Instructions:

Choose *seven* of the eight problems; only write solutions for these seven.
Each problem is worth 10 points; only seven problems will be graded.
Please write neatly.
Please be sure to communicate your reasoning clearly.

1. Let A be a 4×4 real matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = -1$, $\lambda_4 = 3$.
   (a) Find the trace $\mathrm{tr}(A)$ and calculate $\det(e^A)$.
   (b) Is A invertible? Why or why not?
   (c) Is A diagonalizable? Why or why not?
   (d) Does A have a real square root? Does A have a complex square root? Justify.
   (e) Does the nullspace of A equal the orthogonal complement of the row space of A? Why or why not?

2. Consider the 3×5 real matrix

$$A = \begin{pmatrix} 2 & 2 & -1 & 1 & 1 \\ 4 & 4 & -1 & 2 & -1 \\ -4 & -4 & 0 & -2 & 4 \end{pmatrix}.$$

   (a) Find bases for the row space and the column space of A.
   (b) Find the general solution to $A\mathbf{x} = \mathbf{b}$ when $\mathbf{b} = (-1, 1, -4)^\top$.
   (c) Find explicit necessary and sufficient condition(s) on c and d to ensure that $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{R}^5$ when $\mathbf{b} = (-1, c, d)^\top$.

3. Let $\mathbf{u} \in \mathbb{R}^n$ be such that $\mathbf{u}^\top \mathbf{u} = 1$ and let A be the $n \times n$ matrix $I - 2\mathbf{u}\mathbf{u}^\top$, where I is the $n \times n$ identity matrix.
   (a) Is A symmetric? Why or why not?
   (b) Is A orthogonal? Why or why not?
   (c) Calculate $A\mathbf{u}$ and $A\mathbf{w}$, where $\mathbf{w} \in \mathbb{R}^n$ and $\mathbf{u}^\top \mathbf{w} = 0$.
   (d) What are the eigenvalues of A and what are the geometric multiplicities of each eigenvalue?

4. Let W be the subspace of $\mathbb{R}^5$ spanned by

$$\begin{aligned} \mathbf{w}_1 &= (2, 1, 1, 5, -1), \\ \mathbf{w}_2 &= (1, 0, 1, 3, -1), \\ \mathbf{w}_3 &= (-2, -5, 3, -1, -3), \\ \mathbf{w}_4 &= (-1, 1, -2, -4, 2). \end{aligned}$$

   Find bases of W and of $W^\perp$, the orthogonal complement of W.

5. Let A be an $n \times n$ upper triangular complex matrix and let $p(x)$ be its characteristic polynomial. Give a direct proof of the Cayley-Hamilton theorem in this case: $p(A) = 0$.

6. Let A be a 3×3 matrix which has 1 as its only complex eigenvalue.

   (a) List all possible Jordan canonical forms of A. (If two Jordan forms differ only by a permutation of Jordan blocks, only include one of them.)

   (b) If B is another 3×3 matrix which has 1 as its only complex eigenvalue, and if the eigenvalue 1 has geometric multiplicity 2 for both A and B, are A and B necessarily similar? Carefully explain why or why not. What if A and B were 4×4 matrices?

   (c) Find the Jordan canonical form J for $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and a matrix S such that $S^{-1}AS = J$.

7. Let V be a vector space of finite dimension over a field k. Let $\mathrm{End}(V)$ be the vector space of linear transformations $T\colon V \to V$ and let $\mathrm{Bi}(V)$ be the vector space of bilinear forms $B\colon V \times V \to k$.

   (a) Use a basis $v_1, \ldots, v_n$ of V to explicitly define an isomorphism of $\mathrm{End}(V)$ with $\mathrm{Bi}(V)$. Verify that your map is indeed an isomorphism.

   (b) Give an example to show that the isomorphism constructed in part (a) depends on the basis chosen for V.

8. Let V be a real finite dimensional inner product space and let $T\colon V \to V$ be a linear transformation. Assume that $(Tv, w) = (v, Tw)$ for all v, $w \in V$.

   (a) Prove that if λ and μ are distinct eigenvalues of T then the corresponding eigenspaces V_λ and V_μ are orthogonal.

   (b) If W is a subspace of V, prove that $T(W) \subseteq W$ implies that $T(W^\perp) \subseteq W^\perp$.

   (c) Prove that there exists an eigenvector $v_1 \in V$ for T in V with associated (real) eigenvalue λ_1. Do not use a big theorem; prove directly. You may assume the fundamental theorem of algebra however.

   (d) Prove that there exists an orthonormal basis of V consisting of eigenvectors for T.