f(x) = 2$ if $0 \leq x < 1$, $f(1) = 9$, $f(x) = \frac{1}{2}$ if $1 < x \leq 2$. Prove that f is Riemann integrable. Compute $\int_0^2 f(x)\,dx$.

(g) Let X be a metric space with metric ρ. Let $\{x_n\}$ be a sequence of elements of X that converge to a limit $x \epsilon X$. Suppose $y \epsilon X$. Prove that the sequence of real numbers, $\{\rho(x_n, y)\}$ converges to $\rho(x,y)$.

(h) Use a linear approximation to the function $f(x) = \sqrt{x}$ near $x = 9$ to approximate $\sqrt{9.2}$. Estimate the error in your approximation.

(i) Let f be a continuously differentiable on $\mathbb{R}^2$ and suppose that $\gamma(t) =< x(t), y(t) >$ for $0 \leq t \leq 1$ is a level set. Prove that $\gamma'(t)$ is perpendicular to $\nabla f(\gamma(t))$ for all $0 \leq t \leq 1$.

(j) State the implicit function theorem for functions of two variables. Let $f(x,y) = \sin(x+y)+x^2$. Prove that if $|y|$ and $|\epsilon|$ are sufficiently small, there is an x such that $f(x,y) = \epsilon$.

Part II (40 points - choose 4 out of 5)

1. (i) Define the term **radius of convergence** as applied to power series.
   (ii) Find the radius of convergence of this power series:

$$f(x) = \sum_{n=1}^{\infty} 2^n n^2 x^n. \tag{0.1}$$

2. (i) For $f(x) = \cos(2\pi x)$, find a polynomial $g_k(x)$ of order k such that

$$g_k(0) = f(0), \quad g_k'(0) = f'(0), \quad g_k''(0) = f''(0), \quad \ldots \quad g_k^{(k)}(0) = f^{(k)}(0) \tag{0.2}$$

   (ii) Prove that for any x fixed, $\{g_k(x)\}_{k=1}^{\infty}$ is a Cauchy sequence.
   (iii) Prove that

$$\lim_{k\to\infty} \int_{-1}^{1} g_k(x)\,dx = \int_{-1}^{1} f(x)\,dx \tag{0.3}$$

3. If $f(x,y) = e^{-x}\sin(y)$, show that

$$\int_C \nabla f \cdot \mathbf{n}\, ds = 0 \tag{0.4}$$

for any smooth, simple closed curve C in $\mathbb{R}^2$.

4. Suppose that f is a continuous function on $[0,1]$ and that $\sup f = 1$. Prove that there is a point $p\epsilon[0,1]$ such that $f(p) = 1$.

5. Consider the nonlinear function $f(x,y) : \mathbb{R}^2 \to \mathbb{R}^2$, $f = (f_1(x,y), f_2(x,y))$, defined by

$$f_1(x,y) = -\frac{x^2}{2} + \frac{y^3}{3} + 2x + 2, \qquad f_2(x,y) = y^2 - 4x \tag{0.5}$$

(i) Find an affine function $g : \mathbb{R}^2 \to \mathbb{R}^2$ which best approximates f in a neighborhood of the point $(x,y) = (1,1)$.
(ii) Using the Inverse Function Theorem, explain why the map $f(x,y)$ is invertible (bijective) in a sufficiently small neighborhood of the point $(x,y) = (1,1)$.