Duke University
Department of Mathematics

# Qualifying Examination in Linear Algebra

August 2013

## Instructions:

Choose **SEVEN** of the eight problems; only write solutions for these seven.

Please write neatly.

Please be sure to communicate your reasoning clearly.

Good luck!

### Notation:

$\mathbb{R}$ = the field of real numbers.

$\mathcal{N}(M)$ refers to the nullspace of a matrix $M$.

$\mathcal{R}(M)$ refers to the range (columnspace) of a matrix $M$.

$M^T$ is the transpose of a matrix $M$.

$\langle \mathbf{x}, \mathbf{y} \rangle$ denotes an inner product of two vectors.

### Scoring:

Each of the seven problems will count 10 points.

**Examination Problems**

1.

   (a) Show that the $i$th row of an invertible matrix $A$ is orthogonal to the $j$th column of $A^{-1}$ if $i \neq j$.

   (b) Let $M_n(\mathbb{R})$ be the vector space of all real $n \times n$ matrices. Define the *commutator* of matrices $A$ and $B$ in $M_n(\mathbb{R})$ to be

$$[A, B] \equiv AB - BA.$$

      i. Define a transformation $T : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ by $T(X) = [A, X]$, where $A$ is a fixed nonzero matrix in $M_n(\mathbb{R})$. Is $T$ a linear transformation? Justify your answer.

      ii. If $n = 2,013$, what is the numerical value of the trace of $I_{2,013} - [A, B]$. Show your work.

2. Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \pi & e & 0 \\ 0 & 0 & 1 & 13 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) Find bases for the row and column spaces of $A$.

   (b) Determine the nullity of $A$. Is the linear transformation associated with $A$ one-to-one? Justify your answer.

   (c) Let $\mathcal{N}$ be the nullspace of $A$ and let $\mathcal{V}$ be a two dimensional subspace of $\mathbb{R}^5$ such that $\mathcal{V} + \mathcal{N}$ is three dimensional. Determine the dimension of $\mathcal{V} \cap \mathcal{N}$.

3. Let $B$ be an $n \times n$ matrix with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, ..., $\lambda_n = n$.

   (a) Is $B$ invertible? Diagonalizable? Justify your answers.

   (b) Determine the eigenvalues of $B^{-2013}$.

   (c) Let $p(x)$ be the characteristic polynomial of $B$. Given the function

$$f(x) = 2e^x + x^2 p(x),$$

   compute $\det[f(B)]$.

4. Let $Q$ be an orthogonal matrix.

   (a) Is $Q^{2013}$ orthogonal? Justify your answer.

   (b) If the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$ is $\pi/3$, determine the angle between $Q^{2013}\mathbf{x}$ and $Q^{2013}\mathbf{y}$.

   (c) Show that $Q + (1/2)I$ is invertible.

5. Consider the inner product space $\mathbb{R}^3$ with the standard inner product. Let $\mathbf{x} = (1, 2, 1)$, and let $S \subset \mathbb{R}^3$ denote the orthogonal complement of the span of $\mathbf{x}$ ($S$ is the set of all $\mathbf{y} \in \mathbb{R}^3$ such that $\langle \mathbf{y}, \mathbf{x} \rangle = 0$). Compute the matrix $M$ for the linear transformation of $\mathbb{R}^3$ which is the orthogonal projection of $\mathbb{R}^3$ onto $S$.

6. Suppose that $A$ is a real, symmetric $n \times n$ matrix. Suppose $\{e_1, \ldots, e_n\}$ is a given orthogonal basis for $\mathbb{R}^n$.

   (a) Either prove the following or give a counter example: If $\langle e_k, Ae_k \rangle > 0$ for each $k = 1, \ldots, n$, then $A$ is positive definite.

   (b) Suppose $A$ is the matrix
   $$A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}.$$
   Find a real $n \times n$ matrix $B$ such that $A = BB^T$. Hint: observe that $\mathbf{v_1} = (1, 1)$ and $\mathbf{v_2} = (1, -1)$ are eigenvectors of $A$.

7. Find a real $5 \times 5$ matrix $A$ having characteristic polynomial $p(x) = x^2(x + 7)^3$ such that the nullspace of the matrix $A + 7I$ is two-dimensional. Express your answer in Jordan canonical form. Is the answer unique? Explain.

8. Suppose $M$ is a real $n \times n$ matrix. Let $\mathbf{y} \in \mathbb{R}^n$ be a given vector.

   (a) Show that if $\mathbf{y} \in \mathcal{R}(M)$, then $\mathbf{y}$ is orthogonal to $\mathcal{N}(M^T)$.

   (b) Show that if a vector $\mathbf{z} \in \mathbb{R}^n$ is orthogonal to $\mathcal{R}(M)$, then $\mathbf{z} \in \mathcal{N}(M^T)$.