Gergen Lectures 2001

Random Matrix Theory

Lecture 2 This is the second of the Gergen 2001 lectur

My topic is Random Matrix Theory and particu

the use of methods arising from the modern th

of integrable systems in the solution of particular

problems of mathematical and physical interest, such as the universality problem in RMT and Ulam's problem in combinat

~~RMT~~ In the first lecture I gave a very

brief survey of the scope of the modern theory of integ.

Then I described something of the history and remar

broad range of applications of RMT. And then

finally I focused on a particular version of

the Universality Conjecture, which we plan to

prove in this lecture and the next. More precis and ideas from Integ. Systems that are needed in the solution of the Universality Conjecture

- in the next ~~[illegible]~~ lecture, I will show how to assemble all these ideas into a solution of the problem.
- in the 4th lecture, I will consider Ulam's problem.

So recall that a Unitary Ensemble (UE) is the ensemble of $N \times N$ Hermitean matrices $M = (M_{ij})$ with probability distribution

$$(1) \qquad P(M)\,dM = \frac{1}{Z_N} e^{-\operatorname{tr} V(M)}\, dM$$

$$= \frac{1}{Z_N} e^{-\operatorname{tr} V(M)} \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\operatorname{Re} M_{ij} \prod_{i<j} d\operatorname{Im} M_{ij}$$

where $V : \mathbb{R} \to \mathbb{R}$, $V(x) \to \infty$ sufficiently rapidly as $|x| \to \infty$

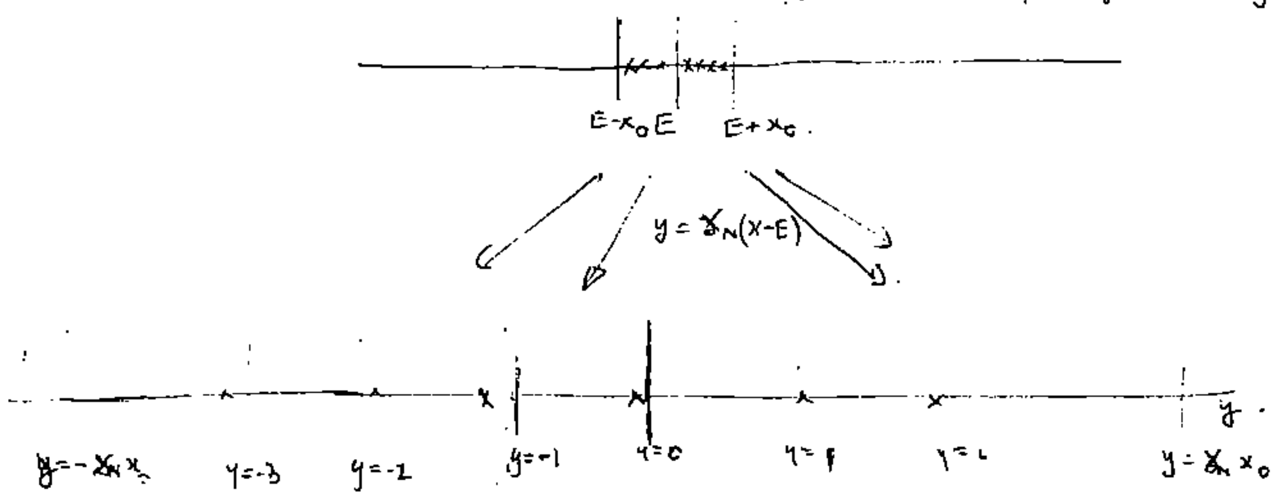
and Z_N is the normalization constant. Of course the probability distribution (1) turns the eigenvalues

$$\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_N(M)$$

Now fix an energy E and look at the eigenvalues in a window $[E-x, E+x]$ about E, and scale tho:

$$x \mapsto y = \gamma_N(x - E)$$

so that the expected # of eigenvalues/y-interval $= 1$,



Now compute

(2) $$\lim_{N\to\infty} P_N\left(\frac{y}{\gamma_N}, E\right) = \lim_{N\to\infty} \text{Prob}\left\{M : M \text{ has no eigenvalues in } \left(E - \frac{y}{\gamma_N}, E + \frac{y}{\gamma_N}\right)\right\}$$

