Duke University

Department of Mathematics

Qualifying Examination in Linear Algebra

August 2014

Instructions:

Choose **SEVEN** of the eight problems; only write solutions for these seven.

Please write neatly.

Please be sure to communicate your reasoning clearly.

Good luck!

Notation:

$\mathbb{R}$ = the field of real numbers.

M^T is the transpose of a matrix M.

$\langle \mathbf{x}, \mathbf{y} \rangle$ denotes an inner product of two vectors.

Scoring:

Each of the seven problems will count 10 points.

Examination Problems

1. Let A_{12} be a real 12×12 matrix having an eignvalue $\lambda = 0$ with a geometric multiplicity of 5.

   (a) Determine the rank of A.

   (b) Suppose that the minimal polynomial of A_{12} is $p_A(\lambda) = \lambda^3(\lambda+4)^2$. Is A_{12} diagonalizable? Justify your answer.

2. Let $f(x) = ((x-1)^2+1)(x+2)^2 + x$ and consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

   Determine the eigenvalues of $f(A)$.

3. Let A be a Hermitian matrix. Is A^2 positive semidefinite? For A skew-Hermitian, would $-A^2$ be positive semidefinite? Justify your answers.

4. Let A be a real $n \times n$ matrix with an eigenvalue λ having algebraic multiplicity n. Show that

$$e^{At} = e^{\lambda t}\left(I \;+\; (A-\lambda I)t \;+\; \cdots \;+\; \frac{(A-\lambda I)^{n-1}}{(n-1)!}\,t^{n-1}\right).$$

5. Describe the set of all eigenvectors for the matrix $A = \begin{pmatrix} -3 & 0 & -15 \\ 10 & 2 & 29 \\ 0 & 0 & 2 \end{pmatrix}$.

6. What sizes are the Jordan blocks of the matrix A in problem 5? Justify your response.

7. Let P and Q be real idempotent square matrices of size n such that $PQ = QP = 0$ and $P+Q$ is the identity. Show that $\mathbb{R}^n$ is the direct sum of the images of P and Q.

8. Let A be a matrix of size $n \times m$ and B a matrix of size $m \times k$. Assume that $AB = 0$. Prove that the ranks of A and B sum to at most m.