Qualifying Exam in Basic Analysis, Spring 2014
Duke University, Mathematics Department
Time Allowed: 3 hours

All answers and statements should be proved; no partial credit is given to answers without justification.

Part I: 6 points each, do all 6 questions

1. State and prove the alternating series test for convergence of series. You may assume the comparison test.

2. Find a solution of the form $u(t) = \sum_{k=1}^{\infty} u_k \sin(kt)$, $\{u_k\}_{k=1}^{\infty} \subset \mathbb{R}$, to the equation
$$-u''(t) + u(t) = 1, \text{ and } u(0) = u(\pi) = 0.$$

3. Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers in $[0,1]$. Let $t_n(x) = 0$ for $x \in [0, q_n)$ and $t_n(x) = \frac{1}{n^2}$ for $x \in [q_n, 1]$. Set $h(x) := \sum_{n=1}^{\infty} t_n(x)$. Prove h is continuous at every *irrational* number.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^3. Prove
$$\lim_{r\to 0} \frac{1}{2\pi} \int_0^{2\pi} (f(a + r\cos(\theta), b + r\sin(\theta)) - f(a,b))d\theta = \frac{f_{xx}(a,b) + f_{yy}(a,b)}{4}.$$

5. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two bounded sequences of real numbers. Prove or give a counter example: $\limsup_{n\to\infty}(x_n + y_n) = \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$.

6. Let $\{f_n\}_{n=1}^{\infty} \subset C([0,1], \mathbb{R})$ be a monotone increasing sequence of continuous functions which converges pointwise to a continuous function g. Prove $f_n \to g$ uniformly.

Part II: 10 points each. Do all 6 questions.

1. Let $g : \mathbb{Z}^2 \to \mathbb{R}$. For which p does the inequality $|g(k)| \leq c(1 + |k|)^{-p}, \forall k$, imply $\sum_{k\in\mathbb{Z}^2} g(k) < \infty$? Justify.

2. Let $\phi : \mathbb{R} \to [-1, 1]$ be C^1. Let $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}$. Set
$$b(x) := \sum_{k=0}^{\infty} a_k \phi(kx).$$
Suppose that b is discontinuous at $x = 1$. Prove $\limsup_{k\to\infty} k^2|a_k| = \infty$.

3. For each $p \geq 1$, define complete normed vector spaces $(l_p, \|\|_p)$ consisting of sequences $s = \{s_k\}_{k=1}^{\infty}$ such that $\|s\|_p := (\sum_{k=1}^{\infty} |s_k|^p)^{1/p} < \infty$. One can show (but don't today) that if $v \in l_1$ and $a \in l_p$, then the new sequence $v * a$ defined by $(v * a)_k = \sum_{j=1}^{\infty} v_j a_{k-j}$ lies in l_p with $\|v * a\|_p \leq \|v\|_1 \|a\|_p$. Suppose that $\|v\|_1 \leq \frac{1}{2}$. Let $f \in l_p$. Show that there exists $a \in l_p$ such that
$$a + v * a = f.$$

4. Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be given by $f(x, y, z, w) = (2e^x + yz^2 - 3, y\cosh(x) - 4x + 2z - w)$. Then $f(0, 1, 2, 3) = (3, 2)$. Show that there exists a smooth function, $g(z, w)$, defined near $(2, 3)$ such that $g(2, 3) = (0, 1)$ and $f(g(z, w), z, w) = (3, 2)$. Compute $dg_{(2,3)}$.

5. Let X denote the space of continuous functions from $[0, 1] \to \mathbb{R}$. Let $\overline{B}_0(R) = \{f \in X : \max|f(x)| \leq R\}$. Is $\overline{B}_0(R)$ compact for $R < \infty$? Justify.

6. Prove that given $f \in C([0,1], \mathbb{R})$ and $\epsilon > 0$, $\exists\{a_j\}_{j=0}^{N}$ (N depending on ϵ, f) such that
$$|\sum_{j=0}^{N} a_j e^{jx} - f(x)| < \epsilon, \forall x \in [0, 1].$$