Duke University
Department of Mathematics

Qualifying Examination in Linear Algebra

January 2014

Instructions:

Choose **SEVEN** of the eight problems; only write solutions for these seven.

Please write neatly.

Please be sure to communicate your reasoning clearly.

Good luck!

Notation:

$\mathbb{R}$ = the field of real numbers.

M^T is the transpose of a matrix M.

$\langle \mathbf{x}, \mathbf{y} \rangle$ denotes an inner product of two vectors.

Scoring:

Each of the seven problems will count 10 points.

Examination Problems

1. Consider the inner product space $\mathbb{R}^3$ with the standard inner product. Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ denote the orthogonal projection of $\mathbb{R}^3$ onto the plane defined by

$$x + 2y - 3z = 0.$$

   (a) What is the kernel of P?

   (b) Compute the matrix M for this linear transformation of $\mathbb{R}^3$.

   (c) Find two vectors which form a basis for the range of P.

2. Let A denote the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

   (a) Find an orthogonal matrix Φ such that $\Phi^{-1}A\Phi$ is a diagonal matrix.

   (b) Compute the matrix $B = e^A$.

3. For each of the following statements, either prove it or give a counter example:

   (a) If A is a real, symmetric $n \times n$ matrix, then all eigenvalues of A are real.

   (b) If all eigenvalues of a real $n \times n$ matrix matrix B are real, then B must be symmetric.

4. Suppose A and B are two real 10×10 matrices. Suppose that the rank of A is 6 and that the rank of B is 4. Justify your answers to each of the following:

   (a) What is the minimum possible rank of the matrix A^2?

   (b) What is the maximum possible rank of the matrix $A(B^T)$?

   (c) If the columns of A are orthogonal to the columns of B, must the rank of the matrix $A + B$ be equal to $6 + 4 = 10$?

5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Find a basis for the eigenspace of A.

6. Show that the matrix in Problem 5 is invertible, but not diagonalizable.

7. Suppose A is a positive definite $n \times n$ real symmetric matrix. Let $\mathbf{e}$ be a unit vector in $\mathbb{R}^n$. Can $\mathbf{e}^T A^{-1} \mathbf{e}$ be non-positive? Justify your answer.

8. Let A denote the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find a Jordan form for A.