For the Case of GUE,

$$P(M) = \frac{1}{Z_N} e^{-\operatorname{tr} M^2} dM$$

a calculation of Gaudin and Mehta using classical special function theory (more later) showed th

(3) $$\gamma_N \sim N^{\frac{1}{2}} \quad \left(= N^{1 - \frac{1}{2m}}\right)$$

(4) $$\lim_{N\to\infty} P_N\left(\frac{y}{\gamma_N}; E\right) = \det(1 - S_y')$$

where S_y' denotes the trace-class operator with kernel

(5) $$S_y'(\xi,\eta) = \frac{\sin \pi(\xi-\eta)}{\pi(\xi-\eta)}$$

acting on $L^2((-y,y))$. The universality conjecture is the claim that (4) is true with the same RHS, for all (suitable) V. The only thing that depends on V is the "setting" for the micro of lecture 1, viz γ_N.

This conjecture was first considered in the Physics literature by Brezin and Zee ('92), and in the mathematical literature, in '97, by Pastur and Scherbina &, in a special case ($V(x) = x^4 + tx^2$) by Its and Scherbina. We will follow the method of D. Kriecherbauer K-T-McLaughlin

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S. Venakides and X. Zhou (DKMVZ). For people who

are interested in the details see DKMVZ, I & II, in CP

and also Orthog. Poly's and Random Matrices: a R-H approach,

P.D. Courant Lecture Notes #3, 1999.

In DKMVZ the authors prove the univ. con

for 2 classes of potentials V

(a) $V(x) = t_{2m} x^{2m} + \cdots + t_0$, $t_{2m} > 0$.

(b) $V(x) = N\, Q(x)$ where

(i) $Q(x)$ is real anal. in a nbhood of $\mathbb{R}$

(ii) $\frac{Q(x)}{\log|x|} \to \infty$ as $|x| \to \infty$.

~~Ra~~ In these lectures, for simplicity, I will only consider the sp

case

(6) $\begin{cases} V(x) = x^{2m} & \text{for some pos. integer } m. \\ E = 0 \end{cases}$

This case contains most (but not all) of the

difficulties: more later.

The proof of (4) proceeds in <u>Steps</u>

<u>Step 1</u> (<u>Weyl integration formula</u>)

Every Hermitean matrix M has a spectral representation

$$M = U \Lambda U^*$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, $\lambda_1 \geq \cdots \geq \lambda_N$, are the eigenvalues of M and U is the matrix of orthonormal eigenvect

if the λ_i's are distinct (which is true on an open dense set of matrices of full measure), then U is uniquely determined as an element of $\underset{\substack{\uparrow \\ \text{unitary} \\ \text{gp}}}{\mathcal{U}(N)}/\mathbb{T}^N$

and we consider the change of variables

$$M = U \Lambda U^* \longmapsto (\Lambda, U) \in \underset{\substack{\uparrow \\ \text{ordered } N\text{-tuple} \\ \lambda_1 > \cdots > \lambda_N}}{(\mathbb{R}_N)^{+1}} \times \mathcal{U}(N)$$

The <u>critical</u> fact is that under this change of vari

the λ_i's and the U_{ij}'s become <u>statistically independe</u>

$$\frac{1}{Z_N} e^{-\operatorname{tr} V(M)} dM = \frac{1}{Z_N} e^{-\sum V(\lambda_i)} \prod_{i<j} (\lambda_i - \lambda_j)^2 d$$

$$\times \quad K(p)\, dp_1 \ldots dp_{N(N-1)}$$

variables p_i describe the U_{ij}'s

Thus if we are interested in computing the expectation of functions (say F), which are invariant under conjugation

$$F(M) = F(UMU^*)$$

so that such functions only depend on (symmetric function of) the eigen- of M, we have (the Weyl integration formula)

$$\operatorname{Exp}(F) = \frac{1}{Z_N} \int_{\lambda_1 \geq \cdots \geq \lambda_N} F(\lambda_1, \ldots, \lambda_N)\, e^{-\sum_i V(\lambda_i)} \prod_{i<j} (\lambda_i - \lambda_j)^2 d^N\lambda \times K(p)\, d^{N(N-1)}p$$

$$= \frac{1}{\mathfrak{Z}_N} \int_{\lambda_1 \geq \cdots \geq \lambda_N} F(\lambda_1, \ldots, \lambda_N) \prod_{i<j} (\lambda_i - \lambda_j)^2 d^N\lambda$$

where the dp_i's $\cong dU_{ij}$'s have been integrated out, $\mathfrak{Z}_N$ is the new norming constant. Thus we are lea- to the distribution on the eigenvalues

