Duke University
Department of Mathematics

Qualifying Examination in Linear Algebra

Fall examination

Saturday, 15 August 2009

Instructions:

Choose **FIVE** of the six problems; only write solutions for these five.

Please write neatly.

Please be sure to communicate your reasoning clearly.

Good luck!

Notation:

$\mathbb{R}$ = the field of real numbers.

Scoring:

Each of the five problems will count 12 points.

Examination Problems

1. (a) Find the matrix for the linear transformation on $\mathbb{R}^3$ given by reflection in the plane $x + y = z$.

   (b) Find an orthonormal basis for the subspace of $\mathbb{R}^4$ orthogonal to $(0, 1, 1, -1)$.

2. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) Find bases for the row and column spaces of $\mathbf{A}$.

   (b) Let $T_{\mathbf{A}} : \mathbb{R}^5 \to \mathbb{R}^3$ be the linear transformation associated with $\mathbf{A}$.

      i. Determine the rank and nullity of $T_{\mathbf{A}}$.

      ii. Is $T_{\mathbf{A}}$ one-to-one? Justify your answer.

      iii. Let V be the kernel of $T_{\mathbf{A}}$ and W a two dimensional subspace of $\mathbb{R}^5$ that intersects V in a 1-dimensional subspace. Compute the dimension of $V + W$.

3. Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{pmatrix} -3 & -13 & -2 \\ 0 & 1 & 0 \\ 4 & 14 & 3 \end{pmatrix}.$$

   Find matrices $\mathbf{B}, \mathbf{J}$ such that $\mathbf{J}$ is in Jordan canonical form and $\mathbf{B}\mathbf{A}\mathbf{B}^{-1} = \mathbf{J}$.

4. Let $\mathbf{A}$ be a Hermitian matrix.

   (a) Show that the eigenvalues of $\mathbf{A}$ are real.

   (b) Show that the eigenvectors of $\mathbf{A}$ from different eigenspaces are orthogonal (with respect to the standard Hermitian inner product).

   (c) Show that $\mathbf{A} + i\mathbf{I}$ is invertible, where $i = \sqrt{-1}$ and $\mathbf{I}$ is the identity matrix.

5. Let V be a finite-dimensional vector space over some field, let V^* be the dual vector space to V, and let V^{**} be the dual vector space to V^*. Construct a map from V to V^{**} that is an isomorphism, and prove that it is an isomorphism.

6. Let $\mathbf{V}$ be a positive definite matrix (real and symmetric).

   (a) Give a one-sentence reason why $\mathbf{V}$ is an invertible matrix.

   (b) State the Spectral Theorem and use it to show that a positive definite matrix $\mathbf{V}$ has a *square root*, i.e., find a matrix $\mathbf{H}$ such that $\mathbf{V} = \mathbf{H}\mathbf{H}^T$.

   (c) Is your answer $\mathbf{H}$ from (b) necessarily an invertible, symmetric matrix? Justify your answer.