Qualifying Exam in Basic Analysis, May 2012
Duke University, Mathematics Department
Time Allowed: 3 hours

All answers and statements should be proved; no partial credit is given to answers without justification.

Part I: 6 points each, do all 7 questions

1. State and prove the Cauchy-Schwartz inequality.

2. Give an example of a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ such that for any $M > 0$, $f_n \to 0$ uniformly on $[-M, M]$, and yet
$$\int_{\mathbb{R}} f_n(x)\, dx \not\to 0.$$
Characterize the intervals $I \subset \mathbb{R}$ such that, under the same hypotheses on $\{f_n\}$ as above, you can guarantee that $\int_I f_n(x)\, dx \to 0$.

3. Let $f : [0, 1] \to \mathbb{R}$. Define what it means for f to be continuous at a point $x_0 \in [0, 1]$. Then give an example of a function that is discontinuous everywhere except at $x_0 = \frac{1}{2}$.

4. Define what it means for a set K in $\mathbb{R}^2$ to be compact. Describe all the possible implications between compactness, boundedness and closedness, and combinations thereof (i.e. which one implies which? any equivalences? etc...), giving proofs and counterexamples as needed, with the exception that you may use Bolzano-Weierstrass' theorem without providing a proof.

5. Let $f \in \mathcal{C}^1([a, b])$ and $f'(x) < 0$ for all $x \in [a, b]$. Prove that f is strictly monotonically descreasing, i.e. $f(x) < f(y)$ for all $x > y$, $x, y \in [a, b]$

6. For $a_1, \ldots, a_n > 0$, let the arithmetic mean be
$$\frac{1}{n}\sum_{i=1}^{n} a_i$$
and the geometric mean be
$$\sqrt[n]{\prod_{i=1}^{n} a_i}\,.$$
Is one always larger than the other? State a theorem and prove it, or provide suitable counterexamples.

7. State Green's formula in the plane.

Part II: 10 points each. Do all 5 questions.

1. Is
$$f(x) = \frac{1}{x\sqrt[4]{1 - \cos\frac{1}{x}}}$$
in $L^1(0, 1)$?

2. Let
$$f(x, y) = \begin{cases} \frac{4y^2 - x^2}{x^2 + 4y^2} \arctan(y\sqrt{x^2 + y^2}) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}.$$
Determine whether
   - f is continuous at $(0, 0)$;
   - f is differentiable at $(0, 0)$;

- f has a minimum or maximum at $(0, 0)$, or neither.

3. Let $f : [0, 1] \to \mathbb{R}$ be monotonically increasing.
   (a) Which type of discontinuities can f have?
   (b) Show that f has at most countably many such discontinuities.

4. Let f_n be a sequence of continuous functions $[0, 1] \to \mathbb{R}$ such that each f_n satisfies $f_n(0) = 1$ and is 1-Lipschitz,, i.e. $|f_n(x) - f_n(y)| \leq |x - y|$ for all $x, y \in [0, 1]$. Show that $\{f_n\}$ has a subsequence that converges uniformly on $[0, 1]$. [*Hint*: Find a subsequence that converges point wise in suitably many points (e.g. a countable dense subset of $[0, 1]$), and then show that it (or a subsequence thereof) converges uniformly on all of $[0, 1]$].

5. Let for $n \geq 0$
$$g_n(x) = \frac{7x}{\sqrt{3x^2 + \frac{1}{(n+1)^2}}} .$$
   Study the point-wise and uniform convergence of the sequence $\{g_n\}$, i.e. determine the subset of $\mathbb{R}$ where the sequence $\{g_n\}$ converges point-wise, and the subset of $\mathbb{R}$ where the convergence is uniform.

   Letting
$$f_n(x) = \int_0^x g_n(t)dt + \frac{7}{3(n+1)} ,$$
   study the point-wise and uniform convergence of $\{f_n\}$.