(For GOE, $\cdots \quad \prod_{i<j} |\lambda_i - \lambda_j|$

GSE, $\cdots \quad \prod_{i<j} |\lambda_i - \lambda_j|^4$

Physicists speak of $\beta = 1, 2, 4$: more later thermodynamics)

From (7), we see ~~from~~ that the probability that eigenvalues are close together is small. One speak of "eigenvalue repulsion". This is a fundamental proper of such ensembles which implies, in particular, tha the eigenvalue spacings are not Poisson

Step 2 (enter orthogonal polynomials : OP's)

Recall that if $d\mu(x)$ is a measure on $\mathbb{R}$ with finite moments

$$\int |x|^q \, d\mu(x) < \infty, \qquad q = 0, 1, 2, \ldots$$

then $d\mu$ generates via the Gramm-Schmidt procedure a unique set of orthonormal polynomials

(8) $$P_n(x) = c_n \pi_n(x) = c_n(x^n + \cdots) \qquad c_n > 0 \qquad n = 0, 1, 2 \ldots$$

(9) $$\int p_n(x)\, p_m(x)\, d\mu(x) = \delta_{n,m}, \quad n,m \geq 0.$$

Let $\{p_n(x) = p_n(x;V)\}_{n\geq 0}$ denote the orthonormal polynomials generated by $d\mu(x) = e^{-V(x)}dx$. The standard fact in the business is that the p_n's satisfy a 3 term recurrence relationship (discrete Schrödinger equation; Jacobi matrix eqtn.)

$$Lp = \begin{pmatrix} a_0 & b_0 & & & \\ b_0 & a_1 & b_1 & & 0 \\ & b_1 & a_2 & b_2 & \\ & 0 & b_2 & \ddots & \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \end{pmatrix} = x \begin{pmatrix} p_0 \\ p_1 \\ \vdots \end{pmatrix} = xp$$

Now it is a curious, helpful and somewhat mysterious fact that these polynomials have a very interesting "functorial interrelationship" with RMT, v

$$\begin{tikzcd}
 & \text{given } V \arrow[dl] \arrow[dr] & \\
e^{-\operatorname{tr} V(M)}dM \arrow[d] & & e^{-V(x)}dx \arrow[d] \\
\text{mean field theory} & & \{p_n\}_{n\geq 0} \arrow[d] \\
 & & {}
\end{tikzcd}$$

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The "random matrix version" of e^{-V} plays the role

of a sort of "$\frac{1}{2}$-quantization" of the "orthogonal

polynomial version" of e^{-V}. In a very concrete way,

mean field theory results imply properties of op's,

On the other hand, results from op's are crucial

in understanding the $N \to \infty$ limit in RMT.

For example, we have the following basic result of

Gaudin and Mehta. Let $\theta > 0$, and consider the

quantity

$$P_N(\theta) = \text{Prob}\{M : M \text{ has no eigenvalues in } (-\theta, \theta)\}$$

$$\left(= P_N(\theta; E=0) \right)$$

Then it turns out that $P_N(\theta)$ can be expressed

<u>explicitly</u> in terms of op's. Indeed:

where K_N is the finite rank (& hence trace class operator) acting on $L^2(-\infty,\infty)$ with kernel

$$K_N(x,y) = \sum_{j=0}^{N-1} \phi_j(x)\,\phi_j(y) \tag{11}$$

where $\phi_j(x) = e^{-\frac{1}{2}V(x)}\,p_j(x)$. Note that

$$\int \phi_j(x)\,\phi_\ell(x)\,dx = \delta_{j\ell}\,, \quad j,\ell \geq 0. \tag{12}$$

The essential technical step in deriving (10) is the so-called "integrating out" lemma of Gaudin.

