Instructions:

Choose *seven* of the eight problems; only write solutions for these seven.
Each problem is worth 10 points; only seven problems will be graded.
Please write neatly.
Please be sure to communicate your reasoning clearly.

1. Let A be a 4×4 matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -1$, $\lambda_4 = 3$.
   (a) Find the trace $\mathrm{tr}(A)$ and the determinant $\det(A)$ of A.
   (b) Must A be invertible? Why or why not?
   (c) Must A be diagonalizable? Why or why not?
   (d) Find the characteristic polynomial of A.

2. Consider the 3×5 real matrix

$$A = \begin{pmatrix} 2 & 2 & 1 & 0 & 2 \\ -2 & -1 & 0 & 0 & -4 \\ 2 & 4 & 3 & 0 & -2 \end{pmatrix}.$$

   (a) Find a basis for the row space of A.
   (b) Find explicit condition(s) on the components (b_1, b_2, b_3) of $\mathbf{b} \in \mathbb{R}^3$ that determine whether or not $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{R}^5$.
   (c) Find a basis for the column space of A.

3. (a) Define what it means for A to be a Hermitian matrix.
   (b) Give a 2×2 example with at least one entry that is not a real number.
   (c) Is your matrix diagonalizable? If so, diagonalize it, if not, say why.
   (d) If A is a $n \times n$ Hermitian matrix, prove that its eigenvalues are real.

4. Let W be the subspace of $\mathbb{R}^5$ spanned by

$$\mathbf{w}_1 = (2, 2, 1, 0, 2),$$
$$\mathbf{w}_2 = (-2, -1, 0, 0, -4),$$
$$\mathbf{w}_3 = (2, 4, 3, 0, -2).$$

   Find a basis for $W^\perp$, the orthogonal complement of W.

5. Consider the 3×3 real matrices

$$A = \begin{pmatrix} 2 & 0 & 5 \\ 0 & 2 & -4 \\ 0 & 0 & 1 \end{pmatrix} ,$$

and

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} .$$

(a) Find the characteristic polynomials of A and B.

(b) Find the minimal polynomials of A and B.

(c) Are A and B similar? Why or why not?

6. Let A be an upper triangular 3×3 matrix with all diagonal entries equal to 2.

(a) What are the possible Jordan canonical forms of A?

(b) Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Which of the possibilities in (a) is the Jordan form of A?

(c) Let A be the matrix from part (b) and let J be its Jordan canonical form. Find a matrix S such that $S^{-1}AS = J$.

(d) For the matrix A from part (b), calculate e^{tA}.

7. Let V be a vector space of finite dimension over a field.

(a) Define the dual V^* and the double dual V^{**} of V.

(b) Define an isomorphism of V with V^{**} that does not depend on a choice of basis.

(c) Show that V is also isomorphic to V^*.

(d) Give an example to show that the isomorphism from V to V^* depends on the basis chosen for V.

8. Let V be a real finite dimensional inner product space and let $Q\colon V \to V$ be a linear transformation. Assume that $Q^\top = Q$, that is, Q is self-adjoint.

(a) Prove that there is an orthonormal basis of V consisting of eigenvectors for Q. (You may assume the result that $Q^\top = Q$ implies that Q is diagonalizable.)

(b) If $(Qv, v) \geq 0$ for all $v \in V$, prove there exists $S\colon V \to V$ such that $Q = S^2$ and $S^\top = S$.