Of course, what we are interested in is

$$\lim_{N\to\infty} P_N\left(\frac{y}{\delta_N}; E=0\right) \quad \boxed{\cancel{= \lim_{N\to\infty} \hat{D}_N\left(\frac{y}{\delta_N}\right)}}$$

and so we see that the question of universa[lity] reduces to a question of the asymptotics of orthog[onal] polynomials. More precisely one needs, (in the case $V(x) = x^{2m}$), the asymptot[ics] of

$$P_N\left(N^{\frac{1}{2m}}x\right), \quad P_{N-1}\left(N^{\frac{1}{2m}}x\right) \quad \text{as } N\to\infty. \tag{13}$$

(Plancherel-Rotach, 1929)

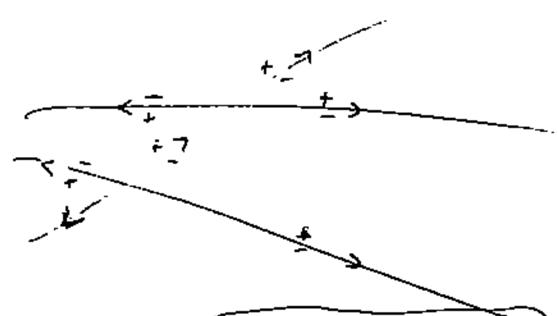
was known in the classical literature only for the Hermite polynomials corresponding to the weight e^{-x^2}. This is precisely the weight that arises in the Gaussian UE, $e^{-\operatorname{tr} M^2} dM$, and this is the reason that Gaudin and Mehta were able compute (4) for GUE. Apart from the special case considered by Its and Bleher $V(x) = x^4 + tx^2$ the papers DKMVZ were the first to prove Plancherel-Rotach asymptotics for general OP's yielding the univ. conj. It is here that various techniques from integral systems begin to play an explicit role.

Step 3 (Enter Riemann-Hilbert problems: RHP's)

The ("double-scaling") analysis in DKMVZ proceeds due expressing OP's in terms of the solution of a RHP, initially introduced by Fokas, Its and Kitaev, and then analyzing this RHP asymptotically using the non-commutative steepest descent method for RHP's introduced by Xin Zhou and P.D in '93. I will say much more about this method in the next lecture.

So what is a RHP?

Let Σ be an oriented contour in $\mathbb{C}$.



Σ

Suppose in addition that for some $k \in \mathbb{N}$ we have a map

$$v : \Sigma \to GL(k, \mathbb{C})$$

st $v, v^{-1} \in L^\infty(\Sigma)$. The RHP (Σ, v)

$m = m(z)$ st

- m is analytic in $\mathbb{C} \setminus \Sigma$

- $m_+(z) = m_-(z)\, v(z)$ a.e. $z \in \Sigma$ (jump condition!)

  where $m_\pm(z) = \lim_{\substack{z' \to z \\ z' \in \pm \text{ side of } \Sigma \text{ at } z}} m(z')$

If in addition, $l = k$ and

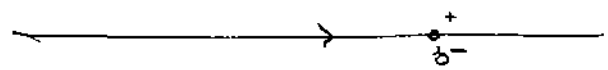
- $m(z) \to I$ as $z \to \infty$

we say that the solution of the RHP is normalized at ∞.

It is a remarkable fact, that an extraordinar # of systems of great mathematical & physical interest can be expressed in terms of the solu of some associated RHP. This includes all the clas integrable systems and provides a gateway for integral methods into an extraordinary variety of a prior unrelated scientific problems.

found for OP's in the following. Recall that $p_k = \ldots$ when π_k is monic, $\pi_k = x^k + \cdots$.

Let $\Sigma = \mathbb{R}$, oriented from L to R



Let $v(z) = \begin{pmatrix} 1 & e^{-V(z)} \\ 0 & 1 \end{pmatrix}$, $z \in \mathbb{R}$. Fix k

$\boxed{\cancel{\text{Then the}}}$ Let $Y = Y^{(k)} = (Y_{ij})_{1\le i, j \le 2}$ be the (unique) 2×2 solution of the RHP (Σ, v)

- $Y(z)$ analytic in $\mathbb{C} \setminus \mathbb{R}$
- $Y_+(z) = Y_-(z)\, v(z)$, $z \in \mathbb{R}$
- $Y(z) \begin{pmatrix} z^{-k} & 0 \\ 0 & z^{k} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as $z \to \infty$

Then

(14) $$\pi_k(x) = Y_{11}(x)$$

and

(15) $$c_{k-1} = \sqrt{\frac{(Y_{21})_1}{-2\pi i}}$$

where

(16) $$Y_{21}(x) = (Y_{21})_1\, x^{k-1} + \cdots$$

<u>Proof</u> Now I haven't proved anything so far in these lectures, so let me prove (14). As you'll see its easy and its fun. I will also prove uniqu[eness] because the method of proof will be important in the 3rd lecture. Consider the first row of the jump condition.

$$(17) \qquad (Y_{11} \;\; Y_{12})_+ = (Y_{11} \;\; Y_{12})_- \begin{pmatrix} 1 & e^{-V(z)} \\ 0 & 1 \end{pmatrix} \qquad , z \in \mathbb{R}$$

Hence $(Y_{11})_+(z) = (Y_{11})_-(z)$ and hence Y_{11} is entire. But

$$(18) \qquad (Y_{11} \;\; Y_{12}) \begin{pmatrix} z^{-k} & 0 \\ 0 & z^{k} \end{pmatrix} = (Y_{11} z^{-k} \;\; Y_{12} z^{k}) \to (1 \;\; 0)$$

as $z \to \infty$ $\therefore$ $Y_{11}(z)$ is a monomial of order k, by Liouville's Theorem: $Y_{11}(z) = z^k + \cdots$. On the other hand, from 17, we see that

$$(Y_{12})_+ = (Y_{12})_- + Y_{11} e^{-V(z)}$$

and hence by the Plemelj formula (by (18) $Y_{12}(z) \to 0$ as $z \to \infty$)

Expanding in powers of $1/z$,

$$Y_{12}(z) = -\frac{1}{z}\frac{1}{2\pi i}\int_{\mathbb{R}} Y_{11}(s)\, e^{-V(s)} \left(1 + \frac{s}{z} + \frac{s^2}{z^2} + \cdots + \frac{s^{k-1}}{z^{k-1}} + \frac{s^k}{z^k} + \cdots\right)$$

But from (18), $Y_{12} = o\left(\frac{1}{z^k}\right)$ as $z \to \infty$

Hence

$$\int_{\mathbb{R}} Y_{11}(s)\, e^{-V(s)}\, s^j\, ds = 0, \quad 0 \le j \le k-1$$

By the construction of the orthogonal polynomials it follows that

$$Y_{11}(s) = \pi_k(s)$$

Insert (17+)

Fokas, Its and Kitaev arrived at their RHP th ~~by~~ their investigation of 2-D quantum gravity, at is an integrable system, as mentioned earlier. A special case of this RHP also appeared ~'94 in work on the Toda rarefaction problem.

In the third lecture we will show how to solve the RHP asymptotically as $N = k \to \infty$

Insert on p17.

Now we prove uniqueness. Note that on $\mathbb{R}$

$$(\det Y)_+ = (\det Y)_- \det v(z) = (\det Y_-)$$

and hence $\det Y(z)$ is entire. But as $z \to \infty$

$$\det Y(z) = \det \left(Y(z) \begin{pmatrix} z^{-k} & 0 \\ 0 & z^k \end{pmatrix} \right) \to 1 \quad \text{as } z \to \infty$$

Hence, again by Liouville, $\det Y(z) \equiv 1$. In particular $Y^{-1}(z)$ is non-singular.
Now let $\tilde{Y}$ be a second solution of the RHP. Set

$$\cancel{\text{Then}} \quad H(z) = \tilde{Y}(z)\, Y(z)^{-1} \qquad \text{Then on } \mathbb{R},$$

$$H_+(z) = \tilde{Y}_+ Y_+^{-1} = (\tilde{Y}_- v)(Y_- v)^{-1} = \tilde{Y}_- v v^{-1} Y_-^{-1}$$
$$= H_-(z)$$

Hence $H(z)$ is entire, and as

$$H(z) = \left(\tilde{Y}(z) \begin{pmatrix} z^{-k} & 0 \\ 0 & z^k \end{pmatrix} \right) \left(Y(z) \begin{pmatrix} z^{-k} & 0 \\ 0 & z^k \end{pmatrix} \right)^{-1}$$

$$\to I \times I = I \quad \text{as } z \to \infty,$$

it follows that $H(z) \equiv I$ $\therefore$
$\tilde{Y}(z) = Y(z)$